

Exponential decay for a neutral one-dimensional viscoelastic equation

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Abstract

In this work we consider a viscoelastic string subject to a delay of neutral type. The delay occurs in the second time derivative. Although delays are known by their destructive nature, here we prove an exponential decay result. We shall use the multiplier method and modify the classical energy by judicious choices of other functionals. This would lead to an appropriate differential inequality which allows us to conclude. It seems that this issue has never been discussed before in the literature.

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1. Introduction

Time-delay systems arouse the curiosity of many researchers in the last three decades. They appear often in real-world engineering problems and population dynamics [14,15,17,36]. Loosely speaking we have a delay whenever the response to an applied force is involved. Models with time-lags in their variables are often referred to as 'Functional Differential Equations' (FDEs) [14,15,17,36]. A special subclass of FDEs is the class of 'Neutral Delay Differential Equations' (NDDEs) [9,14,15,17,18,35,36,40]. It is composed of those differential equations involving derivatives (possibly the highest one) evaluated at previous time.

Researchers have been studying such FDEs not only because they constitute a natural extension of related ordinary differential equations but also and mainly because of their appearance in many real-world problems: heat exchange, ecology, electrodynamics, neutral networks, etc [14,15,17,36-38].

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It has been shown that time-delay may trigger complex behaviors of instability. Even small delays can destabilize a system which was initially stable [2,6-9,12,16]. This phenomenon provokes oscillation, imperfection and discomfort leading sometimes to destructions. To solve this problem, researchers have been trying to stabilize the system by different ways. Although many controls have been devised so far and many methods and results have been found, NDDEs remains a class of differential equations which is not well understood. It is also worth mentioning that besides the fact that systems are very reactive to small delays, on the contrary, they can be stabilized by 'large' neutral delays. In fact, neutral delays are sometimes deliberately inserted into the systems to improve the performance of the structure.

Of concern here is the problem

$$(1.1) \quad \begin{cases} u_{tt}(t, x) - p \int_0^x u_{tt}(t - \tau, y) dy = u_{xx}(t, x) - \int_0^t g(t - s) u_{xx}(s, x) ds \\ \text{in } (0, \infty) \times (0, 1), \\ u(t, x) = 0, t \in (0, \infty), x = 0, 1, \\ u(0, x) = u_0(x), u_t(0, x) = u_1(x), x \in (0, 1), \end{cases}$$

where the delay in the first integral in the equation is of neutral type and the second integral is a viscoelastic term (or a memory term). The (relaxation) function g is a (nonincreasing) continuously differentiable function and p is a real number. The functions $u_0(x)$ and $u_1(x)$ are given initial data and $\tau > 0$.

Second order NDDE appear, in general, in the study of vibrating masses attached to an elastic bar and also (as the Euler equation) in some variational problems [14,15,17,37].

Searching in the literature we were unable to find studies on the asymptotic behavior or stability of this problem. The ones we have found are concerned with either *ordinary* differential equations (see [14,15,18,33-35] and the references therein), or *first* order partial differential equations [26,32]. The few treated *second* order problems are concerned with delays in the state function or its first derivative rather than in the second derivative.

The equation in (1.1) without the neutral delay (*i.e.* $p = 0$) is a one-dimensional viscoelastic equation based on the Boltzmann principle [1]. Under certain assumptions, Boltzmann derived the constitutive relationship between the stress and the strain in the form

$$\sigma(t, x) = a\varepsilon(t, x) + \int_{-\infty}^t g(t - s) [\varepsilon(t, x) - \varepsilon(s, x)] ds.$$

This relation describes adequately viscoelastic materials such as polymers, certain glasses, rubber, concrete and soils. In case the kernel is integrable then this relation becomes

$$\sigma(t, x) = b\varepsilon(t, x) - \int_{-\infty}^t g(t - s)\varepsilon(s, x) ds, \quad b = a + \int_0^{\infty} g(s) ds.$$

Notice that the Newtonian viscosity may result as a limiting case or simply by taking $g = -\delta'$ where δ is the Dirac delta function.

The nonlinear case where u_{xx} is replaced by $\varphi(u_x)_x$ and $\psi(u_x)_x$ in the first and second terms in the right hand side of (1.1) has been investigated by Coleman and Gurtin [4]. The authors discussed the competition between the memory effect and the nonlinear elastic response. Subsequently, several other researchers worked on this issue (see Christensen [3]). We also mention C. M. Dafermos, J. A. Nohel, W. J. Hrusa, H. Engler, M. Renardy, V. J. Mizel, W. A. Day and A. C. Pipkin to cite but a few (see [5,8,19,24]). They have contributed considerably in developing this theory. Then, P. Cannarsa, D. Sforza, M. Fabrizio, B. Lazzari, C. Giorgi, J. E. Munoz Rivera, M. M. Cavalcanti, V. N. Domingos Cavalcanti, P. Martinez, V. Pata and many others took over the study of the (linear) problem. Some of them are cited in [13, 2-23, 27-31]. Roughly and briefly, it has been shown that the memory term produces an extremely weak dissipation but capable of

stabilizing the system. The rate of decay of the (classical) energy is shown to be of the same type as the one of the kernel but does not exceed it. The memory term may also take it over nonlinear sources and therefore prevent blow up in finite time.

