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Oscillation of a first order linear impulsive delay differential equation with continuous and piecewise constant arguments

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Abstract

A class of first order linear impulsive delay differential equation with continuous and piecewise constant arguments is studied. Using a connection between impulsive delay differential equations and non-impulsive delay differential equations sufficient conditions for the oscillation of the solutions are obtained.

Keywords: Oscillation, Delay, Piecewise constant argument, Impulse.

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1. Introduction

In this paper, we consider an impulsive delay differential equation with continuous and piecewise constant arguments of the form

$$x'(t) + a(t)\,x(t) + b(t)x(t-\tau) + c(t)x([t-1]) = 0,\ t \neq t_i,\ t \geq t_0 > 0, \tag{1.1}$$

$$\Delta x(t_i) = b_i x(t_i),\ i = 1, 2, ..., \tag{1.2}$$

where $a \in C([0,\infty),\mathbb{R}),\ b, c \in C([0,\infty),[0,\infty)),\ \tau \in \mathbb{R}^+$ is a fixed constant, $[.]$ denotes the greatest integer function, $\{t_i\}$ is a sequence of real numbers such that $0 < t_0 < t_1 < t_2 < ... < t_j < t_{j+1} < ...$, and $\lim\limits_{i\to\infty} t_i = \infty,\ \Delta x(t_i) = x(t_i^+) - x(t_i^-),\ x(t_i^+) = \lim\limits_{t\to t_i^+} x(t),\ x(t_i^-) = \lim\limits_{t\to t_i^-} x(t),\ b_i \neq 1,\ i = 1, 2, ...,$ are constants.

Since 1980's differential equations with piecewise constant arguments have been attracted great deal of attention of researchers in mathematical and some of the others fields in science. Piecewise constant systems exist in a widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, etc. These kind of equations such as Eq. (1.1) are similar in structure to those found in certain sequential-continuous models of disease dynamics [1]. In 1994, Dai and Singh [2] studied the oscillatory motion of spring-mass systems with subject to piecewise constant forces of the form $f(x[t])$ or $f([t])$. Later, they

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improved an analytical and numerical method for solving linear and nonlinear vibration problems and they showed that a function $f([N(t)]/N)$ is a good approximation to the given continuous function $f(t)$ if N is sufficiently large [3].
In 1984, Cooke and Wiener [4] studied oscillatory and periodic solutions of a linear differential equation with piecewise constant argument and they note that such equations are comprehensively related to impulsive and difference equations. After this work, oscillatory and periodic solutions of linear differential equations with piecewise constant arguments have been dealt with by many authors [5, 6, 7] and the references cited therein. On the other hand, in 1994, the case of studying discontinuous solutions of differential equations with piecewise continuous arguments has been proposed as an open problem by Wiener [8]. Due to this open problem, some impulsive differential equations with piecewise constant arguments have been studied [9, 10, 11]. Moreover, the monographs [12, 13] include many results on the theory of differential equations with piecewise constant arguments.
Now, our aim is to consider the Wiener's open problem for the equation (1.1)-(1.2). Moreover, as we know there is only one work on nonimpulsive delay differential equations with continuous and piecewise constant arguments [14]. In this respect, we obtain sufficient conditions for the existence of oscillatory solutions of Eq. (1.1)-(1.2).

1.1. Definition. It is said that a function x defined on the set $\{-1\} \cup [-\tau, \infty)$ is a solution of Eq. (1.1)-(1.2) if it satisfies the following conditions:
(D_1) $x(t)$ is continuous on $[-\tau, \infty)$ with the possible exception of the points $t_i,\ i = 1, 2, ...$
(D_2) $x(t)$ is right continuous and has left-hand limit at the points $t_i,\ i = 1, 2, ...$
(D_3) $x(t)$ is differentiable and satisfies (1.1) for any $t \in \mathbb{R}^+$, with the possible exception of the points $t_i,\ i = 1, 2, ...$, and $[t] \in [0, \infty)$, where one-sided derivatives exist,
(D_4) $x(t)$ satisfies (1.2) at the points $t_i,\ i = 1, 2, ...$

1.2. Definition. A function $x(t)$ is called oscillatory if it is neither eventually positive nor eventually negative for $t \geq T$ where T is sufficiently large. Otherwise, the solution is called nonoscillatory.

1.3. Remark. In this paper we assume that $-\infty < b_i < 1$ for all $i = 1, 2, ...$ Otherwise, from the impulse conditions (1.2) it is obtained that the solutions are already oscillatory.

