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Time-consistent reinsurance-investment strategy for mean-variance insurers with defaultable security and jumps

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Abstract

This paper studies an optimal reinsurance-investment problem for a mean-variance insurer with defaultable security and jumps. Specially, we assume that the risky asset's price process is described by a geometric Lévy process. By using a game theoretic approach, we establish the extended Hamilton-Jacobi-Bellman system for the post-default case and the pre-default case, respectively. Furthermore, we derive the closed-from expressions for the time-consistent reinsurance-investment strategy and the corresponding value function. Finally, we provide numerical examples to illustrate the impacts of model parameters on the time-consistent strategy.

Keywords: Mean-variance, Proportional reinsurance, Time-consistent strategy, Defaultable bond, Geomeotric Lévy process.

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1. Introduction

In recent years, optimal reinsurance-investment problem for insurance company has gained much attention in the actuarial and finance literature. Many scholars adopt the stochastic control theory and related methodologies to study the optimization problem, see, for example, [1], [2], [7], [12], [16], and references therein. To derive the optimal reinsurance-investment strategies for an insurer, mean-variance criterion is widely adopted by many researchers [1], [2], [3], [10], [17], [18], [19] . However, due to the mean-variance criterion lacks the iterated expectation property, traditional mean-variance problem in a multi-period or continuous time framework are time inconsistent in the sense that Bellman optimality principle does not hold. In fact, time-consistency is important for a rational decision maker, some researcher began to find time-consistent strategy for the mean-variance problem. The main approach is to formulate the problem in a game theoretic framework. For the detailed introduction, we can see [5], [6], [13], [14] and [20].

Moreover, in today's financial markets, the default of one company usually has strong influence on other companies. During the credit crisis, some recent default events have clearly demonstrated this phenomenon. As a result, optimal reinsurance-investment problems with credit or default risk have become an important area of research, such as [4], [8], [9], [15] and references therein. Recently, Zhu et al. [23] investigated the optimal reinsurance-investment strategy for an insurer with CARA utility in a defaultable market. Zhao et al. [22] are the first to present the time-consistent reinsurance-investment strategy for insurer with defaulatable securities under the mean-variance criterion, where the risky asset's price process was described by geometric Brownian motion. In fact, the risky asset's price process is often discontinuous and has jumps. Zeng et al. [21] assumed that the price process of risky asset was modeled by a geometric Lévy process and derived the time-consistent reinsurance and investment strategy for mean-variance insurer with a general jump-diffusion model. However, they did not discuss default security.

In this paper, we consider the same problem as [22] but the surplus of the insurer is modeled by the classical risk model and the price process of risky asset is described by a jump-diffusion process. Similar to [5], we formulate our problem in game theoretic framework to derive the time-consistent strategy. We aim at extending the time-consistent reinsurance-investment strategy to account for not only a defaultable security for insurers, but also discontinuous price process of risky asset.

The rest of the paper is organized as follows. In Section 2, model and assumptions are introduced. Section 3 formulates a time-consistent mean-variance reinsurance-investment problem and provides a verification theorem. In Section 4, the corresponding reinsurance-investment strategy and the optimal value function are derived for the post-default case and the pre-default case, an then, some special cases of our model are presented. In Section 5, we provide some numerical examples to demonstrate the impacts of some model parameters on the optimal time-consistent reinsurance-investment strategy. The final section concludes the paper.

2. Model and assumptions

We consider a complete probability space (Ω, G, Q) that is endowed with a filtration $G = (G_t)_{t\geq 0}$. We denote by Q the risk neutral measure (or a martingale probability measure), which is assumed to be equivalent to the real world measure P. Let τ be a nonnegative random variable on this space which presents the default time of the company issuing the bond. Define a default process by $H_t = 1_{\{\tau\leq t\}}$. It is a nondecreasing right continuous process which makes discrete jumps at a random time τ. The enlarged filtration G_t is assumed to be generated by two Poisson process $\{N_1(t)\}$, $\{N_2(t)\}$ and a standard Brownian motions $\{W(t)\}$. For the sake of convenience, we assume that $\{H_t\}$

is a Poisson process with a constant intensity h^Q, then the martingale default process is given by

$$M^Q(t) = H(t) - \int_0^t (1 - H(u-))h^Q du.$$

Moreover, we assume that a defaultable asset (corporate bond) with a maturity date T_1. The default recovery rate is denoted by $1 - \varsigma$, where ς is called loss rate, which is a constant and takes value between zero and one. As in [4] and [11], we denote the default risk premium by $\frac{1}{\Delta} = \frac{h^Q}{h^P} \geq 1$. Then the dynamics of the defaultable bond price under real world probability measure P follows

$$dp(t, T_1) = p(t-, T_1)[rdt + (1 - H(t))\delta(1 - \Delta)dt - (1 - H(t-))\varsigma dM^P(t)],$$

where $\delta = h^Q\varsigma$ is the risk neutral credit spread and $M^P(t) = H(t) - h^Q \int_0^t \Delta(1 - H(u))du$ is a G-martingale under P.

The price process of the risk-free asset is modeled by

$$dR(t) = rR(t)dt,$$

where $r > 0$ is the constant risk-free interest rate. And a risky asset (e.g. a stock) whose price process is described by

$$dS(t) = S(t)[\mu dt + \sigma dW^P(t) + d\sum_{i=1}^{N_1(t)} Y_i],$$

where μ is the appreciation rate, $\sigma > 0$ represents the volatility, $\{W(t)\}$ is a standard Brownian motion and $N_1(t)$ is a homogeneous Poisson process with intensity λ_1, which denotes the number of the jumps of the risky asset's price occurring during time interval $[0, t]$. Random variables Y_i, $i = 1, 2, \cdots$ are independent and identically distributed, where Y_i stands the ith jump amplitude of the risky asset's price and the first and second moments of Y_i are denoted by $\mu_Y = \mathrm{E}(Y_i)$ and $\sigma_Y^2 = \mathrm{E}(Y_i^2)$. We assume that $\mu > r$ for convention. Moreover, to ensure the price of the risky asset remains positive, we assume that $P(Y_i \geq -1) = 1$ for all $i \geq 1$.

We now describe the surplus process for the insurer. It is modelled by a classical compound Poisson risk process. Then, the reserve denoted by $U(t)$ of an insurer at time t can be given as follows

$$U(t) = u + ct - \sum_{i=1}^{N_2(t)} X_i,$$

where $c > 0$ is the premium rate, u is the initial capital, X_i is the ith claim and $N_2(t)$ is a Poisson process with intensity $\lambda_2 > 0$. All the claims X_i are assumed to be independent and identically distributed positive random variables with common distribution F having a finite first and second moments μ_X and σ_X^2. In addition, we assume that $\{W^P(t)\}$, $\{\sum_{i=1}^{N_2(t)} X_i\}$, $\{\sum_{i=1}^{N_1(t)} Y_i\}$ are independent. Throughout this paper, suppose that the insurance premium is calculated according to the expected value principle, then the premium rate is $c = (1 + \theta)\lambda_2\mu_X$, in which θ is the safety loading of the insurer. The insurer is allowed to purchase proportional reinsurance or acquire new business to control his insurance risk. We denote by $1 - q(t)$ the proportional reinsurance, where $q(t) \in [0, 1]$ is the value of the risk exposure at time t. With the proportional reinsurance being incorporated, the evolution of the reserve process of the insurer becomes

$$U(t, q) = u + \int_0^t ((\theta - \eta) + (1 + \eta)q(s))\lambda_2\mu_X ds - \int_0^t q(s)d\sum_{i=1}^{N_2(s)} X_i.$$

The insurer also has investment opportunities in a risk-free asset, a risky asset and a defaultable bond over the investment horizon $[0,T]$. We assume throughout that $T<T_1$, where T_1 is the maturity time of the defaultable bond. Let $\pi=(q(t),\pi_1(t),\pi_2(t))_{t\in[0,T]}$ be the trading strategy, where $\pi_1(t)$ and $\pi_2(t)$ are denoted by the amounts of wealth invested in the stock market and defaultable bond at time t, respectively. The amounts of wealth invested in risk-free asset at time t is $X(t)-\pi_1(t)-\pi_2(t)$, where $X(t)$ is the wealth of the insurer at time t. Then the surplus process $\{X(t),\ t\in[0,T]\}$ of the insurer is described by

$$\begin{aligned} dX(t) = {} & \pi_1(t)\frac{dS(t)}{S(t)}+\pi_2(t)\frac{dp(t,T_1)}{p(t-,T_1)}+(X(t)-\pi_1(t)-\pi_2(t))\frac{dR(t)}{R(t)}+dU(t,q) \\ = {} & \{rX(t)+(\theta-\eta+(1+\eta)q(t))\lambda_2\mu_X+\pi_1(t)(\mu-r) \\ & +\pi_2(t)(1-H(t))(1-\Delta)\delta\}dt+\sigma\pi_1(t)dW^P(t)-q(t)d\Big(\sum_{i=1}^{N_2(t)}X_i\Big) \\ & +\pi_1(t)d\Big(\sum_{i=1}^{N_1(t)}Y_i\Big)-\pi_2(t)\varsigma dM^P(t), \qquad X(0)=x_0 \end{aligned}$$

with $(1-H(t-))dM^P(t)=dM^P(t)$, where $\eta>\theta$ presents the safety loading of the reinsure.

