Hacettepe Journal of Mathematics and Statistics
Volume 47 (3) (2018), 709 – 720

Estimation of inequality indices based on ranked set sampling

N. Nakhaei Rad*†, G. R. Mohtashami Borzadaran‡ and H. Yari§

Abstract

Measuring the income inequality is a major concern of the economists. Therefore, numerous indices have been devised to show different features of the income inequality. In general, the simple random sampling procedure is commonly utilized to estimate the inequality measures, while the ranked set sampling is a more cost saving method which increases the precision and the efficiency of the inequality estimators. In this paper the advantages of the ranked set sampling when measuring the amount of the income inequality are examined. Through using Monte Carlo simulation technique, this paper proves that the ranked set sampling, increases the precision of inequality indices estimations. In the end, a real income data set is analyzed to illustrate the obtained results.

Keywords: Ranked set sampling, Gini index, Theil index, MLD index, Atkinson index, Monte Carlo simulation, Generalized beta distribution of the first kind, Generalized beta distribution of the second kind, Generalized gamma distribution.

2000 AMS Classification: 62F07

Received : 24.01.2016 *Accepted* : 28.06.2016 *Doi* : 10.15672/HJMS.201613419772

*Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran, Email: najme.rad@gmail.com

†Corresponding Author.

‡Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran, Email: grmohtashami@um.ac.ir

§Iran University of Science and Technology, Tehran, Iran, Email: yari@iust.ac.ir

1. Introduction

Poverty is the product of high levels of income inequality in societies. Moreover, higher levels of income inequality contributes to lower levels of growth. For these reasons, economists have proposed numerous indices to measure the amount of the income inequality in societies. One major sampling scheme for estimating those indices is the simple random sampling (SRS). The main reason for the popularity of this sampling method in economics is simple calculation of inequality indices. However, using this method is not always feasible or appropriate. As a two-phase sampling procedure, the ranked set sampling (RSS) (See [4]) can be used to achieve a balance between cost and efficiency when estimating the amount of the income inequality with respect to SRS. There have been many studies which compared RSS with other sampling methods. RSS is a more efficient sampling method than SRS for estimating the populations' mean and variance (See [8, 13, 14, 16, 17]). Comparing RSS with other two-phase sampling methods shows that unlike them, RSS does not require any particular distributional assumptions. Therefore, when restrictive assumptions are not satisfied using RSS is vindicable; otherwise RSS usually provides less efficient estimators. (See [5, 6]). The concept of RSS is nonparametric, but it has been utilized in parametric cases, too. (See [10, 15]).

Al-Talib and Al-Nasser (2008) introduced a RSS estimator for Gini index based on the Lorenz curve and compared it with SRS only for a specific nonstandard Lorenz curve. Bansal et al. (2013) introduced RSS estimators for Bonferroni index and absolute Lorenz index based on the Benferroni curve and absolute Lorenz curve, respectively and compared their RSS estimations with their SRS estimations when the distribution of the data is known. They considered exponential, Pareto and power distributions to obtain the presented estimators which have close forms of Lorenz curve, Bonferroni curve and absolute Lorenz curve; while in practice, the income distribution is not always known and these curves are not always available in close forms for any income distribution. The four-parameter generalized beta distribution of the first kind (GB_1) and the generalized beta distribution of the second kind (GB_2) and the three-parameter generalized gamma distribution (GG) are the most popular income models which are fitted to income data of many countries in recent years (See [2]). Furthermore, these distributions include the other income distributions as special cases (Figure 1), but their Lorenz curves, Benferroni curves and absolute Lorenz curves are not available in close forms (See [2]). The Gini index is the most commonly used measure of inequality but it has some limitations. Beside Gini index, among the various inequality measures in economics, Theil index (See [3]) is more prominent and practical than the others. In this paper, in addition to the Gini index, Theil index and *mean log deviation* (MLD) (See [3]) index, which are based on entropy concept are chosen. The Atkinson index is also chosen (See [1]) as a weighted inequality measure with the inequality aversion parameter. The Gini index, MLD index and Theil index are respectively as:

$$G = \frac{1}{2\mu}\int_0^\infty \int_0^\infty |x-y|\, f(x)f(y)dxdy, \tag{1.1}$$

$$T = \int_0^\infty \frac{x}{\mu}\log\frac{x}{\mu}dF(x), \tag{1.2}$$

$$MLD = -\int_0^\infty \log\frac{x}{\mu}dF(x), \tag{1.3}$$

where $\mu = E(X)$. The Atkinson family, is defined as below

$$(1.4) \qquad A(\varepsilon) = 1 - \left(\int_0^\infty \left(\frac{x}{\mu} \right)^{1-\varepsilon} dF(x) \right)^{\frac{1}{(1-\varepsilon)}}, \qquad \varepsilon > 0, \varepsilon \neq 1,$$

where ε controls the inequality aversion. Whenε tends to 1, the Atkinson index is obtained as

$$(1.5) \qquad A(1) = 1 - \frac{1}{\mu} \exp \left(\int_0^\infty \log(x) dF(x) \right).$$