1.4. Remark. We assume that $b(t) \not\equiv 0$ or $c(t) \not\equiv 0$. If $b(t) \equiv 0$ and $c(t) \equiv 0$, then Eq. (1.1)-(1.2) reduce to an ordinary differential equation with impulses. The results on the oscillation of impulsive ordinary differential equations can be found in the survey paper [15].

2. Main Results

In this paper we also consider following differential inequalities.

$$\begin{aligned} x'(t) + a(t)x(t) + b(t)x(t-\tau) + c(t)x([t-1]) &\leq 0,\ t \neq t_i,\ t \geq t_0 > 0, \\ \Delta x(t_i) &= b_i x(t_i),\ i = 1, 2, ..., \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} x'(t) + a(t)x(t) + b(t)x(t-\tau) + c(t)x([t-1]) &\geq 0,\ t \neq t_i,\ t \geq t_0 > 0, \\ \Delta x(t_i) &= b_i x(t_i),\ i = 1, 2, ... \end{aligned} \tag{2.2}$$

The main tools for the proofs of our results are following differential equation and inequalities.

$$y'(t) + a(t)y(t) + B(t)y(t-\tau) + C(t)y([t-1]) = 0,\ t \geq t_0 + \max\{\tau, 2\} \tag{2.3}$$

$$y'(t) + a(t)\,y(t) + B(t)y(t-\tau) + C(t)y([t-1]) \le 0,\ \ t \ge t_0 + \max\{\tau, 2\}, \tag{2.4}$$

$$y'(t) + a(t)\,y(t) + B(t)y(t-\tau) + C(t)y([t-1]) \ge 0,\ \ t \ge t_0 + \max\{\tau, 2\}, \tag{2.5}$$

where

$$B(t) = \prod_{t-\tau < t_j \le t} (1-b_j)b(t),\ \ t \ge t_0 + \max\{\tau, 2\}, \tag{2.6}$$

and

$$C(t) = \prod_{[t-1] < t_j \le t} (1-b_j)c(t),\ \ t \ge t_0 + \max\{\tau, 2\}. \tag{2.7}$$

The following theorem is a generalization of Theorem 1 in [16] to impulsive delay differential equations with continuous and piecewise constant arguments.

2.1. Theorem. (i) *Inequality (2.1) has no eventually positive solution if and only if inequality (2.4) has no eventually positive solution.*
(ii) *Inequality (2.2) has no eventually negative solution if and only if inequality (2.5) has no eventually negative solution.*
(iii) *All solutions of the equation (1.1)-(1.2) are oscillatory if and only if all solutions of equation (2.3) are oscillatory.*

Proof. We will prove only (i) since the proofs of (ii) and (iii) are similar. Let $x(t)$ be an eventually positive solution of inequality (2.1) such that $x(t) > 0,\ x(t-\tau) > 0,\ x([t-1]) > 0$ for $t > T \ge t_0 + \max\{\tau, 2\}$, where T is sufficiently large. Set $y(t) = \prod_{T < t_j \le t} (1-b_j)x(t)$. Since $1 - b_j > 0$, it is clear that $y(t) > 0,\ y(t-\tau) > 0$, and $y([t-1]) > 0$ for $t > T$. Now, we will show that $y(t)$ is a solution of inequality (2.4). From (2.6), (2.7), and (2.1) we obtain that for $t \ne t_i$ and $n \le t < n+1$

$$\begin{aligned}
&y'(t) + a(t)\,y(t) + B(t)y(t-\tau) + C(t)y([t-1]) \\
&\quad= \prod_{T<t_j\le t}(1-b_j)x'(t) + a(t)\prod_{T<t_j\le t}(1-b_j)x(t) \\
&\qquad+ \prod_{t-\tau<t_j\le t}(1-b_j)b(t)\prod_{T<t_j\le t-\tau}(1-b_j)x(t-\tau) \\
&\qquad+ \prod_{[t-1]<t_j\le t}(1-b_j)c(t)\prod_{T<t_j\le [t-1]}(1-b_j)x([t-1]) \\
&\quad= \prod_{T<t_j\le t}(1-b_j)\left[x'(t) + a(t)\,x(t) + b(t)x(t-\tau) + c(t)x([t-1])\right] \\
&\quad\le 0.
\end{aligned}$$