2.1. Definition. For any fixed $t\in[0,T]$, a reinsurance-investment strategy $\pi=(q(s),\pi_1(s),\pi_2(s))_{s\in[t,T]}$ is said to be admissible if it satisfies the following conditions
(1) $(\pi_1(s),\pi_2(s),q(s)$ is $G-$ predictable;
(2) $q(s)\geq 0$ for each $s\in[t,T]$ and $\mathrm{E}[\int_t^T(q(s)^2+\pi_1(s)^2+\pi_2(s)^2)ds]<\infty$;
(3) (π,X^π) is a unique strong solution to the stochastic differential equation.

In addition, for any initial condition $(t,x,z)\in[0,T]\times R\times\{0,1\}$, the set of all admissible strategies is denoted by $\Pi(t,x,z)$.

3. Problem formulation and verification theorem

In this section, we consider a mean-variance problem for an insurer choosing the optimal strategy. Suppose that the insurer is allowed to purchase reinsurance (or acquire new business) and invest the stock, the risky-free asset and the defaultable bond. For any fixed $(t,x,z)\in[0,T]\times R\times\{0,1\}$, the insurer aims to obtain an admissible investment reinsurance policy so as to solve the problem as follows

$$\sup_{\pi\in\Pi(t,x,z)} f(t,x,z,\pi)=\sup_{\pi\in\Pi(t,x,z)}\{\mathrm{E}_{t,x,z}[X^\pi(T)]-\frac{\gamma}{2}\mathrm{Var}_{t,x,z}[X^\pi(T)]\}, \tag{3.1}$$

where γ is a risk aversion coefficient of the insurer, $\mathrm{E}_{t,x,z}[\cdot]=\mathrm{E}[\cdot|X^\pi(t)=x,H(t)=z]$ and $\mathrm{Var}_{t,x,z}[\cdot]=\mathrm{Var}[\cdot|X^\pi(t)=x,H(t)=z]$. Due to that problem (3.1) has a non-linear function of expectation of terminal wealth in the variance term, in the sense that the Bellman optimality principle does not allay directly. We find that the problem is time-inconsistent. However, under many situation, time-consistent policy is a basic requirement of rational decision-marking. Similar to [5], we attempt to solve problem (3.1) in a game theoretic framework. That is, we think about our problem as a non-cooperative game, with one player for each time t, where player t is the future incarnation of the insurer at time t.

Based on this, we now give the formal definition of an equilibrium strategy and the equilibrium function for problem (3.1).

3.1. Definition. For any fixed chosen initial state $(t,x,z)\in[0,T]\times R\times\{0,1\}$, consider an admissible strategy $\pi^*(t,x,z)$. Choose four fixed real numbers $\tilde{q}>0,\ \tilde{\pi}_1\in R,\ \tilde{\pi}_2\in R$ and $\tau>0$ and define the following strategy:

$$\pi^\tau(s,\tilde{x},\tilde{z})=\begin{cases}(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{q}), & t\le s<t+\tau,\ \tilde{x}\in R,\ \tilde{z}\in\{0,1\},\\ \pi^*(s,\tilde{x},\tilde{z}), & t+\tau\le s<T,\ \tilde{x}\in R,\ \tilde{z}\in\{0,1\}.\end{cases}\tag{3.2}$$

If

$$\lim_{\tau\to 0}\inf\frac{f(t,x,z,\pi^*)-f(t,x,z,\pi^\tau)}{\tau}\ge 0$$

for all $(\tilde{\pi}_1,\tilde{\pi}_2,\tilde{q})\in R\times R\times R_+$ and $(t,x,z)\in[0,T]\times R\times\{0,1\}$, then we call π^* is an equilibrium strategy and the corresponding equilibrium value function $V(t,x,z)$ is defined by

$$V(t,x,z)=f(t,x,z,\pi^*)=\mathrm{E}_{t,x,z}[X^{\pi^*}(T)]-\frac{\gamma}{2}\mathrm{Var}_{t,x,z}[X^{\pi^*}(T)].\tag{3.3}$$

Note that the equilibrium strategy is time-consistent. Hereafter, we call the equilibrium value function $V(t,x,z)$ and the equilibrium strategy π^* the optimal value function and optimal time-consistent strategy for problem (3.1) in this paper. Before giving the verification theorem, we define the infinitesimal generator. For any $f(t,x,z)\in C^{1,2}([0,T]\times R\times\{0,1\})$, let

$$\begin{aligned}A^\pi f(t,x,z)= \;& f_t(t,x,z)+f_x(t,x,z)[rx+(\theta-\eta)\lambda_2\mu_X+(1+\eta)q(t)\\ & \lambda_2\mu_X+\pi_1(t)(\mu-r)+\pi_2(t)\delta(1-z)]+0.5\sigma^2\pi_1(t)^2\\ & f_{xx}(t,x,z)+\lambda_2\mathrm{E}[f(t,x-qX_1,z)-f(t,x,z)]\\ & +\lambda_1\mathrm{E}[f(t,x+\pi_1(t)Y_1,z)-f(t,x,z)]\\ & +[f(t,x-\pi_2(t)\varsigma,z+1)-f(t,x,z)]h^P(1-z),\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}C^{1,2}([0,T]\times R\times\{0,1\})= \;& \{f(t,x,z)|f(t,x,z)\text{ is continuously differential in }t\\ & \text{and twice continuously differential in }x\}.\end{aligned}$$

Specifically, for the post-default case, i.e., $z=1$, the extended HJB system is given by

$$\sup_{\pi\in\Pi}\{A^\pi W(t,x,1)-A^\pi\frac{\gamma}{2}g(t,x,1)^2+\gamma g(t,x,1)A^\pi g(t,x,1)\}=0,\tag{3.5}$$

$$W(T,x,1)=x,\tag{3.6}$$

$$A^{\pi^*}g(t,x,1)=0,\tag{3.7}$$

$$g(T,x,1)=x,\tag{3.8}$$

where

$$\pi^*=arg\sup_{\pi\in\Pi}\{A^\pi W(t,x,1)-A^\pi\frac{\gamma}{2}g(t,x,1)^2+\gamma g(t,x,1)A^\pi g(t,x,1)\}.$$

When $z=1$, there is not trading for the defaultable bond. In this case, we can use the verification theorem in [21] directly. It is easy to see that if there are two functions $W(t,x,1),\ g(t,x,1)\in C^{1,2}([0,T]\times R)$ satisfying the above extended HJB system, then $V(t,x,1)=W(t,x,1)$, $\mathrm{E}_{t,x,1}[X^{\pi^*}(T)]=g(t,x,1)$, and π^* is the optimal time-consistent reinsurance-investment strategy. For the pre-default case, the verification theorem for the extended HJB system is stated as follows.

3.2. Theorem. *Let $W(t,x,1)$ and $g(t,x,1)$ be solutions (3.5)-(3.8). If there are two functions $W(t,x,0),\ g(t,x,0)\in C^{1,2}$ satisfying the following extended HJB system*

$$\sup_{\pi\in\Pi}\{A^\pi W(t,x,0)-A^\pi\frac{\gamma}{2}g(t,x,0)^2+\gamma g(t,x,0)A^\pi g(t,x,0)\}=0,\tag{3.9}$$

$$W(T,x,0)=x,\tag{3.10}$$

$$A^{\pi^*}g(t,x,0)=0,\tag{3.11}$$

(3.12) $\quad g(T,x,0)=x.$

where

$$\pi^*=arg\sup_{\pi\in\Pi}\{A^\pi W(t,x,0)-A^\pi\frac{\gamma}{2}g(t,x,0)^2+\gamma g(t,x,0)A^\pi g(t,x,0)\},$$

then $V(t,x,0)=W(t,x,0)$, $E_{t,x,0}[X^{\pi^*}(T)]=g(t,x,0)$ *and* π^* *is the optimal time-consistent reinsurance-investment strategy.*

The proof of this theorem is similar to the proof of Theorem 4.1 in [5], Theorem 2.1 in [6], so we omit it here.

4. Solution to the problem

In this section, we will solve the problem (3.1) in the post-default case and pre-default case, respectively. In fact, in the post-default case, i.e., $\tau\le t$, the defaultable bond is not traded. We have $p(t,T_1)=0$, $\tau\le t\le T$. So $\pi_2(t)\equiv 0$ for $\tau\le t$. For the detailed introduction, see [15].