In economics, the nonparametric estimators of the mentioned indices are usually used to measure the income inequality. In this paper, the efficiency of the estimators of these indices based on SRS and RSS is compared using the Monte Carlo simulation technique in nonparametric setting. The organization of this article is as follows: in section two, first the estimators of Gini index, Theil index, MLD index and Atkinson index are presented for both the SRS and RSS. Afterwards, the mentioned indices are expressed in terms of the distributional parameters of GB_1, GB_2 and GG. The third section is dedicated to the simulation study. In the last section, a real data set, consisting of 7200 real gross domestic product (GDP) per capita of 172 countries in 1970-2012 is analyzed.

2. Estimations of the Inequality Indices

In this section, first the SRS and RSS estimators of the mentioned inequality indices are obtained. Then, the mentioned inequality indices are expressed in terms of the distributional parameters of GB_1, GB_2 and GG to derive the *mean square error* (MSE) of their estimators in the simulation study.

Suppose that $X_{SRS} = (X_1, ..., X_n)$is a simple random sample of size n. To obtain a ranked set sample of size $n = rm$, the m number of simple random samples of size m should be chosen and ordered. Then the smallest observation from the first sample and the second smallest observation from the second sample are selected and so on. The vector of observations $X_{RSS} = (X_{1,1}, ..., X_{m,m})$ is a one-cycle RSS of size m. Note that$X_{i,i}$'s are not necessarily ordered. By repeating this procedure r times, $X_{RSS} = \{X_{(i)j},\ i = 1, ..., m,\ j = 1, ..., r\}$, where $X_{(i)j}$ denotes the ith order statistics in jth cycle . The estimators of indices (1.1)-(1.5) based on SRS and RSS are summarized in Table 1. This observational process can be described as follows:

$$\begin{array}{ccccccc}
1: & \mathbf{X_{(1:m)1}} & X_{(2:m)1} & \cdots & X_{(m:m)1} & \rightarrow & X_{1,1} = X_{(1:m)1} \\
2: & X_{(1:m)2} & \mathbf{X_{(2:m)2}} & \cdots & X_{(m:m)2} & \rightarrow & X_{2,2} = X_{(2:m)2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
m: & X_{(1:m)m} & X_{(2:m)m} & \cdots & \mathbf{X_{(m:m)m}} & \rightarrow & X_{m,m} = X_{(m:m)m}
\end{array}$$

Table 1. SRS and RSS estimators of G, T, MLD, $A(\varepsilon)$ and $A(1)$

SRS	RSS
$\hat{G}_{SRS}=\frac{1}{2n^2\bar{x}_{SRS}}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i-x_j\|$	$\hat{G}_{RSS}=\frac{1}{2n^2\bar{x}_{RSS}}\sum_{k=1}^{r}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\|x_{(i)k}-x_{(j)k}\right\|$
$\hat{T}_{SRS}=\frac{1}{n\bar{x}_{SRS}}\sum_{i=1}^{n}x_i\log x_i-\log\bar{x}_{SRS},$	$\hat{T}_{RSS}=\frac{1}{n\bar{x}_{RSS}}\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i)j}\log x_{(i)j}-\log\bar{x}_{RSS},$
$M\hat{L}D_{SRS}=-\frac{1}{n}\sum_{i=1}^{n}\log x_i+\log\bar{x}_{SRS},$	$M\hat{L}D_{RSS}=-\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j}+\log\bar{x}_{RSS}$
$\hat{A}(\varepsilon)_{SRS}=1-\frac{1}{\bar{x}_{SRS}}\left(\frac{1}{n}\sum_{i=1}^{n}x_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}},$	$A\hat{(}\varepsilon)_{RSS}=1-\frac{1}{\bar{x}_{RSS}}\left(\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i)j}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$
$\hat{A}(1)_{SRS}=1-\frac{1}{\bar{x}_{SRS}}\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right),$	$\hat{A}(1)_{RSS}=1-\frac{1}{\bar{x}_{RSS}}\exp\left(\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j}\right)$

where $\bar{x}_{SRS}=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $\bar{x}_{RSS}=\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m}x_{(i)j}$.