So, $y(t)$ is an eventually positive solution of inequality (2.4). On the other hand, from (1.2), we have

$$\begin{aligned}
y(t_i^-) &= \prod_{T<t_j\le t_{i-1}}(1-b_j)x(t_i^-) \\
&= \prod_{T<t_j\le t_i}(1-b_j)x(t_i) \\
&= y(t_i)
\end{aligned}$$

and

$$\begin{aligned}
y(t_i^+) &= \prod_{T<t_j\le t_i}(1-b_j)x(t_i^+)\\
&= \prod_{T<t_j\le t_i}(1-b_j)x(t_i)\\
&= y(t_i).
\end{aligned}$$

So, $y(t)$ is continuous at the impulse points.
Now, let $y(t)$ be an eventually positive solution of inequality (2.4). Then $y(t)>0$, $y(t-\tau)>0$, and $y([t-1])>0$ for $t>T$. We will show that $x(t)$ is an eventually positive solution of inequality (2.1). From (2.6), (2.7), and (2.4) we obtain that for $t\neq t_i$ and $n\le t<n+1$

$$\begin{aligned}
&x'(t)+a(t)x(t)+b(t)x(t-\tau)+c(t)x([t-1])\\
&\quad= \prod_{T<t_j\le t}(1-b_j)^{-1}y'(t)+a(t)\prod_{T<t_j\le t}(1-b_j)^{-1}y(t)\\
&\qquad +b(t)\prod_{T<t_j\le t-\tau}(1-b_j)^{-1}y(t-\tau)+c(t)\prod_{T<t_j\le [t-1]}(1-b_j)^{-1}y([t-1])\\
&\quad= \prod_{T<t_j\le t}(1-b_j)^{-1}\left[y'(t)+a(t)y(t)+B(t)y(t-\tau)+C(t)y([t-1])\right]\\
&\quad\le 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x(t_i^-) &= \prod_{T<t_j\le t_{i-1}}(1-b_j)^{-1}y(t_i^-)\\
&= \prod_{T<t_j\le t_i}(1-b_j)^{-1}(1-b_i)y(t_i)\\
&= (1-b_i)x(t_i)
\end{aligned}$$

and

$$x(t_i^+)=\prod_{T<t_j\le t_i}(1-b_j)^{-1}y(t_i^+)=x(t_i).$$

So, $x(t)$ is an eventually positive solution of inequality (2.1). The proof is complete.

□

Following we give several sufficient conditions for the oscillation of equation (1.1)-(1.2).

2.2. Theorem. *If one of the following conditions be satisfied then every solution of equation (1.1)-(1.2) is oscillatory:*

$$\text{(2.8)}\qquad \limsup_{t\to\infty}\int_{t-l}^{t}\left(\prod_{s-\tau<t_j\le s}(1-b_j)\right)b(s)\exp\left(\int_{s-\tau}^{s}a(u)du\right)ds>1,$$

$$\text{(2.9)}\qquad \limsup_{n\to\infty}\int_{n+1-l}^{n+1}\left(\prod_{n-1<t_j\le s}(1-b_j)\right)c(s)\exp\left(\int_{n-1}^{s}a(u)du\right)ds>1,$$

where $l=\min\{\tau,1\}$.

Proof. Let condition (2.8) or (2.9) is satisfied. We shall prove that the existence of eventually positive (or negative) solutions leads to a contradiction. Let $x(t)$ be an eventually positive solution of equation (1.1)-(1.2). Then $y(t) = \prod\limits_{T<t_j\leq t}(1-b_j)x(t)$ is an eventually positive solution of equation (2.3) such that $y(t)>0,\ y(t-\tau)>0,\ y([t-1])>0$ for $n+1>t\geq n>T$. Taking

$$(2.10)\qquad z(t) = y(t)\exp\left(\int\limits_T^t a(s)ds\right),\ t>T,$$

it is obtained from equation (2.3) that

$$(2.11)\quad z'(t) = -\left\{B(t)z(t-\tau)\exp\left(\int\limits_{t-\tau}^t a(s)ds\right) + C(t)z([t-1])\exp\left(\int\limits_{[t-1]}^t a(s)ds\right)\right\}$$

for $n+1>t\geq n>T$ and $t\neq t_i$. Since $B(t),\ C(t)\geq 0$ for $t\in\mathbb{R}$ and $z(t-\tau),\ z([t-1])\geq 0$ for $n+1>t\geq n>T,$ we get $z(t)$ is nonincreasing for $t>T$.
Now, we consider two cases:

Case 1. $\tau>1$. Then it is clear that $z(t-\tau)\geq z(t-1)$ and $z([t-1])\geq z(t-1)$ for $t>T$. Using (2.11), we obtain that

$$\begin{aligned} 0 &= z'(t) + B(t)z(t-\tau)\exp\left(\int\limits_{t-\tau}^t a(s)ds\right)\\ &+ C(t)z([t-1])\exp\left(\int\limits_{[t-1]}^t a(s)ds\right)\\ &\geq z'(t) + z(t-1)P(t), \qquad (2.12)\end{aligned}$$

where

$$(2.13)\qquad P(t) = B(t)\exp\left(\int\limits_{t-\tau}^t a(s)ds\right) + C(t)\exp\left(\int\limits_{[t-1]}^t a(s)ds\right).$$

Integrating inequality (2.12) from $t-1$ to t, we get

$$z(t)-z(t-1)+\int\limits_{t-1}^t P(s)z(s-1)ds\leq 0.$$

Since $z(t)$ is nonincreasing for $t>T$, from the above inequality, we obtain that

$$z(t)+z(t-1)\left[\int\limits_{t-1}^t P(s)ds-1\right]\leq 0$$

and so, we have

$$\int\limits_{t-1}^t P(s)ds\leq 1.$$

Using (2.13), (2.6), and (2.7), we obtain from the above inequality that

$$\int_{t-1}^{t} \prod_{s-\tau<t_j\le s}(1-b_j)b(s)\exp\left(\int_{s-\tau}^{s} a(u)du\right)ds \le 1, \tag{2.14}$$

and

$$\int_{t-1}^{t} \prod_{[s-1]<t_j\le s}(1-b_j)c(s)\exp\left(\int_{[s-1]}^{s} a(u)du\right)ds \le 1.$$

It is clear that inequality (2.14) contradicts (2.8). On the other hand, integrating inequality (2.12) from n to $n+1$, we get

$$z(n+1)-z(n)+\int_{n}^{n+1} P(s)z(s-1)ds \le 0.$$

Since $z(t)$ is nonincreasing for $t>T$, from the above inequality, we obtain that

$$z(n+1)+z(n)\left[\int_{n}^{n+1} P(s)ds-1\right] \le 0$$

and so, we have

$$\int_{n}^{n+1} P(s)ds \le 1.$$

In view of (2.13), (2.6), and (2.7), we obtain from the above inequality that

$$\int_{n}^{n+1} \prod_{s-\tau<t_j\le s}(1-b_j)b(s)\exp\left(\int_{s-\tau}^{s} a(u)du\right)ds \le 1,$$

and

$$\int_{n}^{n+1} \prod_{[s-1]<t_j\le s}(1-b_j)c(s)\exp\left(\int_{[s-1]}^{s} a(u)du\right)ds \le 1. \tag{2.15}$$

Since $n\le s<n+1$, (2.15) contradicts (2.9).

Case 2. $\tau\le 1$. Then $z(t-\tau)\le z([t-1])$ for $n+1>t\ge n>T$, and from (2.11), we obtain that

$$\begin{aligned} 0 &= z'(t)+B(t)z(t-\tau)\exp\left(\int_{t-\tau}^{t} a(s)ds\right) \\ &+ C(t)z([t-1])\exp\left(\int_{[t-1]}^{t} a(s)ds\right) \\ &\ge z'(t)+z(t-\tau)P(t), \end{aligned} \tag{2.16}$$

where $P(t)$ is defined in (2.13). Integrating inequality (2.16) from $t-\tau$ to t, we get

$$z(t)-z(t-\tau)+\int_{t-\tau}^{t} P(s)z(s-\tau)ds \le 0.$$

Since $z(t)$ is nonincreasing for $t > T$, from the above inequality, we obtain that

$$z(t) + z(t-\tau)\left[\int_{t-\tau}^{t} P(s)ds - 1\right] \le 0$$

and so, we have

$$\int_{t-\tau}^{t} P(s)ds \le 1.$$

Using (2.13), (2.6), and (2.7), we obtain from the above inequality that

$$\int_{t-\tau}^{t}\left(\prod_{s-\tau<t_j\le s}(1-b_j)\right) b(s)\exp\left(\int_{s-\tau}^{s} a(u)du\right) ds \le 1$$

which contradicts (2.8). On the other hand, integrating inequality (2.16) from $n+1-\tau$ to $n+1$, we get

$$z(n+1) - z(n+1-\tau) + \int_{n+1-\tau}^{n+1} P(s)z(s-\tau)ds \le 0.$$

Since $z(t)$ is nonincreasing for $t > T$, from the above inequality, we obtain that

$$\int_{n+1-\tau}^{n+1} P(s)ds \le 1.$$

In view of (2.13), (2.6), and (2.7), we obtain from the above inequality that

$$\int_{n+1-\tau}^{n+1}\left(\prod_{n-1<t_j\le s}(1-b_j)\right) c(s)\exp\left(\int_{n-1}^{s} a(u)du\right) ds \le 1$$

which contradicts (2.9).
If $x(t)$ is an eventually negative solution of equation (1.1)-(1.2), then $-x(t)$ is an eventually positive solution of equation (1.1)-(1.2) and we obtain the same contradiction. So, the proof is complete. □