4.1. Post-default case. Suppose that there exist two functions $V(t,x,1)$ and $g(t,x,1)$ are solutions of (3.5)-(3.8) such that $A^\pi V(t,x,1)-A^\pi\frac{\gamma}{2}g(t,x,1)^2+\gamma g(t,x,1)A^\pi g(t,x,1)$ is concave with respect to $q(t)$ and $\pi_1(t)$ for any admissible strategy π. Then from (3.4) and (3.5), we have

$$\begin{aligned}
&\sup_{\pi\in\Pi}\Big\{V_t(t,x,1)+V_x(t,x,1)[rx+(\theta-\eta)\lambda_2\mu_X+(1+\eta)q(t)\lambda_2\mu_X\\
&+\pi_1(t)(\mu-r)]+\frac{\sigma^2\pi_1(t)^2}{2}(V_{xx}(t,x,1)-\gamma g_x(t,x,1)^2)\\
&+\lambda_2\mathrm{E}[V(t,x-q(t)X_1,1)-\frac{\gamma}{2}g(t,x-q(t)X_1,1)(g(t,x-q(t)X_1,1)\\
&-2g(t,x,1))]+\lambda_1\mathrm{E}[V(t,x+\pi_1(t)Y_1,1)-\frac{\gamma}{2}g(t,x+\pi_1(t)Y_1,1)\\
&\quad(g(t,x+\pi_1(t)Y_1,1)-2g(t,x,1))]-(\lambda_1+\lambda_2)[V(t,x,1)\\
&\quad+\frac{\gamma}{2}g(t,x,1)^2]\Big\}=0.
\end{aligned}\tag{4.1}$$

Given the structure of the (3.7) and (4.1), as well as the boundary conditions of $V(t,x,1)$ and $g(t,x,1)$ given by (3.6) and (3.8), it is natural to assume that

$$V(t,x,1)=A(t)x+\frac{B(t)}{\gamma},\ A(T)=1,\ B(T)=0\tag{4.2}$$

$$g(t,x,1)=a(t)x+\frac{b(t)}{\gamma},\ a(T)=1,\ b(T)=0.\tag{4.3}$$

We have the partial derivatives

$$V_t(t,x,1)=A'(t)(x)+\frac{B'(t)}{\gamma},\ V_x(t,x,1)=A(t),\ V_{xx}(t,x,1)=0,$$

$$g_t(t,x,1)=a'(t)(x)+\frac{b'(t)}{\gamma},\ g_x(t,x,1)=a(t),\ g_{xx}(t,x,1)=0.$$

Plugging (4.2), (4.3) and the above partial derivatives into (4.1) yields

$$\begin{aligned}
&\sup_{\pi\in\Pi}\Big\{A'(t)x+\frac{B'(t)}{\gamma}+A(t)[rx+(\theta-\eta)\lambda_2\mu_X+\eta\lambda_2\mu_X q(t)+\pi_1(t)\\
&\cdot(\mu-r+\lambda_1\mu_Y)]-0.5\gamma a(t)^2[q(t)^2\lambda_2\sigma_X^2+\pi_1(t)^2(\sigma^2+\lambda_1\sigma_Y^2)]\Big\}=0.
\end{aligned}\tag{4.4}$$

Then, the first-order maximization conditions for the optimal strategy $(q^*(t),\ \pi_1^*(t),\ \pi_2^*(t))$ are

$$\eta\lambda_2\mu_X A(t) - \gamma\lambda_2\sigma_X^2 a(t)^2 q^*(t) = 0,$$
$$(\mu - r)A(t) + \lambda_1\mu_Y A(t) - \gamma a(t)^2\pi_1^*(t)(\sigma^2 + \lambda_1\sigma_Y^2) = 0.$$

Therefore, we have

$$q^*(t) = \frac{\eta\mu_X}{\gamma\sigma_X^2}\frac{A(t)}{a(t)^2},\ \pi_1^*(t) = \frac{\mu - r + \lambda_1\mu_Y}{\gamma(\sigma^2 + \lambda_1\sigma_Y^2)}\frac{A(t)}{a(t)^2}. \tag{4.5}$$

Inserting (4.5) into (4.4) and (3.4), we obtain

$$A'(t)x + \frac{B'(t)}{\gamma} + rA(t)x + (\theta - \eta)\lambda_2\mu_X A(t) + \beta\frac{A(t)^2}{2\gamma a(t)^2} = 0, \tag{4.6}$$

and

$$a'(t)x + \frac{b'(t)}{\gamma} + ra(t)x + (\theta - \eta)\lambda_2\mu_X a(t) + \beta\frac{A(t)^2}{\gamma a(t)^2} = 0, \tag{4.7}$$

where $\beta = \dfrac{\lambda_2\eta^2\mu_X^2}{\sigma_X^2} + \dfrac{(\mu - r + \lambda_1\mu_Y)^2}{\sigma^2 + \lambda_1\sigma_Y^2}$.

By matching coefficients, we decompose (4.6) and (4.7) into

$$A'(t) + rA(t) = 0,\ A(T) = 1, \tag{4.8}$$

$$\frac{B'(t)}{\gamma} + (\theta - \eta)\lambda_2\mu_X A(t) + \beta\frac{A(t)^2}{2\gamma a(t)^2},\ B(T) = 0, \tag{4.9}$$

$$a'(t) + ra(t) = 0,\ a(T) = 1, \tag{4.10}$$

and

$$\frac{b'(t)}{\gamma} + (\theta - \eta)\lambda_2\mu_X a(t) + \beta\frac{A(t)^2}{\gamma a(t)^2},\ b(T) = 0. \tag{4.11}$$

Solving the equations (4.8) and (4.11), respectively, we have

$$A(t) = e^{r(T-t)}, \tag{4.12}$$

and

$$a(t) = e^{r(T-t)}. \tag{4.13}$$

By plugging (4.12) and (4.13), then

$$B(t) = \frac{\gamma(\theta - \eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \frac{\beta(T-t)}{2}, \tag{4.14}$$

$$b(t) = \frac{\gamma(\theta - \eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \beta(T-t). \tag{4.15}$$

Under the above discuss, we have the following Proposition without proof.

4.1. Proposition. *For the time-consistent mean-variance reinsurance investment policy selection problem in the post-default case, the optimal time-consistent strategy is*

$$q^*(t) = \frac{\eta\mu_X}{\gamma\sigma_X^2}e^{-r(T-t)},\ \pi_1^*(t) = \frac{\mu - r + \lambda_1\mu_Y}{\gamma(\sigma^2 + \lambda_1\sigma_Y^2)}e^{-r(T-t)},\ \pi_2^*(t) = 0. \tag{4.16}$$

The optimal value function is given by

$$V(t, x, 1) = e^{r(T-t)}x + \frac{(\theta - \eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \frac{\beta(T-t)}{2\gamma}. \tag{4.17}$$

The expectation and variance of the terminal surplus of the insurer are

$$E_{t,x,1}[X^{\pi^*}(T)] = xe^{r(T-t)} + \frac{(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-t)}-1) + \sqrt{Var_{t,x,1}[X^{\pi^*}]\beta(T-t)},$$

and

$$Var_{t,x,1}[X^{\pi^*}(T)] = \frac{\beta(T-t)}{\gamma^2}.$$

4.2. Period before default. In this subsection, we will address the pre-default case, i.e., $z=0$. Suppose that there are two functions $V(t,x,0)$ and $g(t,x,0)$ satisfying the conditions given in Theorem 3.1 such that $A^\pi V(t,x,0) - A^\pi \frac{\gamma}{2} g(t,x,0)^2 + \gamma g(t,x,0) A^\pi g(t,x,0)$ is concave w.r.t. $q(t)$, $\pi_1(t)$ and $\pi_2(t)$ for any admissible strategy. From (3.4) and Theorem 3.1, then (3.9) can be rewritten as

$$\begin{aligned}
&\sup_{\pi\in\Pi}\Big\{V_t(t,x,0)+V_x(t,x,0)[rx+(\theta-\eta)\lambda_2\mu_X+(1+\eta)q(t)\lambda_2\mu_X\\
&+\pi_2(t)\delta+\pi_1(t)(\mu-r)]+\frac{\sigma^2\pi_1(t)^2}{2}(V_{xx}(t,x,0)-\gamma g_x(t,x,0)^2)\\
&+\lambda_2\mathrm{E}[V(t,x-q(t)X,0)-\frac{\gamma}{2}g(t,x-q(t)X,0)(g(t,x-q(t)X,0)\\
&-2g(t,x,0))]+\lambda_1\mathrm{E}[V(t,x+\pi_1(t)Y,0)-\frac{\gamma}{2}g(t,x+\pi_1(t)Y,0)\\
&(g(t,x+\pi_1(t)Y,0)-2g(t,x,0))]-(\lambda_1+\lambda_2)[V(t,x,0)\\
&\qquad+\frac{\gamma}{2}g(t,x,0)^2]+[V(t,x-\varsigma\pi_2(t),1)-V(t,x,0)]h^P\\
&\qquad-\frac{\gamma}{2}[g(t,x-\gamma\pi_2(t),1)-g(t,x,0)]^2h^P\Big\}=0.
\end{aligned}\tag{4.18}$$