McDonald and Ransom (2008) have obtained the Gini index and the Theil index forGB_1, GB_2 and GGdistributions. In the following, the MLD and the Atkinson indices are obtained for these specific distributions.

The cumulative distribution function (cdf) of $GB_1(a,b,p,q)$ is:

$$F_{GB_1}(x;a,b,p,q)=\frac{B_{\left(\frac{x}{b}\right)^a}(p,q)}{\beta(p,q)},$$

where $B_x(p,q)=\int_0^x t^{p-1}(1-t)^{q-1}dt$. The indices (1.3)-(1.5) of GB_1 are obtained as:

$$MLD_{GB_1}=\frac{1}{a}\left(\varphi(p+q)-\varphi(p)\right)+\log\frac{\beta\left(p+\frac{1}{a},q\right)}{\beta(p,q)},$$

$$A(\varepsilon)_{GB_1}=1-\frac{\beta^{1-\frac{1}{1-\varepsilon}}(p,q)\,\beta^{\frac{1}{1-\varepsilon}}\left(p+\frac{1-\varepsilon}{a},q\right)}{\beta\left(p+\frac{1}{a},q\right)},\qquad \varepsilon\neq 1,$$

$$A(1)_{GB_1}=1-\frac{\beta(p,q)}{b\beta\left(p+\frac{1}{a},q\right)}\exp\left(\frac{\varphi(p)}{a}-\frac{\varphi(p+q)}{a}+\log b\right),$$

respectively, where $\varphi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ and $\Gamma'(z)=\frac{d}{dx}\Gamma(x)$.

The cumulative distribution function of $GB_2(a,b,p,q)$ is:

$$F_{GB_2}(x;a,b,p,q)=1-\frac{B_{\left(1+\left(\frac{x}{b}\right)^a\right)^{-1}}(p,q)}{\beta(p,q)}.$$

Indices (1.3)-(1.5) are

$$MLD_{GB_2}=-\frac{1}{a}\left(\varphi(p)-\varphi(q)\right)+\log\frac{\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}{\Gamma(p)\Gamma(q)},$$

$$A(\varepsilon)_{GB_2}=1-\frac{\Gamma^{\frac{1}{1-\varepsilon}}\left(p+\frac{1-\varepsilon}{a}\right)\Gamma^{\frac{1}{1-\varepsilon}}\left(q-\frac{1-\varepsilon}{a}\right)\Gamma^{1-\frac{1}{1-\varepsilon}}(p)\Gamma^{1-\frac{1}{1-\varepsilon}}(q)}{\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)},\quad \varepsilon\neq 1,$$

$$A(1)_{GB_2}=1-\frac{\Gamma(p)\Gamma(q)}{b\Gamma\left(p+\frac{1}{a}\right)\Gamma\left(q-\frac{1}{a}\right)}\exp\left(\frac{\varphi(p)}{a}-\frac{\varphi(q)}{a}+\log b\right),$$

respectively.

The cumulative distribution function of $GG(a,b,p)$ is:

$$F_{GG}(x;a,b,p)=\Gamma_{\left(\frac{x}{b}\right)^a}(p,1),$$

where $\Gamma_x(\alpha,\lambda)=\int_0^x \frac{t^{\alpha-1}e^{\frac{-t}{\lambda}}}{\lambda^\alpha\Gamma(\alpha)}dt$. Its (1.3)-(1.5) indices are derived as follows:

$$MLD_{GG}=-\frac{1}{a}\varphi\left(p\right)+\log\frac{\Gamma\left(p+\frac{1}{a}\right)}{\Gamma(p)},$$
$$A(\varepsilon)_{GG}=1-\frac{\Gamma^{\frac{1}{1-\varepsilon}}\left(p+\frac{1-\varepsilon}{a}\right)\Gamma^{1-\frac{1}{1-\varepsilon}}\left(p\right)}{\Gamma\left(p+\frac{1}{a}\right)},\quad \varepsilon\neq 1,$$
$$A(1)_{GG}=1-\frac{\Gamma\left(p\right)}{b\Gamma\left(p+\frac{1}{a}\right)}\exp\left(\frac{\varphi\left(p\right)}{a}+\log b\right).$$

According to the Figure 1, these indices can be found for other distributions related to the GB_1,GB_2 and GG.