In the next section, we shall introduce the functionals we will work with. Section 3 is devoted to the statement and proof of our result.

2. Preliminaries

In this section we prepare some material which will be used to prove our result. The existence and uniqueness for this type of problems and even more general ones has been discussed in [10,11,39,40]. Classical solutions exist in

$$D(A) := \{u \in H^2(0, 1) : u(0) = u(1) = 0\}$$

where $Aw := w''$, for regular initial data.

Let us define the functionals

$$(2.1) \quad \mathcal{E}(t) := \frac{1}{2} \left\{ \left\| u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right\|^2 + \left(1 - \int_0^t g(s) ds \right) \|u_x\|^2 + \int_0^1 (g \square u_x) dx \right\},$$

$$(2.2) \quad \phi_1(t) := - \int_0^1 \theta(x) \left[u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right] \times \left[u_x(t) - \int_0^t g(t-s) u_x(s) ds \right] dx,$$

$$(2.3) \quad \phi_2(t) := \int_0^1 \left[u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right] u dx,$$

$$(2.4) \quad \phi_3(t) := - \int_0^1 \left[u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right] \left(\int_0^t g(t-s)(u(t) - u(s)) ds \right) dx,$$

and

$$(2.5) \quad \phi_4(t) := e^{-\beta t} \int_0^1 \int_{t-\tau}^t e^{\beta(s+\tau)} |u_t|^2(s) ds dx, \quad t \geq 0$$

for some $\beta > 0$ (to be determined), where $\|\cdot\|$ stands for the L^2 -norm,

$$(g \square u_x)(t) := \int_0^t g(t-s) |u_x(t) - u_x(s)|^2 ds,$$

and $\theta(x)$ is a continuously differentiable function satisfying $\theta(0) = 0$ and $\theta(1) = 2$ (we may take simply $\theta(x) = 2x$).

Observe that the functional $\mathcal{E}(t)$, we shall work with is different from the 'classical' energy

$$(2.6) \quad E(t) := \frac{1}{2} [\|u_t\|^2 + \|u_x\|^2], \quad t \geq 0.$$

This latter functional (2.6) is not convenient to handle because of the appearance of higher order terms in its derivative which are difficult to estimate.

The second functional $\phi_1(t)$ is useful to eliminate some undesirable boundary terms which arise in the derivative of $\mathcal{E}(t)$. The functionals $\phi_2(t)$ and $\phi_3(t)$ will provide us with $-\|u_x\|^2$ and $-\|u_t\|^2$, respectively, when differentiated. As for $\phi_4(t)$, it will be used to take care of the neutral delay term. Assume that g is a nonnegative nonincreasing function such that

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0$$

for some positive constants ξ_1 and ξ_2 ,

$$(2.7) \quad \bar{g} := \int_0^\infty g(s) ds < 1$$

and let, for $t_* > 0$

$$(2.8) \quad g_* := \int_0^{t_*} g(s) ds > 0.$$

These conditions may be relaxed considerably. One may consider the larger classes of relaxation functions in [13,20-22,23,27-31].

2.1. Lemma. *The derivative of $\phi_1(t)$, along solutions of (1.1), is estimated as follows:*

$$\begin{aligned} \phi'_1(t) &\leq -[u_x(1) - \int_0^t g(t-s)u_x(s,1)ds]^2 + \left[\bar{\theta}' + \frac{\bar{\theta}}{2}(1+p)g(t)\right] \|u_x\|^2 \\ &+ \left[\left(p + \frac{1}{2}\right)\bar{\theta}' + \bar{\theta}\left(p + \delta + \frac{g(t)}{2}\right)\right] \|u_t\|^2 + \frac{p}{4}[\bar{\theta}' + \bar{\theta}(1+2g(t)+4\delta)] \\ &\times \|u_t(t-\tau)\|^2 + \left[\bar{\theta}' \int_0^t g(s)ds + \frac{\bar{\theta}}{4\delta} g(0) \xi_1(1+p)\right] \int_0^1 (g \square u_x) dx, \quad t \geq 0 \end{aligned}$$

for some $\delta > 0$ and where $\bar{\theta} = \max_{0 \leq x \leq 1} \theta(x)$.