2.3. Corollary. *Assume that $b(t) \ne 0$, $c(t) \equiv 0$ and that*

$$\limsup_{t\to\infty} \int_{t-\tau}^{t}\left(\prod_{s-\tau<t_j\le s}(1-b_j)\right) b(s)\exp\left(\int_{s-\tau}^{s} a(u)du\right) ds > 1. \tag{2.17}$$

Then every solution of Eq. (1.1)-(1.2) is oscillatory.

2.4. Remark. If $b(t) \ne 0$ and $c(t) \equiv 0$, then Eq. (1.1)-(1.2) reduces to a delay differential equation with impulses. Condition (2.17) is similar to hypothesis of Theorem $3'$ in [16]. The difference between the hypotheses occurs because of the right continuity of the solution instead of left continuity.

More results on the oscillation of impulsive delay differential equations can be found in the survey paper [17].

2.5. Corollary. *Assume that* $b(t) \equiv 0,\ c(t) \neq 0$ *and that*

$$(2.18) \qquad \limsup_{n\to\infty} \int_{n}^{n+1} \left(\prod_{n-1<t_j\le s} (1-b_j) \right) c(s) \exp\left(\int_{n-1}^{s} a(u)du \right) ds > 1.$$

Then every solution of Eq. (1.1)-(1.2) is oscillatory.

2.6. Remark. If $b(t) \equiv 0,\ c(t) \neq 0$, then Eq. (1.1)-(1.2) reduces to an impulsive differential equation with piecewise constant argument. Eq. (1.1)-(1.2) with $b(t) \equiv 0$, and $t_i = i,\ i = 1, 2, \ldots$ has been investigated in [9]. So, Corollary 2.5 is a generalization of Theorem 4 in [9].
Moreover, in [9], a difference equation is a main tool for the proofs. Similarly, in the other works such as [4, 5, 6, 7, 10, 11, 18, 19] the relation between difference equations and differential equations with piecewise constant arguments are underlined. Here, because of the existence of continuous argument, we have diffuculty to obtain related difference equation. So, we apply another technique which is worked for delay differential equations.

2.7. Theorem. *If one of the following conditions is satisfied then every solution of equation (1.1)-(1.2) is oscillatory:*

$$(2.19) \qquad \liminf_{t\to\infty} \int_{t-l}^{t} \left(\prod_{s-\tau<t_j\le s} (1-b_j) \right) b(s) \exp\left(\int_{s-\tau}^{s} a(u)du \right) ds > \frac{1}{e},$$

$$(2.20) \qquad \liminf_{n\to\infty} \int_{n+1-l}^{n+1} \left(\prod_{n-1<t_j\le s} (1-b_j) \right) c(s) \exp\left(\int_{n-1}^{s} a(u)du \right) ds > \frac{1}{e},$$

where $l = \min\{\tau, 1\}$.

Proof. Let condition (2.19) or (2.20) is satisfied. We shall prove that the existence of eventually positive (or negative) solutions leads to a contradiction. Let $x(t)$ be an eventually positive solution of equation (1.1)-(1.2). Then $y(t) = \prod_{T<t_j\le t} (1-b_j)x(t)$ is an eventually positive solution of equation (2.3) such that $y(t) > 0,\ y(t-\tau) > 0,\ y([t-1]) > 0$ for $n+1 > t \ge n > T$. Using the same arguments in the proof of Theorem 2.2, we obtained that $z(t)$ defined in (2.10) is nonincreasing for $t > T$. We consider two cases:

Case 1. $\tau > 1$. Dividing inequality (2.12) by $z(t)$, and then integrating from $t-1$ to t, it is obtained that

$$(2.21) \qquad \ln \frac{z(t-1)}{z(t)} \ge \int_{t-1}^{t} P(s) \frac{z(s-1)}{z(s)} ds,$$

where $P(t)$ is defined in (2.13). Since $e^x \ge ex$ for $x \in \mathbb{R}$, we obtain that

$$\begin{aligned} \frac{z(t-1)}{z(t)} &\ge \exp\left(\int_{t-1}^{t} P(s) \frac{z(s-1)}{z(s)} ds \right) \\ &\ge e \left(\int_{t-1}^{t} P(s) \frac{z(s-1)}{z(s)} ds \right). \qquad (2.22) \end{aligned}$$

Let $u(t) = \dfrac{z(t-1)}{z(t)}$. Since $z(t)$ is nonincreasing for $t > T$, $\liminf\limits_{t\to\infty} u(t) \ge 1$.