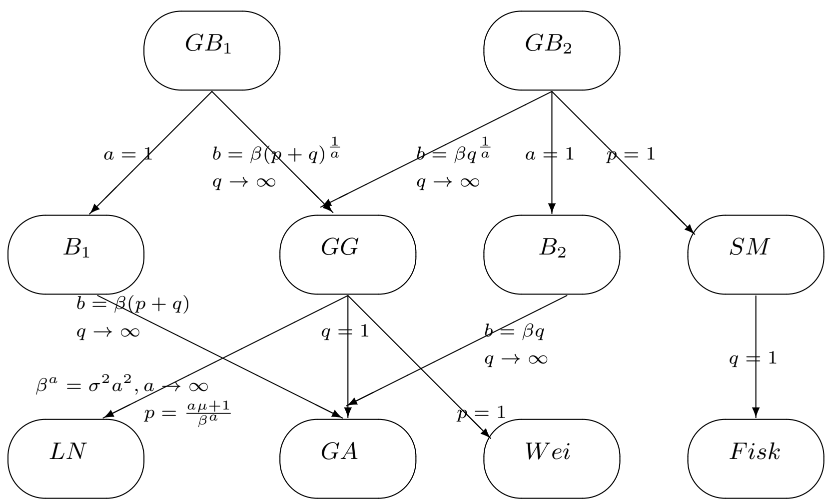

Figure 1. Some related distributions of GB_1 and GB_2

3. Simulation Study

In this section, the performance of the estimators of the Gini, Theil, MLD and Atkinson indices based on SRS and RSS procedures is compared. The inverse cdf simulation technique is used in order to generate data from GB_1, GB_2 and GG distributions. Simple random samples of sizes $10, 30, 45, 60, 80, 100$ and ranked set samples with $(m,r)=(2,5),(3,10),(3,15),(4,15),(4,20),(5,20)$ are considered, where m is the set size and r is the number of cycles. In this paper, since perfect ranking is considered, m is chosen up to 5 to reduce the ranking error. To analyze the sensitivity of the results, the simulation is performed for the mentioned distributions with some different selections of the parameters. The MSE and the bias of estimators in Section 2 were computed for both of the sampling procedures, using the Monte Carlo method with 10000 replications.

The relative efficiency (RE) is calculated for all the estimators as follows

$$RE(\hat{\theta}_{SRS}, \hat{\theta}_{RSS}) = \frac{MSE(\hat{\theta}_{RSS})}{MSE(\hat{\theta}_{SRS})}.$$

The results are presented in the Figures 2-4. The bias of estimators are summarized in Table 1-12 in the Appendix. It can be seen that RSS estimations perform better than the SRS estimations in all cases. The results are independent from both the type of distribution and values of parameters. Also, it is observed that RE decreases as n increases which means for a large sample size, the RSS estimators are much better than their counterparts in the SRS scheme.

Remark: In general, Cowell and Flachaire (2007) studied the effect of the extreme values (both large and small incomes) on Gini index, Atkinson index and generalized entropy family (Theil index and MLD index) using bootstrap method based on some distributions with heavy or light tail.

4. Real Data

To illustrate the results obtained in the previous sections, a real data set consisting of 7200 GDP (million dollars) per capita of 172 countries in 1970-2012 has been considered (the data were extracted from http://www.unctadstat.unctad.org). The maximum likelihood estimations of the parameters, the P-value of the *Kolmogorov-Smirnov* (KS) test and the *Akaike information criterion* (AIC) for each distribution are calculated and their results are summarized in Table 2. It is concluded that all the distributions can be fitted on the data; but according to the AIC, GB_2 provides the best fit to the data. Simple random samples of sizes $3600, 2400, 1800, 1440$ and ranked set samples with $(m, r) = (2, 1800), (3, 800), (4, 450), (5, 288)$ are chosen from the data set. The estimation of Gini, Theil, MLD and Atkinson indices based on RSS and SRS are summarized in Table 4. Comparing the results in Table 4 with the values of indices from Table 3, which are estimated based on GB_2 distribution, shows that the inequality indices' estimations based on the RSS are slightly closer to the GB_2 than their counterparts.