Proof. Clearly, for $t \geq 0$

$$\begin{aligned} \phi'_1(t) &= -\int_0^1 \theta(x) \left(u_{xx} - \int_0^t g(t-s)u_{xx}(s)ds\right) \left(u_x - \int_0^t g(t-s)u_x(s)ds\right) dx \\ &- \int_0^1 \theta(x) \left(u_t - p \int_0^x u_t(t-\tau, y)dy\right) \left(u_{xt} - g(0)u_x - \int_0^t g'(t-s)u_x(s)ds\right) dx. \end{aligned}$$

To facilitate the estimation later, we write

$$-g(0)u_x - \int_0^t g'(t-s)u_x(s)ds = -g(t)u_x - \int_0^t g'(t-s)[u_x(s) - u_x(t)]ds, \quad t \geq 0.$$

Therefore, for $t \geq 0$, we have

$$\begin{aligned} \phi'_1(t) &= -\frac{1}{2} \int_0^1 \theta(x) \frac{\partial}{\partial x} \left(u_x - \int_0^t g(t-s)u_x(s)ds\right)^2 dx - \frac{1}{2} \int_0^1 \theta(x) \frac{\partial |u_t|^2}{\partial x} dx \\ &+ g(t) \int_0^1 \theta(x) u_t u_x dx + \int_0^1 \theta(x) u_t \int_0^t g'(t-s)[u_x(s) - u_x(t)]ds dx \\ &+ p \int_0^1 \theta(x) u_{xt} \int_0^x u_t(t-\tau, y)dy dx - pg(t) \int_0^1 \theta(x) u_x \int_0^x u_t(t-\tau, y)dy dx \\ &- p \int_0^1 \theta(x) \left(\int_0^x u_t(t-\tau, y)dy\right) \left(\int_0^t g'(t-s)[u_x(s) - u_x(t)]ds\right) dx. \end{aligned}$$

Applying integration by parts and Young inequality, we get

$$\begin{aligned} \phi'_1(t) &\leq -\frac{1}{2} \left[\theta(x) \left(u_x - \int_0^t g(t-s)u_x(s)ds\right)\right]_0^1 - \frac{1}{2} [\theta(x)|u_t|^2]_0^1 \\ &+ \frac{1}{2} \int_0^1 \theta'(x) \left\{ \left(1 - \int_0^t g(s)ds\right) u_x - \int_0^t g(t-s)[u_x(s) - u_x(t)]ds \right\}^2 dx \\ &+ \frac{1}{2} \int_0^1 \theta'(x) |u_t|^2 dx + g(t) \frac{\bar{\theta}}{2} \|u_t\|^2 + g(t) \frac{\bar{\theta}}{2} \|u_x\|^2 + \bar{\theta} \delta \|u_t\|^2 \\ &+ \frac{\bar{\theta}}{4\delta} \int_0^1 \left(\int_0^t g'(t-s)[u_x(s) - u_x(t)]ds\right)^2 dx + p \left[\theta(x)u_t \int_0^x u_t(t-\tau, y)dy\right]_0^1 \\ &- p \int_0^1 \theta'(x) u_t \int_0^x u_t(t-\tau, y)dy dx - p \int_0^1 \theta(x) u_t(t-\tau) u_t dx \\ &- pg(t) \int_0^1 \theta(x) u_x \int_0^x u_t(t-\tau, y)dy dx \\ &- p \int_0^1 \theta(x) \left(\int_0^x u_t(t-\tau, y)dy\right) \left(\int_0^t g'(t-s)[u_x(s) - u_x(t)]ds\right) dx, \quad t \geq 0. \end{aligned}$$

Further estimations lead to

$$\begin{aligned} \phi'_1(t) &\leq \frac{\bar{\theta}'}{2} \left\| \left(1 - \int_0^t g(s)ds\right) u_x - \int_0^t g(t-s)[u_x(s) - u_x(t)]ds \right\|^2 \\ &- \left[u_x(1) - \int_0^t g(t-s)u_x(s,1)ds \right]^2 + \left(\frac{\bar{\theta}'}{2} + g(t) \frac{\bar{\theta}}{2} + \bar{\theta} \delta \right) \|u_t\|^2 \\ &+ \frac{g(t)\bar{\theta}}{2} \|u_x\|^2 + \frac{\bar{\theta}}{4\delta} \left(\int_0^t -g'(s)ds\right) \left(\int_0^t -g'(t-s) \|u_x(t) - u_x(s)\|^2 ds\right) \\ &+ p \bar{\theta}' \|u_t\|^2 + \frac{p\bar{\theta}'}{4} \|u_t(t-\tau)\|^2 + p\bar{\theta} \|u_t\|^2 + \frac{p\bar{\theta}}{4} \|u_t(t-\tau)\|^2 \\ &+ g(t) \frac{p\bar{\theta}}{2} [\|u_x\|^2 + \|u_t(t-\tau)\|^2] + p\bar{\theta} \delta \|u_t(t-\tau)\|^2 \\ &+ \frac{p\bar{\theta}}{4\delta} \left(\int_0^t -g'(s)ds\right) \left(\int_0^t -g'(t-s) \|u_x(t) - u_x(s)\|^2 ds\right), \quad t \geq 0 \end{aligned}$$