Assume that $\liminf\limits_{t\to\infty} u(t) = +\infty$. Then integrating inequality (2.12) from $t - \dfrac{1}{2}$ to t, we have

$$z(t) - z(t-\frac{1}{2}) + \int\limits_{t-\frac{1}{2}}^{t} P(s)z(s-1)ds \le 0.$$

Since $z(t)$ is nonincreasing, from the above inequality, we obtain that

$$z(t) - z(t-\frac{1}{2}) + z(t-1)\int\limits_{t-\frac{1}{2}}^{t} P(s)ds \le 0. \tag{2.23}$$

Dividing inequality (2.23) by $z(t)$ and $z(t-\frac{1}{2})$, we get

$$1 - \frac{z(t-\frac{1}{2})}{z(t)} + \frac{z(t-1)}{z(t)}\int\limits_{t-\frac{1}{2}}^{t} P(s)ds \le 0, \tag{2.24}$$

and

$$\frac{z(t)}{z(t-\frac{1}{2})} - 1 + \frac{z(t-1)}{z(t-\frac{1}{2})}\int\limits_{t-\frac{1}{2}}^{t} P(s)ds \le 0, \tag{2.25}$$

respectively. Now, from (2.24) we obtain

$$\liminf_{t\to\infty}\frac{z(t-\frac{1}{2})}{z(t)} = +\infty$$

which contradicts with (2.25). So, $\liminf\limits_{t\to\infty} u(t)$ is finite.
If $\liminf\limits_{t\to\infty} u(t) = w,\ w \ge 1$ is finite, then inequality (2.22) implies that

$$\lim_{t\to\infty}\,\inf\int\limits_{t-1}^{t} P(s)ds \le \frac{1}{e}.$$

In view of (2.13), (2.6), and (2.7), we obtain from the above inequality that

$$\lim_{t\to\infty}\,\inf\int\limits_{t-1}^{t}\left(\prod_{s-\tau<t_j\le s}(1-b_j)\right)b(s)\exp\left(\int\limits_{s-\tau}^{s} a(u)du\right)ds \le \frac{1}{e},$$

which contradicts the hypothesis (2.19).
Now, dividing inequality (2.12) by $z(t)$, and then integrating from n to $n+1$, it is obtained that

$$\ln\frac{z(n)}{z(n+1)} \ge \int\limits_{n}^{n+1} P(s)\frac{z(s-1)}{z(s)}ds,$$

where $P(t)$ is defined in (2.13). Since $e^x \ge ex$ for $x \in \mathbb{R}$, we obtain that

$$\frac{z(n)}{z(n+1)} \ge e\left(\int\limits_{n}^{n+1} P(s)\frac{z(s-1)}{z(s)}ds\right). \tag{2.26}$$

Define $v(n) = \dfrac{z(n)}{z(n+1)}$. Since $z(t)$ is nonincreasing for $t > T$, $\liminf_{n\to\infty} v(n) \geq 1$. By doing the same calculations with first part of the proof, we get that $\liminf_{n\to\infty} v(n)$ is finite. Therefore, from the inequality (2.26), we have

$$\lim_{n\to\infty}\inf \int_{n}^{n+1} \left(\prod_{n-1<t_j\leq s} (1-b_j) \right) c(s) \exp\left(\int_{n-1}^{s} a(u)du \right) ds \leq \frac{1}{e},$$

which contradicts (2.20).