Similar to the post-default case, we assume that

$$V(t,x,0)=\tilde{A}(t)x+\frac{\tilde{B}(t)}{\gamma},\tag{4.19}$$

with the boundary conditions

$$\tilde{A}(T)=1,\ \tilde{B}(T)=0,\tag{4.20}$$

and

$$g(t,x,0)=\tilde{a}(t)x+\frac{\tilde{b}(t)}{\gamma},\tag{4.21}$$

with the boundary conditions

$$\tilde{a}(T)=1,\ \tilde{b}(T)=0.\tag{4.22}$$

Substituting (4.19) and (4.21) into (4.18), we have after simplification

$$\begin{aligned}
&\sup_{\pi\in\Pi}\Big\{\tilde{A}'(t)x+\frac{\tilde{B}'(t)}{\gamma}+\tilde{A}(t)\Big[rx+(\theta-\eta+\eta q(t))\lambda_2\mu_X+\pi_2(t)\delta\\
&+\pi_1(t)(\mu-r+\lambda_1\mu_Y)\Big]-0.5\gamma\tilde{a}(t)^2\Big[q(t)^2\lambda_2\sigma_X^2+\pi_1(t)^2(\sigma^2\\
&+\lambda_1\sigma_Y^2)\Big]+\Big[A(t)(x-\varsigma\pi_2(t))+\frac{B(t)}{\gamma}-\tilde{A}(t)x-\frac{\tilde{B}(t)}{\gamma}\Big]h^P\\
&-0.5\gamma\Big[a(t)(x-\varsigma\pi_2(t))+\frac{b(t)}{\gamma}-\tilde{a}(t)x-\frac{\tilde{b}(t)}{\gamma}\Big]^2h^P\Big\}=0.
\end{aligned}\tag{4.23}$$

Differentiating (4.23) with respect to $q(t)$, $\pi_1(t)$ and $\pi_2(t)$ gives

$$q^*(t)=\frac{\eta\mu_X}{\gamma\sigma_X^2}\frac{\tilde{A}(t)}{\tilde{a}(t)^2},\ \pi_1^*(t)=\frac{\mu-r+\lambda_1\mu_Y}{\gamma(\sigma^2+\lambda_1\sigma_Y^2)}\frac{\tilde{A}(t)}{\tilde{a}(t)^2},\tag{4.24}$$

and

$$\pi_2^*(t)= \frac{x}{\varsigma}-\frac{1}{\gamma\varsigma}e^{-r(T-t)}+\frac{b(t)-\tilde{b}(t)-\gamma x\tilde{a}(t)}{\gamma\varsigma}e^{-r(T-t)} +\frac{\tilde{A}(t)\delta}{\gamma\varsigma^2h^P}e^{-2r(T-t)}. \tag{4.25}$$

Notice that

$$\begin{aligned}
&\tilde{A}(t)(\eta\lambda_2\mu_X q^*(t)+\pi_1^*(t)(\mu-r+\lambda_1\mu_Y))-0.5\gamma\tilde{a}(t)^2(q^*(t)^2\lambda_2\sigma_X^2+\pi_1^*(t)^2\\
&\cdot(\sigma^2+\lambda_1\sigma_Y^2))\\
&=\tilde{A}(t)\Big(\frac{\lambda_2\eta^2\mu_X^2}{\gamma\sigma_X^2}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}+\frac{(\mu-r+\lambda_1\mu_Y)^2}{\gamma(\sigma^2+\lambda_1\sigma_Y^2)}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}\Big)\\
&-0.5\gamma\tilde{a}(t)^2\Big(\Big(\frac{\eta\mu_X}{\gamma\sigma_X^2}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}\Big)^2\lambda_2\sigma_X^2+\Big(\frac{\mu-r+\lambda_1\mu_Y}{\gamma(\sigma^2+\lambda_1\sigma_Y^2)}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}\Big)^2(\sigma^2+\lambda_1\sigma_Y^2)\Big)\\
&=\frac{\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2}\Big(\frac{\lambda_2\eta^2\mu_X^2}{\sigma_X^2}+\frac{(\mu-r+\lambda_1\mu_Y)^2}{\sigma^2+\lambda_1\sigma_Y^2}\Big)=\frac{\beta\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{a}(t)(\eta\lambda_2\mu_X q^*(t)+\pi_1^*(t)(\mu-r+\lambda_1\mu_Y))\\
&=\tilde{a}(t)\Big(\frac{\lambda_2\eta^2\mu_X^2}{\gamma\sigma_X^2}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}+\frac{(\mu-r+\lambda_1\mu_Y)^2}{\gamma(\sigma^2+\lambda_1\sigma_Y^2)}\frac{\tilde{A}(t)}{\tilde{a}(t)^2}\Big)\\
&=\frac{\tilde{A}(t)}{\gamma\tilde{a}(t)}\Big(\frac{\lambda_2\eta^2\mu_X^2}{\sigma_X^2}+\frac{(\mu-r+\lambda_1\mu_Y)^2}{\sigma^2+\lambda_1\sigma_Y^2}\Big)=\frac{\beta\tilde{A}(t)}{\gamma\tilde{a}(t)}.
\end{aligned}$$

By putting (4.24),(4.25) into (4.23) and (3.4), we get

$$\begin{aligned}
&\tilde{A}'(t)x+\frac{\tilde{B}'(t)}{\gamma}+\tilde{A}(t)(rx+(\theta-\eta)\lambda_2\mu_X)+\tilde{A}(t)\Big(\frac{\delta x}{\varsigma}-\frac{\delta}{\gamma\varsigma}e^{-r(T-t)}\\
&+\delta\frac{(b(t)-\tilde{b}(t))}{\gamma\varsigma}e^{-r(T-t)}-\frac{\delta x}{\varsigma}\tilde{a}(t)e^{-r(T-t)}+\frac{\tilde{A}(t)\delta^2}{\gamma\varsigma^2h^P}e^{-2r(T-t)}\Big)+\frac{\beta\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2}\\
&+\Big(e^{r(T-t)}\Big(\frac{1}{\gamma}e^{-r(T-t)}-\frac{(b(t)-\tilde{b}(t))}{\gamma}e^{-r(T-t)}-\frac{\tilde{A}(t)\delta}{\gamma\varsigma h^P}e^{-2r(T-t)}\\
&+x\tilde{a}(t)e^{-r(T-t)}\Big)+\frac{B(t)-\tilde{B}(t)}{\gamma}-\tilde{A}(t)x\Big)h^P-\frac{1}{2}\gamma\Big(e^{r(T-t)}\Big(\frac{1}{\gamma}e^{-r(T-t)}\\
&-\frac{(b(t)-\tilde{b}(t))}{\gamma}e^{-r(T-t)}-\frac{\tilde{A}(t)\delta}{\gamma\varsigma h^P}e^{-2r(T-t)}+x\tilde{a}(t)e^{-r(T-t)}\Big)+\frac{b(t)-\tilde{b}(t)}{\gamma}\\
&-\tilde{a}(t)x\Big)^2h^P\\
&=\tilde{A}'(t)x+\frac{\tilde{B}'(t)}{\gamma}+\tilde{A}(t)(rx+(\theta-\eta)\lambda_2\mu_X)+\tilde{A}(t)\frac{\delta x}{\varsigma}-\frac{\delta}{\gamma\varsigma}\tilde{A}(t)e^{-r(T-t)}\\
&+\delta\frac{(b(t)-\tilde{b}(t))}{\gamma\varsigma}\tilde{A}(t)e^{-r(T-t)}-\frac{\delta x}{\varsigma}\tilde{A}(t)\tilde{a}(t)e^{-r(T-t)}+\frac{\tilde{A}(t)^2\delta^2}{\gamma\varsigma^2h^P}e^{-2r(T-t)}\\
&+\frac{\beta\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2}+\frac{h^P}{\gamma}-\frac{(b(t)-\tilde{b}(t))}{\gamma}h^P-\frac{\tilde{A}(t)\delta}{\gamma\varsigma}e^{-r(T-t)}+x\tilde{a}(t)h^P-\tilde{A}(t)h^Px\\
&+\frac{B(t)-\tilde{B}(t)}{\gamma}h^P-\frac{1}{2}\gamma\Big(\frac{1}{\gamma}-\frac{\tilde{A}(t)\delta}{\gamma\varsigma h^P}e^{-r(T-t)}\Big)^2h^P\\
&=\Big[\tilde{A}'(t)+\Big(r+\frac{\delta}{\varsigma}-\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)}-h^P\Big)\tilde{A}(t)+h^P\tilde{a}(t)\Big]x+\frac{\tilde{B}'(t)}{\gamma}+\frac{\beta\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2}\\
&+(\theta-\eta)\lambda_2\mu_X\tilde{A}(t)+\frac{1}{\gamma}\Big(\frac{\delta\tilde{A}(t)e^{-r(T-t)}}{2\varsigma h^P}-1+b(t)-\tilde{b}(t)\Big)\frac{\delta\tilde{A}(t)}{\varsigma}e^{-r(T-t)}\\
&+\frac{h^P}{\gamma}\Big(B(t)-\tilde{B}(t)+\tilde{b}(t)-b(t)+\frac{1}{2}\Big)=0,
\end{aligned}$$