or

$$\begin{aligned} \phi'_1(t) \leq & - \left[u_x(1) - \int_0^t g(t-s)u_x(s,1)ds \right]^2 + \left[\bar{\theta}' + (1+p)\frac{\bar{\theta}}{2}g(t) \right] \|u_x\|^2 \\ & + \left[\left(\frac{1}{2}+p\right)\bar{\theta}' + g(t)\frac{\bar{\theta}}{2} + (p+\delta)\bar{\theta} \right] \|u_t\|^2 + \frac{p}{4} \left[\bar{\theta}' + \bar{\theta}(1+2g(t)+4\delta) \right] \\ & \times \|u_t(t-\tau)\|^2 + \left[\bar{\theta}' \left(\int_0^t g(s)ds \right) + \frac{(p+1)\bar{\theta}}{4\delta}g(0)\xi_1 \right] \int_0^1 (g \square u_x) dx, \quad t \geq 0. \end{aligned}$$

■

2.2. Lemma. *The derivative of $\phi_2(t)$, along solutions of (1.1), satisfies*

$$\begin{aligned} \phi'_2(t) \leq & (1 + \delta_1 p) \|u_t\|^2 + \frac{p}{4\delta_1} \|u_t(t-\tau)\|^2 - (1 - \bar{g} - \delta) \|u_x\|^2 \\ & + \frac{1}{4\delta} \left(\int_0^t g(s)ds \right) \int_0^1 (g \square u_x) dx, \quad t \geq 0 \end{aligned}$$

for some $\delta, \delta_1 > 0$.

Proof. It is easy to see that, from (2.3) and (1.1), we have

$$\begin{aligned} \phi'_2(t) &= \int_0^1 u_t \left[u_t - p \int_0^x u_t(t-\tau, y) dy \right] dx + \int_0^1 u \left[u_{tt} - \int_0^x u_{tt}(t-\tau, y) dy \right] dx \\ &= \|u_t\|^2 - p \int_0^1 u_t \int_0^x u_t(t-\tau, y) dy dx + \int_0^1 u \left[u_{xx} - \int_0^t g(t-s)u_{xx}(s) ds \right] dx \\ &\leq (1 + \delta_1 p) \|u_t\|^2 + \frac{p}{4\delta_1} \|u_t(t-\tau)\|^2 - \left(1 - \int_0^t g(s)ds \right) \|u_x\|^2 \\ &\quad + \int_0^1 u_x \int_0^t g(t-s)[u_x(s) - u_x(t)] ds dx \\ &\leq (1 + \delta_1 p) \|u_t\|^2 + \frac{p}{4\delta_1} \|u_t(t-\tau)\|^2 - (1 - \bar{g}) \|u_x\|^2 \\ &\quad + \delta \|u_x\|^2 + \frac{1}{4\delta} \left(\int_0^t g(s)ds \right) \int_0^1 (g \square u_x) dx, \end{aligned}$$

for $t \geq 0$ and some $\delta, \delta_1 > 0$. The proof is complete. ■

2.3. Lemma. *The rate of change of the functional $\phi_3(t)$, fulfills the estimate*

$$\begin{aligned} \phi'_3(t) \leq & \delta(1 - g_*)^2 \bar{g} \|u_x\|^2 + (\delta + p\delta_2 \bar{g}^2 - g_*) \|u_t\|^2 \\ & + p \left(\delta + \frac{1}{4\delta_2} \right) \|u_t(t-\tau)\|^2 + \frac{1}{2} \left(2\bar{g} + \frac{1+2g(0)C_p \xi_1}{2\delta} \right) \int_0^1 (g \square u_x) dx, \quad t \geq t_* \end{aligned}$$

for some positive constants δ and δ_2 .