Case 2. $\tau \leq 1$. Since the proof is similar to proof of *Case 1*, we shall give the sketch of the proof. Dividing inequality (2.16) by $z(t)$, and then integrating from $t-\tau$ to t, it is obtained that

$$(2.27) \qquad \frac{z(t-\tau)}{z(t)} \geq e\left(\int_{t-\tau}^{t} P(s)\frac{z(s-\tau)}{z(s)}ds \right).$$

Using the similar arguments in *Case 1*, we get that $\liminf_{t\to\infty} \dfrac{z(t-\tau)}{z(t)}$ is finite. So, from inequality (2.27), we have

$$\lim_{t\to\infty}\inf \int_{t-\tau}^{t} \left(\prod_{s-\tau<t_j\leq s} (1-b_j) \right) b(s) \exp\left(\int_{s-\tau}^{s} a(u)du \right) ds \leq \frac{1}{e},$$

which contradicts (2.19).
Moreover, dividing inequality (2.16) by $z(t)$, and then integrating from $n+1-\tau$ to $n+1$, it is obtained that

$$(2.28) \qquad \frac{z(n+1-\tau)}{z(n+1)} \geq e\left(\int_{n+1-\tau}^{n+1} P(s)\frac{z(s-\tau)}{z(s)}ds \right).$$

By using the similar arguments in *Case 1*, we get that $\liminf_{n\to\infty} \dfrac{z(n+1-\tau)}{z(t)}$ is finite. So, from inequality (2.28), we have

$$\lim_{n\to\infty}\inf \int_{n+1-\tau}^{n+1} \left(\prod_{n-1<t_j\leq s} (1-b_j) \right) c(s) \exp\left(\int_{n-1}^{s} a(u)du \right) ds \leq \frac{1}{e},$$

which contradicts (2.20). So, the proof is complete. □

2.8. Corollary. *Assume that $b(t) \neq 0$, $c(t) \equiv 0$ and that*

$$\lim_{t\to\infty}\inf \int_{t-\tau}^{t} \left(\prod_{s-\tau<t_j\leq s} (1-b_j) \right) b(s) \exp\left(\int_{s-\tau}^{s} a(u)du \right) ds > \frac{1}{e}.$$

Then every solution of Eq. (1.1)-(1.2) is oscillatory.

2.9. Corollary. *Assume that $b(t) \equiv 0$, $c(t) \neq 0$ and that*

$$\lim_{n\to\infty}\inf \int_{n}^{n+1} \left(\prod_{n-1<t_j\leq s} (1-b_j) \right) c(s) \exp\left(\int_{n-1}^{s} a(u)du \right) ds > \frac{1}{e}.$$

Then every solution of Eq. (1.1)-(1.2) is oscillatory.

Now, we give some examples to illustrate our results. Note that previous results in the litarature can not be applied following differential equations to obtain existence of oscillatory solutions.

2.10. Example. Let us consider the following differential equation

$$(2.29)\quad \begin{cases} x'(t) + \pi x(t - \frac{1}{2}) + c(t)x([t-1]) = 0, \; t \neq n, \; n = 1, 2, ..., \; t \geq 0, \\ x(n^+) - x(n^-) = -x(n^+), \; n = 1, 2, ..., \end{cases}$$

where $c(t) \geq 0$ is any continuous function. It can be shown that the hypotheses of Theorem 2.2 as well as Theorem 2.7 are satisfied. So, all solutions of Eq. (2.29) are oscillatory. The solution $x(t)$ of the equation (2.29) with the initial condition $\phi(t) = \sin \pi t, -1/2 \leq t \leq 0$ is demonstrated in Figure 1.

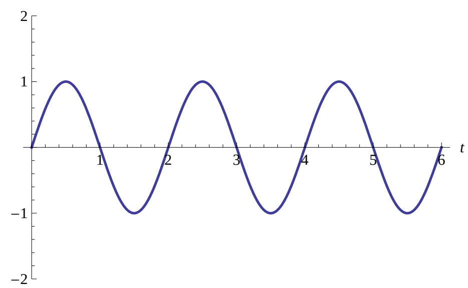

Figure 1. The solution $x(t)$ of the equation (2.29) with the initial condition $\phi(t) = \sin \pi t, -1/2 \leq t \leq 0$.

2.11. Example. Consider the following differential equation

$$(2.30)\quad \begin{cases} x'(t) + x(t) + \pi x(t - \frac{5}{2}) + e^t x([t-1]) = 0, \; t \neq t_n, \; n = 1, 2, ..., \; t \geq 0, \\ x(t_n^+) - x(t_n^-) = -2^n x(t_n^+), \; n = 1, 2, ..., \end{cases}$$

where $\{t_n\}_{n=1}^{\infty}$ is an increasing sequence such that $\lim_{n \to \infty} t_n = \infty$.