and

$$
\begin{aligned}
&\tilde{a}'(t)x+\frac{\tilde{b}'(t)}{\gamma}+\tilde{a}(t)(rx+(\theta-\eta)\lambda_2\mu_X+q^*(t)\lambda_2\mu_X+q^*(t)\lambda_2\mu_X\eta\\
&+\pi_1^*(t)(\mu-r))+\tilde{a}(t)\Big(\frac{\delta x}{\varsigma}-\frac{\delta}{\gamma\varsigma}e^{-r(T-t)}+\delta\frac{(b(t)-\tilde{b}(t))}{\gamma\varsigma}e^{-r(T-t)}\\
&-\frac{\delta x}{\varsigma}\tilde{a}(t)e^{-r(T-t)}+\frac{\tilde{A}(t)\delta^2}{\gamma\varsigma^2h^P}e^{-2r(T-t)}\Big)+\lambda_2\mathrm{E}\Big(\tilde{a}(t)(x-q^*(t)X_1)+\frac{\tilde{b}(t)}{\gamma}-\tilde{a}(t)x\\
&-\frac{\tilde{b}(t)}{\gamma}\Big)+\lambda_1\mathrm{E}\Big(\tilde{a}(t)(x+\pi_1^*(t)Y_1)+\frac{\tilde{b}(t)}{\gamma}-\tilde{a}(t)x-\frac{\tilde{b}(t)}{\gamma}\Big)+\Big(a(t)(x-\pi_2^*(t)\varsigma)\\
&+\frac{b(t)}{\gamma}-\tilde{a}(t)x-\frac{\tilde{b}(t)}{\gamma}\Big)h^P\\
&=\tilde{a}'(t)x+\frac{\tilde{b}'(t)}{\gamma}+\tilde{a}(t)(rx+(\theta-\eta)\lambda_2\mu_X)+\frac{\beta\tilde{A}(t)}{\gamma\tilde{a}(t)}+\tilde{a}(t)\Big(\frac{\delta x}{\varsigma}-\frac{\delta}{\gamma\varsigma}e^{-r(T-t)}\\
&+\delta\frac{(b(t)-\tilde{b}(t))}{\gamma\varsigma}e^{-r(T-t)}-\frac{\delta x}{\varsigma}\tilde{a}(t)e^{-r(T-t)}+\frac{\tilde{A}(t)\delta^2}{\gamma\varsigma^2h^P}e^{-2r(T-t)}\Big)+\Big(e^{r(T-t)}\\
&\cdot\Big(\frac{1}{\gamma}e^{-r(T-t)}-\frac{(b(t)-\tilde{b}(t))}{\gamma}e^{-r(T-t)}-\frac{\tilde{A}(t)\delta}{\gamma\varsigma h^P}e^{-2r(T-t)}+x\tilde{a}(t)e^{-r(T-t)}\Big)\\
&+\frac{b(t)-\tilde{b}(t)}{\gamma}-\tilde{a}(t)x\Big)h^P\\
&=\Big[\tilde{a}'(t)+\Big(r+\frac{\delta}{\varsigma}-\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)}\Big)\tilde{a}(t)\Big]x+\frac{\tilde{b}'(t)}{\gamma}+\frac{\beta\tilde{A}(t)}{\gamma\tilde{a}(t)}+(\theta-\eta)\lambda_2\mu_X\tilde{a}(t)\\
&+\frac{1}{\gamma}\Big(\frac{\delta\tilde{A}(t)e^{-r(T-t)}}{\varsigma h^P}-1+b(t)-\tilde{b}(t)\Big)\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)}-\frac{\delta\tilde{A}(t)}{\gamma\varsigma}e^{-r(T-t)}\\
&+\frac{h^P}{\gamma}=0.
\end{aligned}
$$

By separating the variables with x and without x, respectively, we have the following ordinary differential equations:

$$
\tilde{a}'(t)+(r+\frac{\delta}{\varsigma}-\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)})\tilde{a}(t)=0,\ \tilde{a}(T)=1, \tag{4.26}
$$

$$
\tilde{A}'(t)+(r+\frac{\delta}{\varsigma}-\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)}-h^P)\tilde{A}(t)+h^P\tilde{a}(t)=0,\ \tilde{A}(T)=1, \tag{4.27}
$$

$$
\begin{aligned}
&\frac{\tilde{B}'(t)}{\gamma}+(\theta-\eta)\lambda_2\mu_X\tilde{A}(t)+\frac{1}{\gamma}\Big(\frac{\delta\tilde{A}(t)e^{-r(T-t)}}{2\varsigma h^P}-1+b(t)-\tilde{b}(t)\Big)\\
&\frac{\delta\tilde{A}(t)}{\varsigma}e^{-r(T-t)}+\frac{\beta\tilde{A}(t)^2}{2\gamma\tilde{a}(t)^2}+\frac{h^P}{\gamma}\Big(B(t)-\tilde{B}(t)+\tilde{b}(t)-b(t)+\frac{1}{2}\Big)\\
&=0,\ \tilde{b}(T)=0,\ \tilde{B}(T)=0,
\end{aligned} \tag{4.28}
$$

and

$$
\begin{aligned}
&\frac{\tilde{b}'(t)}{\gamma}+(\theta-\eta)\lambda_2\mu_X\tilde{a}(t)+\frac{1}{\gamma}\Big(\frac{\delta\tilde{A}(t)e^{-r(T-t)}}{\varsigma h^P}-1+b(t)-\tilde{b}(t)\Big)\\
&\frac{\delta\tilde{a}(t)}{\varsigma}e^{-r(T-t)}+\frac{\beta\tilde{A}(t)}{\gamma\tilde{a}(t)}+\frac{h^P}{\gamma}-\frac{\delta\tilde{A}(t)}{\gamma\varsigma}e^{-r(T-t)}=0,\ \tilde{b}(T)=0.
\end{aligned} \tag{4.29}
$$

Note that (4.26) is a Bernoulli differential equation, we can easily derive the explicit solution

$$
\tilde{a}(t)=e^{r(T-t)}. \tag{4.30}
$$

Putting (4.32) into (4.29), we have

$$
\tilde{A}(t)=e^{-r(T-t)}. \tag{4.31}
$$

From (4.14), (4.15), (4.30) and (4.31), we have

$$\tilde{B}(t) = e^{h^P t} N(t),\ \tilde{b}(t) = e^{\frac{\delta}{\varsigma} t} M(t), \tag{4.32}$$

where

$$\begin{aligned} M(t) = \quad & \frac{\gamma\varsigma(\theta-\eta)\lambda_2\mu_X}{\delta + r\varsigma}\Big(e^{r(T-t)-\frac{\delta}{\varsigma}t} - e^{-\frac{\delta}{\varsigma}T}\Big) + \Big(\frac{\varsigma\beta}{\delta} + \frac{\varsigma h^P}{\delta} + \frac{\delta}{\varsigma h^P} - 2\Big) \\ & \cdot\Big(e^{-\frac{\delta}{\varsigma}t} - e^{-\frac{\delta}{\varsigma}T}\Big) + \frac{\delta}{\varsigma}\int_t^T b(s) e^{-\frac{\delta}{\varsigma}s} ds, \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} N(t) = \quad & \frac{\gamma(\theta-\eta)\lambda_2\mu_X}{r}\Big(e^{rT-(r+h^P)t} - e^{-h^P T}\Big) \\ & + \frac{1}{2h^P}\Big(\frac{2\delta}{\varsigma} - \frac{\delta^2}{\varsigma^2 h^P} - \beta - h^P\Big)\Big(e^{-h^P T} - e^{-h^P t}\Big) \\ & - \int_t^T \Big[\Big(h^P - \frac{\delta}{\varsigma}\Big)\Big(b(s) - \tilde{b}(s)\Big) - h^P B(s)\Big] e^{-h^P s} ds. \end{aligned} \tag{4.34}$$

Based on the above derivation, we can derive the following theorem.

4.2. Theorem. *For the mean-variance problem (3.1) in the pre-default case, the optimal time-consistent strategy is given by*

$$q^*(t) = \frac{\eta\mu_X}{\gamma\sigma_X^2} e^{-r(T-t)},\ \pi_1^*(t) = \frac{\mu - r + \lambda_1\mu_Y}{\gamma(\sigma^2 + \lambda_1\sigma_Y^2)} e^{-r(T-t)},$$

and

$$\pi_2^*(t) = \frac{1}{\gamma\varsigma}\Big(\frac{\delta}{\varsigma h^P} + b(t) - \tilde{b}(t) - 1\Big)e^{-r(T-t)}. \tag{4.35}$$

The equilibrium value function is given by

$$V(t,x,0) = xe^{r(T-t)} + e^{h^P t}\frac{N(t)}{\gamma}, \tag{4.36}$$

where $b(t)$, $\tilde{b}(t)$ and $N(t)$ are given by (4.15), (4.32) and (4.34), respectively.