Proof. From the definition of $\phi_3(t)$, it appears that

$$\begin{aligned} \phi'_3(t) &= - \int_0^1 \left(u_{xx} - \int_0^t g(t-s)u_{xx}(s)ds \right) \left(\int_0^t g(t-s)(u(t) - u(s))ds \right) dx \\ &\quad - \int_0^1 (u_t - p \int_0^x u_t(t-\tau, y) dy) \left[\left(\int_0^t g(s)ds \right) u_t + \int_0^t g'(t-s)(u(t) - u(s))ds \right] dx \\ &\leq \int_0^1 \left(u_x - \int_0^t g(t-s)u_x(s)ds \right) \left(\int_0^t g(t-s)(u_x(t) - u_x(s))ds \right) dx \\ &\quad - \left(\int_0^t g(s)ds \right) \|u_t\|^2 + \delta \|u_t\|^2 + \frac{g(0)}{4\delta} C_p \int_0^1 (-g' \square u_x) dx + \delta p \|u_t(t-\tau)\|^2 \\ &\quad + \frac{g(0)}{4\delta} C_p \int_0^1 (-g' \square u_x) dx + \left(\int_0^t g(s)ds \right)^2 p\delta_2 \|u_t\|^2 + \frac{p}{4\delta_2} \|u_t(t-\tau)\|^2, \quad t \geq 0 \end{aligned}$$

where C_p is the Poincaré constant, for some positive constants δ and δ_2 . Therefore,

$$\begin{aligned} \phi'_3(t) \leq & \delta \left(1 - \int_0^t g(s)ds \right)^2 \bar{g} \|u_x\|^2 + \left(\frac{1}{4\delta} + \bar{g} \right) \int_0^1 (g \square u_x) dx + (\delta + p\delta_2 \bar{g}^2 - g_*) \\ & \times \|u_t\|^2 + p \left(\delta + \frac{1}{4\delta_2} \right) \|u_t(t-\tau)\|^2 + \frac{g(0)}{2\delta} C_p \xi_1 \int_0^1 (g \square u_x) dx, \quad t \geq t_*. \end{aligned}$$

This finishes the proof. ■

2.4. Lemma. *The derivative of $\phi_4(t)$ is equal to*

$$\phi'_4(t) = -\beta \phi_4(t) + e^{\beta\tau} \|u_t\|^2 - \|u_t(t-\tau)\|^2, \quad t \geq 0.$$

Proof. Straightforward. ■

2.5. Theorem. Let $u_0(0) \in H^1(0, 1)$ and $u_1 \in L^\infty([-\tau, 0]; L^2(0, 1))$. There exists a $\bar{p} < 1$ (depending on ξ_2) such that if $0 < p \leq \bar{p}$, then solutions of (1.1) decay to zero in an exponential manner. That is, there exist two positive constants M and c such that

$$E(t) := \frac{1}{2}(\|u_t\|^2 + \|u_x\|^2) \leq Me^{-ct}, \quad t \geq 0.$$

Proof. First, let us evaluate the derivative of $\mathcal{E}(t)$. Multiplying the equation in (1.1) by the expression

$$u_t(t) - p \int_0^x u_t(t - \tau, y) dy$$

we find, for $t \geq 0$

(2.9)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 [u_t(t) - p \int_0^x u_t(t - \tau, y) dy]^2 dx + \frac{1}{2} \frac{d}{dt} \|u_x\|^2 \\ &= - \int_0^1 u_t \int_0^t g(t-s) u_{xx}(s) ds dx - p \int_0^1 u_{xx} \int_0^x u_t(t - \tau, y) dy dx \\ & \quad + p \int_0^1 \left(\int_0^x u_t(t - \tau, y) dy \right) \left(\int_0^t g(t-s) u_{xx}(s) ds \right) dx \\ &= \int_0^1 u_{tx} \int_0^t g(t-s) u_x(s) ds dx + p \int_0^1 u_t(t - \tau) u_x dx - p u_x(1) \int_0^1 u_t(t - \tau, y) dy \\ & \quad - p \int_0^1 u_t(t - \tau) \int_0^t g(t-s) u_x(s) ds dx + p \left(\int_0^1 u_t(t - \tau, y) dy \right) \int_0^t g(t-s) u_x(s, 1) ds. \end{aligned}$$

The first term in the right hand side of (2.9) is equal to

$$\begin{aligned} \int_0^1 u_{tx} \int_0^t g(t-s) u_x(s) ds dx &= \frac{1}{2} \int_0^1 (g' \square u_x) dx - \frac{1}{2} g(t) \|u_x\|^2 \\ & \quad - \frac{1}{2} \frac{d}{dt} \left\{ \int_0^1 (g \square u_x) dx - \left(\int_0^t g(s) ds \right) \|u_x\|^2 \right\}, \quad t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{E}'(t) &\leq \frac{1}{2} \int_0^1 (g' \square u_x) dx - \frac{1}{2} g(t) \|u_x\|^2 + p \delta_3 \|u_x\|^2 \\ & \quad + p \delta_3 \left(\int_0^t g(s) ds \right) \int_0^1 (g \square u_x) dx + \frac{p}{2 \delta_3} \|u_t(t - \tau)\|^2 \\ & \quad + \frac{p}{2} \|u_t(t - \tau)\|^2 + \frac{p}{2} \left[u_x(1) - \int_0^t g(t-s) u_x(s, 1) ds \right]^2, \quad t \geq 0 \end{aligned}$$

for some $\delta_3 > 0$, or better

$$(2.10) \quad \begin{aligned} \mathcal{E}'(t) &\leq - \left(\frac{\xi_2}{2} - p \delta_3 \bar{g} \right) \int_0^1 (g \square u_x) dx - \left(\frac{1}{2} g(t) - p \delta_3 \right) \|u_x\|^2 \\ & \quad + \frac{p}{2} \left(1 + \frac{1}{\delta_3} \right) \|u_t(t - \tau)\|^2 + \frac{p}{2} \left[u_x(1) - \int_0^t g(t-s) u_x(s, 1) ds \right]^2, \quad t \geq 0. \end{aligned}$$