Table 2. The values of $\hat{a}, \hat{b}, \hat{p}, \hat{q}$ and the corresponding AIC and the P-value of the KS test

distribution	$\hat{a}$	$\hat{b}$	$\hat{p}$	$\hat{q}$	P-value	AIC
GB_1	0.0276	15192536.1287	28.1417	6.7575	0.2315	160252.952
$\mathbf{GB_2}$	**0.0619**	**15159394.5785**	**61.4665**	**99.9399**	**0.7099**	**159133.6884**
GG	0.3970	28644.3062	0.9581	——–	0.5627	159408.5400

Table 3. The estimates of the indices G, T, MLD, $A(0,1)$ and $A(1)$ in the data set

index	G	T	MLD	A(0,1)	A(1)
GB_2	0.4618	0.4183	0.5267	0.2812	0.9653

Table 4. The estimations of G, T, MLD,$A(0,1)$ and $A(1)$ based on the SRS and RSS

r	m	n	G_{SRS}	G_{RSS}	T_{SRS}	T_{RSS}	MLD_{SRS}	MLD_{RSS}	$A(0,1)_{SRS}$	$A(0,1)_{RSS}$	$A(1)_{SRS}$	$A(1)RSS$
1800	2	3600	0.3806	0.4307	0.3961	0.4066	0.4905	0.4973	0.2069	0.2322	0.9547	0.9568
800	3	2400	0.3827	0.4009	0.3527	0.3828	0.4819	0.4909	0.2091	0.2208	0.9551	0.9548
450	4	1800	0.3741	0.4126	0.3368	0.3831	0.4162	0.4918	0.1990	0.2211	0.9263	0.9551
288	5	1440	0.3818	0.4019	0.2602	0.2433	0.4030	0.4468	0.1681	0.2156	0.9205	0.9491

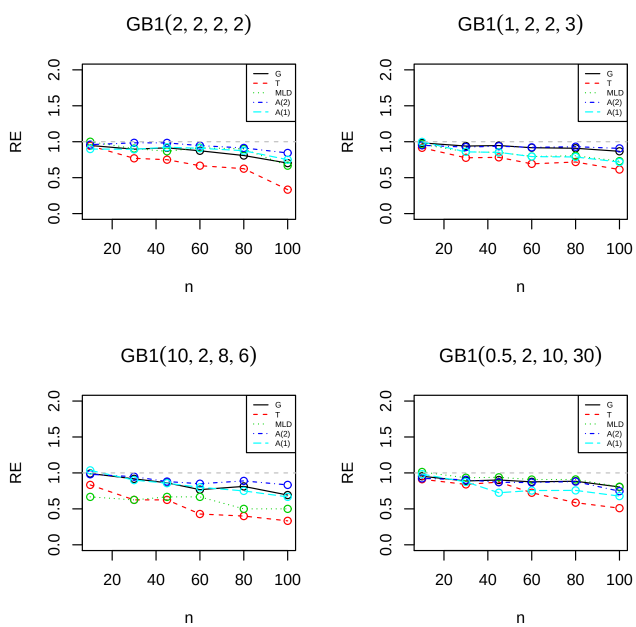

Figure 2. The plot of RE of the estimations of G, T, MLD, $A(2)$ and $A(1)$ versus the sample size n

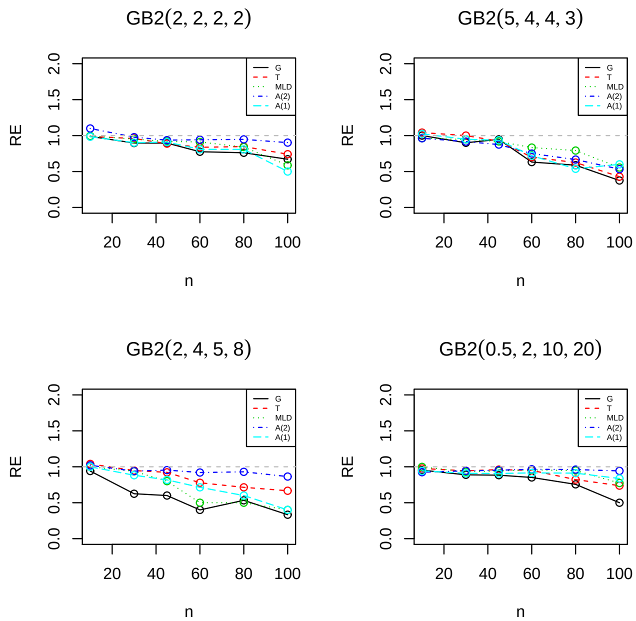

Figure 3. The plot of RE of the estimations of G, T, MLD, $A(2)$ and $A(1)$ versus the sample size n