4.3. Proposition. *According to the definition of the optimal value function given by (3.3) and Theorem 4.2, we obtain*

$$Var_{t,x,0}[X^{\pi^*}(T)] = \frac{J(t)}{\gamma} + \frac{K(t)}{\gamma^2},$$

and

$$\begin{aligned} E_{t,x,0}[X^{\pi^*}(T)] = \quad & xe^{r(T-t)} + \frac{(\theta-\eta)\lambda_2\mu_X}{r}\Big(e^{r(T-t)} - 1\Big) \\ & + \frac{I(t)\Big(-J(t) + \sqrt{J(t)^2 + 4K(t)Var_{t,x,0}[X^{\pi^*}(T)]}\Big)}{2K(t)}, \end{aligned}$$

where

$$\begin{aligned} J(t) = \quad & 2H(t) - \frac{2(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - e^{-h^P(T-t)}) - \frac{2(\theta-\eta)\lambda_2\mu_X}{r(r+h^P)}(e^{r(T-t)} \\ & - e^{-h^P(T-t)}) + \frac{2(\theta-\eta)\lambda_2\mu_X}{rh^P}(1 - e^{-h^P(T-t)}), \end{aligned}$$

$$\begin{aligned} K(t) = \quad & 2I(t) + \frac{1}{h^P}\Big(\frac{2\delta}{\varsigma} - \frac{\delta^2}{\varsigma^2 h^P} - \beta - h^P\Big)(1 - e^{-h^P(T-t)}) \\ & - 2e^{h^P t}\int_t^T [(h^P - \frac{\delta}{\varsigma})(I(s) - \beta(T-s)) + \frac{h^P}{2}\beta(T-s)] e^{-h^P s} ds, \end{aligned}$$

$$H(t)=\frac{(\theta-\eta)\lambda_2\mu_X}{r}\Big(e^{r(T-t)}-1\Big),$$

$$I(t)=\Big(\frac{\varsigma\beta}{\delta}+\frac{\varsigma h^P}{\delta}+\frac{\delta}{\varsigma h^P}-2\Big)\Big(1-e^{-\frac{\delta}{\varsigma}(T-t)}\Big)+\frac{\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T\beta(T-s)e^{-\frac{\delta}{\varsigma}s}ds.$$

Proof. According to the definition of the optimal value function $V(t,x,z)$ and Theorem 3.2 , we obtain

$$\mathrm{E}_{t,x,0}[X^{\pi^*}(T)]=e^{r(T-t)}x+\frac{e^{\frac{\delta}{\varsigma}t}M(t)}{\gamma},\tag{4.37}$$

and

$$\mathrm{V}ar_{t,x,0}[X^{\pi^*}(T)]=\frac{2}{\gamma^2}(e^{\frac{\delta}{\varsigma}t}M(t)-e^{h^Pt}N(t)),\tag{4.38}$$

where $M(t)$ and $N(t)$ are given by (4.33) and (4.34).

Notice that

$$\begin{aligned}
&\frac{2}{\gamma^2}e^{\frac{\delta}{\varsigma}t}M(t)\\
&=\frac{2\varsigma(\theta-\eta)\lambda_2\mu_X}{\gamma(\delta+r\varsigma)}(e^{r(T-t)}-e^{-\frac{\delta}{\varsigma}(T-t)})-\frac{2}{\gamma^2}\Big(\frac{\varsigma\beta}{\delta}+\frac{\varsigma h^P}{\delta}+\frac{\delta}{\varsigma h^P}-2\Big)(e^{-\frac{\delta}{\varsigma}(T-t)}\\
&-1)+\frac{2\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T\Big(\frac{(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{rT-(r+\frac{\delta}{\varsigma})s}-e^{-\frac{\delta}{\varsigma}s})+\frac{\beta(T-s)}{\gamma^2}e^{-\frac{\delta}{\varsigma}s}\Big)ds\\
&=\frac{2\varsigma(\theta-\eta)\lambda_2\mu_X}{\gamma(\delta+r\varsigma)}(e^{r(T-t)}-e^{-\frac{\delta}{\varsigma}(T-t)})-\frac{2}{\gamma^2}\Big(\frac{\varsigma\beta}{\delta}+\frac{\varsigma h^P}{\delta}+\frac{\delta}{\varsigma h^P}-2\Big)(e^{-\frac{\delta}{\varsigma}(T-t)}\\
&-1)+\frac{2\delta(\theta-\eta)\lambda_2\mu_X}{\gamma r(\delta+r\varsigma)}(e^{r(T-t)}-e^{-\frac{\delta}{\varsigma}(T-t)})+\frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{-\frac{\delta}{\varsigma}(T-t)}-1)\\
&+\frac{2\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T\frac{\beta(T-s)}{\gamma^2}e^{-\frac{\delta}{\varsigma}s}ds\\
&=\frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{r(T-t)}-1)+\frac{2}{\gamma^2}\Big(\frac{\varsigma\beta}{\delta}+\frac{2\varsigma h^P}{\delta}+\frac{\delta}{\varsigma h^P}-2\Big)(1-e^{-\frac{\delta}{\varsigma}(T-t)}\\
&+\frac{2\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T\frac{\beta(T-s)}{\gamma^2}e^{-\frac{\delta}{\varsigma}s}ds\\
&=\frac{2H(t)}{\gamma}+\frac{2I(t)}{\gamma^2},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{\gamma^2}e^{h^Pt}N(t)\\
&=\frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{r(T-t)}-e^{-h^P(T-t)})+\frac{1}{\gamma^2h^P}\Big(\frac{2\delta}{\varsigma}-\frac{\delta^2}{\varsigma^2h^P}-\beta-h^P\Big)(1\\
&-e^{-h^P(T-t)})+\frac{2e^{h^Pt}}{\gamma^2}\int_t^T(h^P-\frac{\delta}{\varsigma})(I(s)-\beta(T-s))e^{-h^Ps}ds+\frac{e^{h^Pt}}{\gamma^2}\\
&\cdot h^P\int_t^T\beta(T-s)e^{-h^Ps}ds+\frac{2e^{h^Pt}}{\gamma}\int_t^T h^P\frac{(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-s)}-1)e^{-h^Ps}ds\\
&=\frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{r(T-t)}-e^{-h^P(T-t)})+\frac{1}{\gamma^2h^P}\Big(\frac{2\delta}{\varsigma}+\frac{\delta^2}{\varsigma^2h^P}-\beta-h^P\Big)(1\\
&-e^{-h^P(T-t)})+\frac{2e^{h^Pt}}{\gamma^2}\int_t^T\Big((h^P-\frac{\delta}{\varsigma})(I(s)-\beta(T-s))+\frac{h^P}{2}\beta(T-s)\Big)e^{-h^Ps}ds\\
&+\frac{2h^Pe^{h^Pt}}{\gamma}\int_t^T\frac{(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-s)}-1)e^{-h^Ps}ds
\end{aligned}$$

$$= \frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r}(e^{r(T-t)} - e^{-h^P(T-t)}) + \frac{1}{\gamma^2 h^P}\Big(\frac{2\delta}{\varsigma} + \frac{\delta^2}{\varsigma^2 h^P} - \beta - h^P\Big)(1$$
$$-e^{-h^P(T-t)}) + \frac{2e^{h^P t}}{\gamma^2}\int_t^T \Big((h^P - \frac{\delta}{\varsigma})(I(s) - \beta(T-s)) + \frac{h^P}{2}\beta(T-s)\Big)e^{-h^P s}ds$$
$$+\frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r(r+h^P)}(e^{r(T-t)} - e^{-h^P(T-t)}) + \frac{2(\theta-\eta)\lambda_2\mu_X}{\gamma r h^P}(e^{-h^P(T-t)} - 1).$$

Then we have

$$\mathrm{V}ar_{t,x,0}[X^{\pi^*}(T)] = \frac{J(t)}{\gamma} + \frac{K(t)}{\gamma^2},$$

which leads that

$$\frac{1}{\gamma} = \frac{-J(t) + \sqrt{J(t)^2 + 4K(t)\mathrm{V}ar_{t,x,0}[X^{\pi^*}(T)]}}{2K(t)}.$$

Then putting the expression for $\frac{1}{\gamma}$ into (4.37) yields

$$\begin{aligned}\mathrm{E}_{t,x,0}[X^{\pi^*}(T)] = \ & xe^{r(T-t)} + \frac{(\theta-\eta)\lambda_2\mu_X}{r}\Big(e^{r(T-t)} - 1\Big) \\ & + \frac{I(t)\Big(-J(t) + \sqrt{J(t)^2 + 4K(t)Var_{t,x,0}[X^{\pi^*}(T)]}\Big)}{2K(t)}.\end{aligned}$$