Now, we consider the functional

$$L(t) := \mathcal{E}(t) + \sum_{i=1}^4 \lambda_i \phi_i(t), \quad t \geq 0, \quad \lambda_i \geq 0, \quad i = 1, 2, 3, 4.$$

In view of (2.10) and Lemma 2.1 to Lemma 2.4, we get

$$\begin{aligned} L'(t) &\leq - \left(\frac{\xi_2}{2} - p \delta_3 \bar{g} \right) \int_0^1 (g \square u_x) dx - \left(\frac{1}{2} g(t) - p \delta_3 \right) \|u_x\|^2 + \frac{p}{2} \left(1 + \frac{1}{\delta_3} \right) \\ & \quad \times \|u_t(t - \tau)\|^2 + \frac{p}{2} \left[u_x(1) - \int_0^t g(t-s) u_x(s, 1) ds \right]^2 + \lambda_1 \left[\bar{\theta}' + \frac{\bar{\theta}}{2} (1+p) g(t) \right] \\ & \quad \times \|u_x\|^2 - \lambda_1 \left[u_x(1) - \int_0^t g(t-s) u_x(s, 1) ds \right]^2 + \lambda_1 \left[\left(p + \frac{1}{2} \right) \bar{\theta}' + \bar{\theta} \left(p + \delta + \frac{g(t)}{2} \right) \right] \\ & \quad \times \|u_t\|^2 + \lambda_1 \frac{p}{4} [\bar{\theta}' + \bar{\theta} (1 + 2g(t) + 4\delta)] \|u_t(t - \tau)\|^2 + \lambda_1 \left[\bar{\theta}' \bar{g} + \frac{\bar{\theta}}{4\delta} g(0) \xi_1 (1+p) \right] \\ & \quad \int_0^1 (g \square u_x) dx + \lambda_2 (1+p \delta_1) \|u_t\|^2 + \frac{\lambda_2 p}{4\delta_1} \|u_t(t - \tau)\|^2 - \lambda_2 (1 - \bar{g} - \delta) \|u_x\|^2 \\ & \quad + \frac{\lambda_2 \bar{g}}{4\delta} \int_0^1 (g \square u_x) dx + \lambda_3 \delta (1 - g_*)^2 \bar{g} \|u_x\|^2 + \lambda_3 (\delta + p \delta_2 \bar{g}^2 - g_*) \|u_t\|^2 \\ & \quad + \lambda_3 p \left(\delta + \frac{1}{4\delta_2} \right) \|u_t(t - \tau)\|^2 + \frac{\lambda_3}{2} \left(2\bar{g} + \frac{1+2g(0)C_p \xi_1}{2\delta} \right) \int_0^1 (g \square u_x) dx \\ & \quad - \beta \lambda_4 \phi_4(t) + \lambda_4 e^{\beta \tau} \|u_t\|^2 - \lambda_4 \|u_t(t - \tau)\|^2, \end{aligned}$$

for $t \geq t_*$. Arranging different terms there, we find

$$\begin{aligned}
 (2.11) \quad L'(t) &\leq -\left\{\frac{\xi_2}{2} - \delta_3 \bar{g} p - \lambda_1 \left[\bar{\theta}' \bar{g} + \frac{\bar{\theta}}{4\delta} g(0) \xi_1 (1+p)\right] - \frac{\lambda_2}{4\delta} \bar{g} - \frac{\lambda_3}{2}\right. \\
 &\times \left(2\bar{g} + \frac{1+2g(0)C_p \xi_1}{2\delta}\right)\left\} \int_0^1 (g \square u_x) dx - \left\{\frac{1}{2}g(t) - \lambda_1 \left[\bar{\theta}' + \frac{\bar{\theta}}{2}(1+p)g(t)\right]\right. \\
 &\quad \left. - p\delta_3 + \lambda_2(1 - \bar{g} - \delta) - \lambda_3 \delta (1 - g_*)^2 \bar{g}\right\} \|u_x\|^2 + \left(\frac{p}{2} - \lambda_1\right) \\
 &\times \left[u_x(1) - \int_0^t g(t-s)u_x(s,1)ds\right]^2 + \left\{\lambda_1 \left[\left(p + \frac{1}{2}\right)\bar{\theta}' + \bar{\theta} \left(p + \delta + \frac{g(t)}{2}\right)\right]\right. \\
 &\quad \left. + \lambda_2(1 + \delta_1 p) + \lambda_4 e^{\beta\tau} + \lambda_3(\delta + p\delta_2 \bar{g}^2 - g_*)\right\} \|u_t\|^2 + \left\{\frac{p}{2} \left(1 + \frac{1}{\delta_3}\right) + \frac{\lambda_2 p}{4\delta_1}\right. \\
 &\quad \left. + \lambda_1 \frac{p}{4} [\bar{\theta}' + \bar{\theta}(1 + 2g(t) + 4\delta)] + \lambda_3 p \left(\delta + \frac{1}{4\delta_2}\right) - \lambda_4\right\} \|u_t(t-\tau)\|^2 \\
 &\quad - \beta \lambda_4 \phi_4(t), \quad t \geq t_*.
 \end{aligned}$$