It is clear that $a(t) = 1, \; b(t) = \pi, \; c(t) = e^t, \; \tau = \frac{5}{2}$ and $b_n = -2^n$. It can be shown that the hypotheses of Theorem 2.2 as well as Theorem 2.7 are satisfied. So, all solutions of Eq. (2.30) are oscillatory. The solution $x(t)$ of the equation (2.30) with $\{t_n\} = 2n, n = 1, 2, ...,$ and the initial condition $\phi(t) = \exp(-t) \sin \pi t, -5/2 \leq t \leq 0$ is demonstrated in Figure 2.

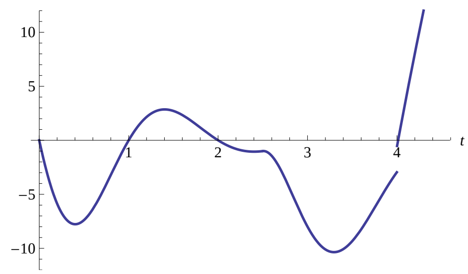

Figure 2. The solution $x(t)$ of the equation (2.30) with the initial condition $\phi(t) = \exp(-t)\sin \pi t$, $-5/2 \leq t \leq 0$.

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References

[1] Busenberg, S. and Cooke, K.L. *Models of vertically transmitted diseases with sequential-continuous dynamics*, In: Nonlinear Phenomena in Mathematical Sciences, V. Lakshmikantham (Ed.), Academic Press, 179-187, 1982.

[2] Dai, L. and Singh, M.C. *On oscillatory motion of spring mass systems subject to piesewise constant forces*, J. Sound Vib. 173, 217-233, 1994.

[3] Dai, L. and Singh, M.C. *An analytical and numerical method for solving linear and nonlinear vibration problems*, Int. J. Solids Struct. 34, 2709-2731, 1997.

[4] Cooke, K.L. and Wiener, J. *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. 99, 265-297, 1984.

[5] Aftabizadeh, A.R. and Wiener, J. *Oscillatory properties of first order linear functional differential equations*, Applicable Anal. 20, 165-187, 1985.

[6] Aftabizadeh, A.R., Wiener, J. and Ming Xu, J. *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument*, Proc. of American Math. Soc. 99, 673-679, 1987.

[7] Shen, J.H. and Stavroulakis, I.P. *Oscillatory and nonoscillatory delay equations with piecewise constant argument*, J. Math. Anal. Appl. 248, 385-401, 2000.

[8] Wiener, J. *Generalized Solutions of Functional Differential Equations*, World Scientific, Singapore, 1994.

[9] Karakoç, F., Bereketoglu H. and Seyhan, G. *Oscillatory and periodic solutions of impulsive differential equations with piecewise constant argument*, Acta Appl. Math. 110, 499-510, 2010.

[10] Bereketoglu, H., Seyhan G. and Ogun, A. *Advanced impulsive differential equations with piecewise constant arguments*, Math. Model. Anal. 15, 175-187, 2010.

[11] Karakoç, F., Ogun Unal, A. and Bereketoglu, H. *Oscillation of nonlinear impulsive differential equations with piecewise constant arguments*, E. J. Qualitative Theory of Diff. Equ. No. 49, 1-12, 2013.

[12] Gyori, I. and Ladas, G. *Oscillation Theory of Delay Differential Equations*, Oxford University Press, 1991.

[13] Akhmet, M. *Nonlinear Hybrid Continuous/Discrete-Time Models*, Atlantis Press, Springer, 2011.

[14] Gopalsamy, K., Gyori I. and Ladas, G. *Oscillations of a class of delay equations with continuous and piecewise constant arguments*, Funkcialaj Ekvacioj, 32, 395-406, 1989.

[15] Agarwal, R.P., Karakoç, F. and Zafer, A. *A survey on oscillation of impulsive ordinary differential equations*, Advances in Difference Equations, Volume 2010, Article ID 354841, 52 pages, doi:10.1155/2010/354841.

[16] Yan J. and Kou, C. *Oscillation of solutions of impulsive delay differential equations*, J. Math. Anal. Appl. 254, 358-370, 2001.

[17] Agarwal, R.P. and Karakoç, F. *A survey on oscillation of impulsive delay differential equations*, Comput. Math. Appl. 60, 1648-1685, 2010.

[18] Chiu, K.S. and Pinto, M. *Periodic solutions of differential equations with a general piecewise constant argument and applications*, E. J. Qualitative Theory of Diff. Equ., No. 46, 1-19, 2010.

[19] Chiu, K.S. and Jeng, J.C. *Stability of oscillatory solutions of differential equations with general piecewise constant arguments of mixed type*, Mathematische Nachrichten 288(10), 1085-1097, 2015.