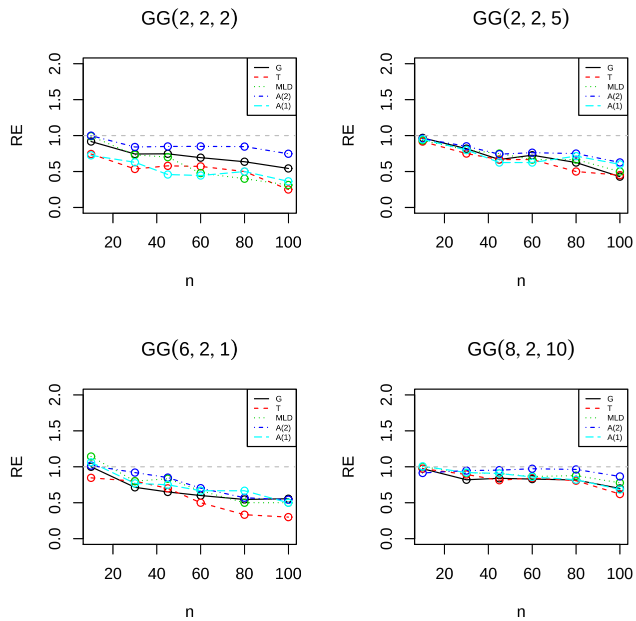

Figure 4. The plot of RE of the estimations of G, T, MLD, $A(2)$ and $A(1)$ versus the sample size n

5. Conclusion

Comparing the RSS to SRS, in terms of both different sample sizes and various underlyingþ income distributions indicates that the ranked set estimations of income inequality indices outperforms the traditional simple random estimations of them. In addition, as sample size increases, RSS estimators become more efficient than SRS estimators.

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6. Appendix

Table 1. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_1(2, 2, 2, 2)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00221	-0.00185	0.00117	0.04225	0.00035	-0.00237	-0.00098	0.00222	0.04457	0.00137
3	10	-0.00260	-0.00075	0.00323	0.05039	0.00236	-0.00253	-0.00026	0.00385	0.05187	0.00297
3	15	-0.00264	-0.00032	0.00367	0.05158	0.00322	-0.00246	-0.00002	0.00406	0.05253	0.00359
4	15	-0.00259	-0.00077	0.00438	0.05103	0.00407	-0.00231	-0.00055	0.00467	0.05178	0.00436
4	20	-0.00274	-0.00098	0.00523	0.05205	0.00365	-0.00220	-0.00082	0.00544	0.05260	0.00386
5	20	-0.00266	-0.00038	0.00497	0.05340	0.00472	-0.00217	-0.00027	0.00512	0.05384	0.00487

Table 2. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_1(1, 2, 2, 3)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.01162	-0.00858	-0.01251	0.07942	-0.01351	-0.00930	-0.00563	-0.00809	0.07126	-0.00966
3	10	-0.00395	-0.00261	0.00389	0.03669	-0.00431	-0.00246	-0.00094	0.00130	0.03098	-0.00198
3	15	-0.00233	-0.00156	0.00221	-0.02622	0.00256	-0.00138	-0.00052	0.00060	-0.02247	0.00109
4	15	-0.00254	-0.00164	0.00259	0.02394	0.00271	-0.00179	-0.00093	0.00144	0.02086	0.00163
4	20	-0.00205	-0.00124	0.00202	-0.01986	0.00210	-0.00150	-0.00075	0.00121	-0.01758	0.00133
5	20	-0.00104	-0.00052	0.00075	0.01476	0.00095	-0.00060	-0.00016	0.00014	0.01272	0.00035

Table 3. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_1(10, 2, 8, 6)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-2.8e-05	-2.5e-05	-3.1e-05	0.00024	-3.1e-05	-1.9e-05	-1.6e-05	-2.2e-05	0.00025	-2.1e-05
3	10	-9.3e-06	-1.0e-05	-1.2e-05	0.00028	-1.4e-05	-3.6e-06	-4.3e-06	-6.3e-06	0.00029	-8.9e-06
3	15	-5.4e-06	-4.2e-06	-8.6e-06	0.00028	-5.2e-06	-1.7e-06	-5.7e-07	-4.9e-06	0.00029	-1.5e-06
4	15	-5.7e-06	-8.3e-06	-1.2e-06	0.00028	5.2e-06	-3.2e-06	-5.8e-06	1.2e-06	0.00028	3.7e-06
4	20	-5.7e-06	-1.0e-05	4.5e-06	0.00029	-3.7e-06	-3.2e-06	-8.8e-06	6.3e-06	0.00029	-1.8e-06
5	20	-4.5e-06	-5.8e-06	2.5e-06	0.00029	9.1e-06	-2.6e-06	-4.4e-06	3.9e-06	0.00030	1.0e-05