Our objective is to end up with $L'(t) \leq -CL(t)$, $t \geq t_*$ for some positive constant C . Observe that, if all the coefficients in (2.11) are negative (except the one of the boundary term which may be nonpositive), then

$$(2.12) \quad L'(t) \leq -C_1 \left\{ \|u_t\|^2 + \|u_x\|^2 + \int_0^1 (g \square u_x) dx + \|u_t(t-\tau)\|^2 + \phi_4(t) \right\}, \quad t \geq t_*,$$

for some $C_1 > 0$. Moreover, using Young inequality and Poincaré inequality, it is easily seen that the right hand side in (2.12) is smaller than or equal to $-C_2 L(t)$ for some $C_2 > 0$. That is,

$$L'(t) \leq -C_2 L(t), \quad t \geq t_*.$$

This implies that

$$(2.13) \quad L(t) \leq L(t_*) e^{-C_2(t-t_*)} \leq L(0) e^{-C_2(t-t_*)}, \quad t \geq t_*.$$

Let us first, forget about the first term in the right hand side of (2.11), ignore δ and ignore $g(t)$ as we may push t_* to the right until we obtain a small enough $g(t)$. We need to have

$$\begin{cases} p\delta_3 + \lambda_1 \bar{\theta}' < \lambda_2(1 - \bar{g}), \\ \frac{p}{2} \leq \lambda_1, \\ \lambda_1 \left[\left(p + \frac{1}{2}\right)\bar{\theta}' + \bar{\theta} p\right] + \lambda_2(1 + \delta_1 p) + \lambda_4 e^{\beta\tau} < \lambda_3(g_* - p\delta_2 \bar{g}^2), \\ \frac{p}{2} \left(1 + \frac{1}{\delta_3}\right) + \lambda_1 \frac{p}{4} (\bar{\theta}' + \bar{\theta}) + \frac{\lambda_2 p}{4\delta_1} + \frac{\lambda_3 p}{4\delta_2} < \lambda_4. \end{cases}$$

To fix ideas, let $\theta(x) = 2x$, $\delta_1 = \delta_3 = 1$, $\lambda_1 = p/2$, $\lambda_2 = 4p/(1 - \bar{g})$. Then, we need

$$\left(2p + \frac{1}{2}\right)p + \frac{4p(1+p)}{1-\bar{g}} + \lambda_4 e^{\beta\tau} < \lambda_3(g_* - p\delta_2 \bar{g}^2)$$

and

$$4p + p^2 + \frac{4p^2}{1-\bar{g}} + \frac{\lambda_3 p}{\delta_2} < 4\lambda_4.$$

Combining these two inequalities, we see that it is possible to select λ_3 and λ_4 provided that

$$g_* - p\delta_2 \bar{g}^2 - \frac{p}{4\delta_2} > 0$$

that is

$$4p\bar{g}^2\delta_2^2 - 4g_*\delta_2 + p < 0.$$

This requires $p < g_*/\bar{g}$ and δ_2 between the two positive roots of

$$4p\bar{g}^2x^2 - 4g_*x + p = 0.$$

Notice that as t_* is large then g_*/\bar{g} is close to 1.

Note that $e^{\beta\tau}$ may be ignored as it can be made very close to 1 by considering β small enough. Therefore, all the inequalities are satisfied for p small, namely when

$$\begin{cases} p(1 + \bar{\theta}') < \lambda_2(1 - \bar{g}), \\ p \left[\left(p + \frac{1}{2}\right) \bar{\theta}' + \bar{\theta}p \right] < \frac{\lambda_3}{6} g_*, \\ p \left[1 + \frac{p}{4}(\bar{\theta}' + \bar{\theta}) \right] < \frac{\lambda_4}{3}. \end{cases}$$

Next, we select δ small enough and t_* large enough so that all the coefficients in (2.11) are negative (except the first one in the right side of (2.11)).