□

4.4. Remark. If the surplus of the insurer follows the classical risk model perturbed by a diffusion: $U(t) = u + ct + \sigma_0 W_0^P(t) - \sum_{i=1}^{N_2(t)} X_i$, where σ_0 is a positive constant and $W_0^P(t)$ is an standard Brownian motion dependent of $W^P(t)$. Here $\sigma_0 W_0^P(t)$ can be regarded as the uncertainty associated with the insurer's surplus at time t. Denote the correlation coefficient between $W_0^P(t)$ and $W^P(t)$ by ρ, i.e., $\mathrm{E}(W_0^P(t)W^P(t)) = \rho t$, then the wealth process $X(t)$ of the insurer takes the form

$$\begin{aligned}dX(t) = \ & \{rX(t) + (\theta - \eta + (1+\eta)q(t))\lambda_2\mu_X + \pi_1(t)(\mu - r) \\ & + \pi_2(t)(1 - H(t))(1-\Delta)\delta\}dt + \sigma\pi_1(t)dW^P(t) + q(t)\sigma_0 dW_0^P(t) \\ & - q(t)d(\sum_{i=1}^{N_2(t)} X_i) + \pi_1(t)d(\sum_{i=1}^{N_1(t)} Y_i) - \pi_2(t)\varsigma dM^P(t).\end{aligned}$$

Denote $\beta_0 = \frac{\eta^2\lambda_2^2\mu_X^2(\sigma^2+\lambda_1\sigma_Y^2) + (\mu - r + \lambda_1\mu_Y)^2(\sigma_0^2 + \lambda_2\sigma_X^2) - 2\rho\sigma_0\sigma\eta\lambda_2\mu_X(\mu - r + \lambda_1\mu_Y)}{(\sigma^2+\lambda_1\sigma_Y^2)(\sigma_0^2+\lambda_2\sigma_X^2) - \rho^2\sigma_0^2\sigma^2}$. Similar to the derivation of the original model, we have the optimal time-consistent strategy

$$\begin{aligned} q^*(t) &= \frac{[\eta\lambda_2\mu_X(\sigma^2+\lambda_1\sigma_Y^2) - \rho\sigma_0\sigma(\mu - r + \lambda_1\mu_Y)] \vee 0}{\gamma((\sigma^2+\lambda_1\sigma_Y^2)(\sigma_0^2+\lambda_2\sigma_X^2) - \rho^2\sigma_0^2\sigma^2)}e^{-r(T-t)}, \\ \pi_1^*(t) &= \frac{[(\mu - r + \lambda_1\mu_Y)(\sigma_0^2+\lambda_2\sigma_X^2) - \eta\lambda_2\mu_X\rho\sigma_0\sigma] \vee 0}{\gamma((\sigma^2+\lambda_1\sigma_Y^2)(\sigma_0^2+\lambda_2\sigma_X^2) - \rho^2\sigma_0^2\sigma^2)}e^{-r(T-t)}, \\ \pi_2^*(t) &= \Big[\frac{1}{\gamma\varsigma}(\frac{\delta}{\varsigma h^P} + b_0(t) - \tilde{b}_0(t) - 1)e^{-r(T-t)}\Big]\mathbf{1}_{\{\tau > t\}}, \end{aligned}$$

and the equilibrium value function

$$\begin{aligned} V(t,x,1) &= e^{r(T-t)}x + \frac{(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \frac{\beta_0(T-t)}{2\gamma}, \\ V(t,x,0) &= e^{r(T-t)}x + e^{h^P t}\frac{N_0(t)}{\gamma}, \end{aligned}$$

where

$$b_0(t) = \frac{\gamma(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \beta_0(T-t),\ \tilde{b}_0(t) = e^{\frac{\delta}{\varsigma}t}M_0(t),$$

$$M_0(t) = \frac{\gamma\varsigma(\theta-\eta)\lambda_2\mu_X}{\delta+r\varsigma}\Big(e^{r(T-t)-\frac{\delta}{\varsigma}t} - e^{-\frac{\delta}{\varsigma}T}\Big) + \Big(\frac{\varsigma\beta}{\delta} + \frac{\varsigma h^P}{\delta} + \frac{\delta}{\varsigma h^P} - 2\Big) \cdot\Big(e^{-\frac{\delta}{\varsigma}t} - e^{-\frac{\delta}{\varsigma}T}\Big) + \frac{\delta}{\varsigma}\int_t^T b_0(s)e^{-\frac{\delta}{\varsigma}s}ds,$$

and

$$\begin{aligned} N_0(t) = \; & \frac{\gamma(\theta-\eta)\lambda_2\mu_X}{r}\Big(e^{rT-(r+h^P)t} - e^{-h^P T}\Big) + \frac{1}{2h^P}\Big(\frac{2\delta}{\varsigma} - \frac{\delta^2}{\varsigma^2 h^P} - \beta - h^P\Big) \\ & \cdot\Big(e^{-h^P T} - e^{-h^P t}\Big) - \int_t^T \Big(h^P - \frac{\delta}{\varsigma}\Big)\Big(b_0(s) - \tilde{b}_0(s)\Big)e^{-h^P s}ds \\ & - h^P\int_t^T \Big(\frac{(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-s)} - 1) + \frac{\beta_0(T-s)}{2\gamma}\Big)e^{-h^P s}ds. \end{aligned}$$

Compare with the original model, if we let $\rho = 0$, we can readily get the result of Proposition 4.1 and Theorem 4.2.

4.5. Remark. From Proposition 4.1 and Theorem 4.2, we know that the optimal time-consistent reinsurance strategy and optimal amount invested in the risky asset are not dependent on the default event. Moreover, as described in [22], the parameters of the risky asset and the risk model have influences on the optimal amount of the insurer's money that is invested in the defaultable bond. In the following remark, we present some special cases of our model.

4.6. Remark. Consider some special cases of the mean-variance problem (3.1) in the pre-default case. We have the optimal reinsurance investment strategy, respectively. Our model's results in the previous section will be reduced to the following special cases.

Case (1) : investment-only model. Suppose that the insurer can not purchase reinsurance or acquire new business, i. e., $q(t) \equiv 1$, $\forall t \in [0, T]$. Then the dynamics of the wealth process $X(t)$ corresponding to an admissible strategy π follows

$$\begin{aligned} dX(t) = \; & [rX(t) + (\theta+1)\lambda_2\mu_X + \pi_1(t)(\mu - r) + \pi_2(t)(1-H(t))(1-\Delta)\delta]dt \\ & + \sigma\pi_1(t)dW^p(t) - d(\sum_{i=1}^{N_2(t)} X_i) + \pi_1(t)d(\sum_{i=1}^{N_1(t)} Y_i) - \pi_2(t)\xi dM^p(t), \end{aligned}$$

Similar to the derivation of the original model, the optimal time-consistent strategy is given by

$$\pi_1^*(t) = \frac{\mu - r + \lambda_1\mu_Y}{\gamma(\sigma^2 + \lambda_1\sigma_Y^2)}e^{-r(T-t)},\ \pi_2^*(t) = \frac{1}{\gamma\varsigma}(\frac{\delta}{\varsigma h^P} + b_1(t) - \tilde{b}_1(t) - 1)e^{-r(T-t)}.$$

where

$$b_1(t) = \frac{\gamma\theta\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + \beta_1(T-t),\ \beta_1 = \frac{(\mu - r + \lambda_1\mu_Y)^2}{\sigma^2 + \lambda_1\sigma_Y^2},$$

and

$$\tilde{b}_1(t) = \frac{\gamma\varsigma\theta\lambda_2\mu_X}{\delta + r\varsigma}\Big(e^{r(T-t)} - e^{-\frac{\delta}{\varsigma}(T-t)}\Big) + \Big(\frac{\varsigma\beta_1}{\delta} + \frac{\varsigma h^P}{\delta} + \frac{\delta}{\varsigma h^P} - 2\Big) \cdot\Big(1 - e^{-\frac{\delta}{\varsigma}(T-t)}\Big) + \frac{\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T b_1(s)e^{-\frac{\delta}{\varsigma}s}ds.$$

Case (2): No insurance case. We consider that there is no insurance business. In this case, the wealth process (2.5) can be described by

$$\begin{aligned} dX(t) = \; & [rX(t) + \pi_1(t)(\mu - r) + \pi_2(t)(1-H(t))(1-\Delta)\delta]dt + \sigma\pi_1(t)dW^P(t) \\ & + \pi_1(t)d(\sum_{i=1}^{N_1(t)} Y_i) - \pi_2(t)\varsigma dM^P(t). \end{aligned}$$

By some calculations, we have that the optimal dollar amounts allocated to the risky asset is given by the expression in (4.24) and the optimal amounts allocated to the defaultable bond is

$$\pi_2^*(t) = \frac{1}{\gamma\varsigma}(\frac{\delta}{\varsigma h^P} + b_2(t) - \tilde{b}_2(t) - 1)e^{-r(T-t)},$$

where

$$b_2(t) = \beta_1(T-t),$$

and

$$\tilde{b}_2(t) = (\frac{\varsigma\beta_1}{\delta} + \frac{\delta}{\varsigma h^P} + \frac{\varsigma h^P}{\delta} - 2)(1 - e^{-\frac{\delta}{\varsigma}(T-t)}) + \int_t^T \beta_1(T-s)e^{-\frac{\delta}{\varsigma}s}ds.$$