Table 4. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_1(0.5, 2, 10, 30)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.03597	-0.01413	-0.00665	0.10434	-0.01499	-0.02849	-0.01008	-0.00189	0.11182	-0.01087
3	10	-0.01279	-0.00609	0.00444	0.12564	-0.00758	-0.00775	-0.00362	0.00738	0.13068	-0.00497
3	15	-0.00812	-0.00376	0.00860	0.13040	-0.00442	-0.00478	-0.00204	0.01057	0.13375	-0.00268
4	15	-0.00706	-0.00442	0.01009	0.13112	0.00052	-0.00479	-0.00331	0.01133	0.13339	0.00059
4	20	-0.00551	-0.00376	0.01240	0.13604	-0.00017	-0.00387	-0.00302	0.01326	0.13768	0.00060
5	20	-0.00334	-0.00245	0.01014	0.13659	0.00017	-0.00194	-0.00185	0.01083	0.13799	0.00082

Table 5. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_2(2, 2, 2, 2)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.03694	-0.03095	-0.03113	0.07518	-0.06033	-0.03027	-0.02507	-0.02513	0.08325	-0.05531
3	10	-0.01399	-0.01489	-0.01116	0.09366	-0.03431	-0.01025	-0.01128	-0.00747	0.09912	-0.03112
3	15	-0.00988	-0.00657	-0.00726	0.10025	-0.02367	-0.00778	-0.00366	-0.00454	0.10409	-0.02132
4	15	-0.00642	-0.01029	-0.00822	0.10231	-0.02415	-0.00436	-0.00817	-0.00650	0.10472	-0.02267
4	20	-0.00460	-0.00630	-0.00368	0.11329	-0.01815	-0.00353	-0.00510	-0.00261	0.11492	-0.01722
5	20	-0.00454	-0.01023	-0.01072	0.10686	-0.01971	-0.00210	-0.00923	-0.00986	0.10828	-0.01893

Table 6. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_2(5, 4, 4, 3)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00931	-0.00300	-0.01204	0.01070	-0.00840	-0.00641	-0.00260	-0.01127	0.01163	-0.00668
3	10	-0.00330	-0.00254	-0.00806	0.01240	-0.00733	-0.00140	-0.00218	-0.00693	0.01300	-0.00556
3	15	-0.00212	-0.00216	-0.00629	0.01273	-0.00451	-0.00081	-0.00209	-0.00504	0.01314	-0.00301
4	15	-0.00166	-0.00188	-0.00521	0.01272	-0.00280	-0.00090	-0.00170	0.00380	0.01296	-0.00145
4	20	-0.00127	-0.00162	0.00323	0.01320	-0.00139	-0.00074	-0.00130	0.00202	0.01337	-0.00100
5	20	-0.00073	-0.00129	0.00196	0.01321	0.00067	-0.00030	-0.00098	0.00113	0.01335	0.00043

Table 7. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_2(2, 4, 5, 8)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00276	-0.00564	-0.00685	0.03321	-0.00789	-0.00184	-0.00393	-0.00518	0.03031	-0.00630
3	10	-0.00096	-0.00255	-0.00280	0.03729	-0.00523	-0.00037	-0.00148	-0.00175	0.03539	-0.00421
3	15	-0.00060	-0.00143	-0.00141	0.03802	-0.00352	-0.00020	-0.00066	-0.00067	0.03671	-0.00281
4	15	-0.00051	-0.00178	-0.00112	0.03774	-0.00254	-0.00027	-0.00131	-0.00069	0.03699	-0.00213
4	20	-0.00040	-0.00146	-0.00039	0.03952	-0.00221	-0.00023	-0.00116	-0.00010	0.03899	-0.00194
5	20	-0.00021	0.00136	-0.00139	0.03876	-0.00197	-0.00007	-0.00111	-0.00115	0.03833	-0.00174

Table 8. The bias of the Gini, Theil, MLD and Atkinson indices for $GB_2(0.5, 2, 10, 20)$

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.06493	-0.07821	-0.07300	0.08027	-0.14257	-0.05355	-0.06800	-0.06152	0.06888	-0.13442
3	10	-0.02375	-0.03937	-0.02808	0.10300	-0.09633	-0.01577	-0.03269	-0.02083	0.09502	-0.09087
3	15	-0.01581	-0.01882	-0.01885	0.11344	-0.07466	-0.01024	-0.01329	-0.01346	0.10786	-0.07063
4	15	-0.01230	-0.02545	-0.02230	0.11695	-0.07627	-0.00872	-0.02133	-0.01887	0.11337	-0.07371
4	20	-0.00937	-0.01537	-0.01256	0.13841	-0.06240	-0.00692	-0.01307	-0.01041	0.13597	-0.06077
5	20	-0.00645	-0.02530	-0.02803	0.12224	-0.06855	-0.00421	-0.02323	-0.02623	0.12000	-0.06708