The first term in the right hand side of (2.11) is negative if $Ap < \xi_2/2$ where A is a positive constant which is known after our previous selections. This requires a restriction on p in terms of ξ_2 . See the remark after the proof regarding the range of p .

Clearly

$$\left\| u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right\|^2 \leq 2L(t), \quad t \geq t_*.$$

Therefore,

$$\begin{aligned} \|u_t\| &\leq \left\| u_t(t) - p \int_0^x u_t(t - \tau, y) dy \right\| + p \|u_t(t - \tau)\| \\ &\leq \sqrt{2L(0)} e^{-C_2(t-t_*)/2} + p \|u_t(t - \tau)\| \\ &\leq C_3 e^{-C_2(t-t_*)/2} + p \|u_t(t - \tau)\|, \quad C_3 = \sqrt{2L(0)}, \quad t \geq t_*. \end{aligned}$$

This leads to

$$(2.14) \quad \|u_t\| \leq C_4 e^{-a(t-t_*)} + e^{-b(t-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\|, \quad t \geq t_*$$

for some $a, b > 0$ and $C_4 > C_3$ (to be determined). Indeed, notice first, that (2.14) holds at t_* . Assume that there exists $\bar{t} > t_*$ such that

$$(2.15) \quad \|u_t(\bar{t})\| = C_4 e^{-a(\bar{t}-t_*)} + e^{-b(\bar{t}-t_*)} \sup_{t_* - \tau \leq \sigma \leq \bar{t}_*} \|u_t(\sigma)\|$$

and

$$\|u_t(t)\| < C_4 e^{-a(t-t_*)} + e^{-b(t-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\|, \quad t \in [t_*, \bar{t}).$$

Suppose that $\bar{t} - \tau \in [t_*, \bar{t})$ i.e. $\bar{t} \geq t_* + \tau$, then

$$\begin{aligned} \|u_t(\bar{t})\| &\leq C_3 e^{-C_2(\bar{t}-t_*)/2} + p \|u_t(\bar{t} - \tau)\| \\ &< C_3 e^{-C_2(\bar{t}-t_*)/2} + p \left[C_4 e^{-a(\bar{t}-\tau-t_*)} + e^{-b(\bar{t}-\tau-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\| \right] \\ &= (C_3 + pC_4 e^{a\tau}) e^{-a(\bar{t}-t_*)} + p e^{b\tau} e^{-b(\bar{t}-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\| \end{aligned}$$

provided that $a \leq C_2/2$. To reach a contradiction with (2.15), we need to have

$$C_3 + pC_4 e^{a\tau} \leq C_4 \quad \text{and} \quad p e^{b\tau} \leq 1.$$

As $p < 1$, we can always pick $C_4 = \frac{C_3}{1 - p e^{a\tau}}$ with $p e^{a\tau} < 1$ and $p e^{b\tau} \leq 1$. That is, a and b exist. Therefore,

$$\|u_t\| \leq \frac{C_3}{1 - p e^{a\tau}} e^{-a(t-t_*)} + e^{-b(t-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\|$$

for $t \geq t_*$ with the parameters as above. If $t_* < \bar{t} < t_* + \tau$ i.e. $\bar{t} - \tau < t_*$, then

$$\|u_t(\bar{t} - \tau)\| \leq \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\|.$$

Therefore,

$$\|u_t(\bar{t})\| \leq C_3 e^{-C_2(\bar{t}-t_*)/2} + p e^{b(\bar{t}-t_*)} e^{-b(\bar{t}-t_*)} \|u_t(\bar{t} - \tau)\|.$$

If $p e^{b(\bar{t}-t_*)} \leq 1$ which holds if $p e^{b\tau} \leq 1$, then

$$\begin{aligned} \|u_t(\bar{t})\| &\leq C_3 e^{-a(\bar{t}-t_*)} + e^{-b(\bar{t}-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\| \\ &< C_4 e^{-a(\bar{t}-t_*)} + e^{-b(\bar{t}-t_*)} \sup_{t_* - \tau \leq \sigma \leq t_*} \|u_t(\sigma)\| \end{aligned}$$

and we reach again a contradiction. The assertion in the theorem is proved for $t \geq t_*$ and it holds also on $[0, t_*]$. ■

2.6. Remark. For the last part of the proof of the theorem we required $0 < p < 1$ and before that we have imposed some smallness conditions on p . In fact, p may be very close to one. This depends on ξ_2 ; if ξ_2 is large enough then these conditions are minor and not very restrictive. That is, p need not be very small. The reader will notice easily that we did not manage any effort in improving these conditions. It would be interesting to determine an "optimal" or "critical" number p^* beyond which we get some kind of instability.

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