Table 9. The bias of the Gini, Theil, MLD and Atkinson indices for GG(2, 2, 2)

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.01996	-0.00597	0.00735	0.08164	-0.00382	-0.01585	-0.00446	-0.00693	0.08283	-0.00427
3	10	-0.00729	-0.00227	0.00704	0.08279	-0.00325	-0.00529	-0.00150	0.00698	0.08356	0.00338
3	15	-0.00449	-0.00186	0.00644	0.08269	0.00396	-0.00310	-0.00134	0.00611	0.08258	0.00226
4	15	-0.00340	-0.00185	0.00557	0.07945	0.00298	-0.00242	-0.00108	0.00568	0.08074	0.00263
4	20	-0.00242	-0.00137	0.00444	0.07963	0.00139	-0.00141	-0.00074	0.00466	0.08149	0.00113
5	20	-0.00219	-0.00091	0.00628	0.07736	0.00139	-0.00123	-0.00063	0.00304	0.08020	0.00168

Table 10. The bias of the Gini, Theil, MLD and Atkinson indices for GG(6, 2, 1)

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00570	-0.00158	0.00181	0.02033	-0.00092	-0.00396	-0.00082	0.00169	0.02165	0.00172
3	10	-0.00195	-0.00058	0.00157	0.02470	0.00163	-0.00083	-0.00052	0.00151	0.02386	0.00138
3	15	-0.00119	-0.00029	0.00160	0.02489	0.00097	-0.00045	-0.00023	0.00155	0.02435	0.00103
4	15	-0.00109	-0.00050	0.00143	0.02443	0.00121	-0.00062	-0.00035	0.00126	0.02405	0.00109
4	20	-0.00085	-0.00063	0.00108	0.02490	0.00102	-0.00051	-0.00044	0.00084	0.02462	0.00097
5	20	-0.00044	-0.00031	0.00092	0.02541	0.00097	-0.00016	-0.00031	0.00084	0.02519	0.00055

Table 11. The bias of the Gini, Theil, MLD and Atkinson indices for GG(2, 2, 5)

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00500	-0.00232	-0.00216	0.02421	-0.00234	-0.00369	0.00032	-0.00127	0.02247	-0.00148
3	10	-0.00165	-0.00095	-0.00035	0.02730	-0.00100	-0.00081	-0.00042	0.00020	0.02619	-0.00046
3	15	-0.00095	-0.00057	0.00024	0.02755	-0.00031	-0.00041	-0.00022	0.00061	0.02682	0.00004
4	15	-0.00104	-0.00074	0.00067	0.02718	0.00036	-0.00065	-0.00053	0.00090	0.02671	0.00059
4	20	-0.00084	-0.00077	0.00110	0.02783	0.00021	-0.00056	-0.00062	0.00127	0.02749	0.00037
5	20	-0.00039	-0.00040	0.00081	0.02807	0.00071	-0.00016	-0.00028	0.00094	0.02779	0.00084

Table 12. The bias of the Gini, Theil, MLD and Atkinson indices for GG(8, 2, 10)

		SRS					RSS				
m	r	G	T	MLD	A(2)	A(1)	G	T	MLD	A(2)	A(1)
2	5	-0.00016	-7.6e-05	-6.3e-05	0.00081	-8.8e-05	-0.00010	-4.89e-05	-9.1e-05	0.00077	-6.0e-05
3	10	-0.00005	-3.0e-05	-1.7e-05	0.00081	-4.6e-05	-0.00002	-1.2e-05	-3.5e-05	0.00079	-2.8e-05
3	15	-0.00003	-1.7e-05	-8.0e-06	0.00080	-2.1e-05	-0.00001	-5.9e-06	-1.9e-05	0.00078	-1.0e-05
4	15	-0.00003	-2.3e-05	3.3e-06	0.00081	4.9e-06	-0.00001	-1.6e-05	-3.8e-06	0.00080	-2.2e-06
4	20	-0.00002	-2.6e-05	1.4e-05	0.00082	-2.1e-05	-0.00001	-2.1e-05	8.8e-06	0.00081	-5.3e-06
5	20	-0.00001	-1.4e-05	6.1e-06	0.00081	-1.0e-05	-8.4e-06	-1.0e-05	2.0e-06	0.00081	1.7e-05