Case 3: If the price process of the risky asset has no jumps (i.e., $\lambda_1 = 0$), then the wealth process $X(t)$ reduces to

$$\begin{aligned} dX(t) = \quad & [rX(t) + [\theta - \eta + (1+\eta)q(t)]\lambda_2\mu_X + \pi_2(t)(1 - H(t))(1-\Delta)\delta \\ & + \pi_1(t)(\mu - r)]dt + \sigma\pi_1(t)dW^P(t) - q(t)d(\sum_{i=1}^{N_2(t)} X_i) - \pi_2(t)\varsigma dM^P(t). \end{aligned}$$

We find that the optimal time-consistent investment strategy becomes

$$\pi_1(t) = \frac{\mu - r}{\gamma\sigma^2}e^{-r(T-t)}, \ \pi_2(t) = \frac{1}{\gamma\varsigma}(\frac{\delta}{\varsigma h^P} + b_3(t) - \tilde{b}_3(t) - 1)e^{-r(T-t)}.$$

Due to the optimal time-consistent reinsurance strategy $q^*(t)$ is independent of the parameters of the risky asset, so it is the same as the first expression in (4.24). where

$$b_3(t) = \frac{\gamma(\theta-\eta)\lambda_2\mu_X}{r}(e^{r(T-t)} - 1) + (\frac{\lambda_2\eta^2\mu_X^2}{\sigma_X^2} + \frac{(\mu-r)^2}{\sigma^2})(T-t),$$

and

$$\begin{aligned} \tilde{b}_3(t) = \quad & \frac{\gamma\varsigma(\theta-\eta)\lambda_2\mu_X}{r\varsigma + \delta}(e^{r(T-t)} - e^{-\frac{\delta}{\varsigma}(T-t)}) + (\frac{\lambda_2\varsigma\eta^2\mu_X^2}{\delta\sigma_X^2} + \frac{\varsigma(\mu-r)^2}{\delta\sigma^2} + \frac{\delta}{\varsigma h^P} \\ & + \tfrac{\varsigma h^P}{\delta} - 2)(1 - e^{-\frac{\delta}{\varsigma}(T-t)}) + \frac{\delta}{\varsigma}e^{\frac{\delta}{\varsigma}t}\int_t^T b_3(s)e^{-\frac{\delta}{\varsigma}s}ds. \end{aligned}$$

5. Sensitivity analysis

In this section, we investigate some sensitivity analysis of the optimal reinsurance-investment strategy for the time-consistent mean-variance problem in the case of pre-default. Throughout the numerical analysis, unless otherwise stated, the basic parameters are given by $\theta = 0.1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 0.8$, $\mu_X = 0.5$, $\sigma_X = 0.8$, $\mu_Y = 0.4$, $\sigma_Y = 0.5$, $\mu = 0.2$, $\sigma = 0.3$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\delta = 0.01$, $\varsigma = 0.6$, $h^P = 0.003$, $T = 6$ and $t = 0$.

(1) The impact of model parameters on the optimal reinsurance strategy $q^*(t)$.

According to (4.16), we can obtain that $\frac{\partial q^*(t)}{\partial \eta} > 0$, $\frac{\partial q^*(t)}{\partial \mu_X} > 0$, $\frac{\partial q^*(t)}{\partial t} > 0$ $\frac{\partial q^*(t)}{\partial \gamma} < 0$, $\frac{\partial q^*(t)}{\partial \sigma_X^2} < 0$ and $\frac{\partial q^*(t)}{\partial r} < 0$, which indicate that $q^*(t)$ decreases with respect to risk aversion coefficient of insurer γ, the claims' second-order moment σ_X^2 and the risk-free interest rate r, but increases with the safety loading of the reinsure η, time t and the expectation of claims μ_X. That is, as γ, σ_X^2 or r increases, the insurer will purchase more reinsurance or acquire less new business to reduce the risk exposure. Furthermore, we can find that the insurer should increase the exposure to insurance risk by purchasing less reinsurance or acquiring more new business while η, μ_X and t increases.

(2) The impact of model parameters on the optimal investment strategy $\pi_1^*(t)$.

From the expression of $\pi_1^*(t)$ in (4.16), we know that it is only dependent with the parameters of risk-free asset and risky asset. Differentiating $\pi_1^*(t)$ with respect to r, we

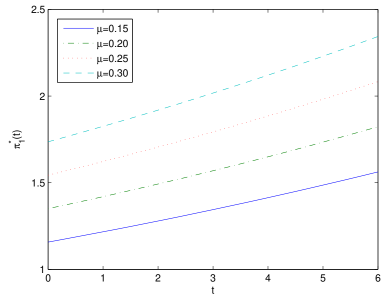

Figure 1. The impact of parameter μ on $\pi_1^*(t)$

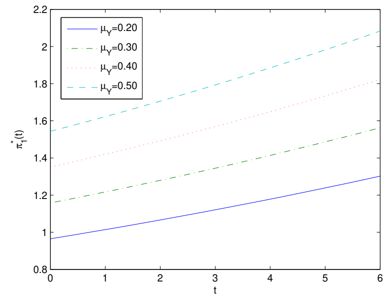

Figure 2. The impact of parameter μ_Y on $\pi_1^*(t)$

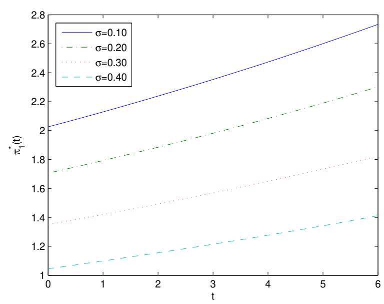

Figure 3. The impact of parameter σ on $\pi_1^*(t)$

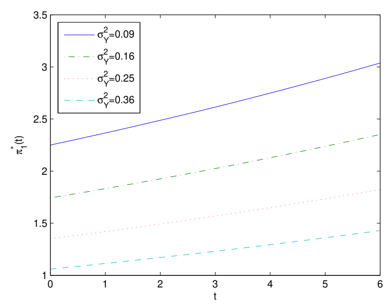

Figure 4. The impact of parameter σ_Y^2 on $\pi_1^*(t)$

have $\frac{\partial \pi_1^*(t)}{\partial r} = -\frac{(\mu - r + \lambda_1 \mu_Y)(T-t)+1}{\gamma(\sigma^2 + \lambda_1 \sigma_Y^2)} e^{-r(T-t)} < 0$, which implies that the cost of borrowing and lending will be higher as the interest rate increases. Thus, the insurer should invest less money in the risky asset. According to $\pi_1^*(t)$, we obtain that $\frac{\partial \pi_1^*(t)}{\partial \mu} > 0$ and $\frac{\partial \pi_1^*(t)}{\partial \mu_Y} <$ 0, which shows that the optimal investment strategy $\pi_1^*(t)$ increases with μ and μ_Y. In other words, the insurer will invest more money in the risky asset with the increase of μ and μ_Y, which can be demonstrated in Fig. 1 and Fig. 2. We also note that $\frac{\partial \pi_1^*(t)}{\partial \sigma} < 0$ and $\frac{\partial \pi_1^*(t)}{\partial \sigma_Y^2} < 0$, which shows that $\pi_1^*(t)$ decrease with the volatility of the risky asset's price σ, and the second-order moment of each jump amplitude of the risky asset's price σ_Y^2. The insurer should reduce investment in the risky asset when σ^2 or σ_Y^2 becomes larger to hedge the risk. This is showed in Fig. 3 and Fig. 4. In addition, we derive that $\frac{\partial \pi_1^*(t)}{\partial \gamma} < 0$ and $\frac{\partial \pi_1^*(t)}{\partial t} > 0$. They implies that the optimal investment strategy $\pi_1^*(t)$ increases with time t, while decreases with γ. As we show that the more risk aversion the insurer is, the smaller the amount of the money is invested in the risky asset.

(3) The impact of model parameters on the optimal investment strategy $\pi_2^*(t)$.

Figs. 5-10 show the impact of some model parameters on the optimal investment in the defaultable bond $\pi_2^*(t)$. From Fig. 5, we know that the optimal money invested in the defaultable bond increases with respect to the credit spread δ and decreases with

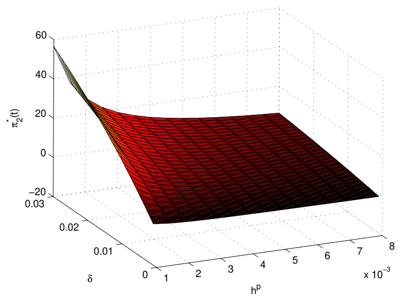

Figure 5. The impact of parameter δ, h^P on $\pi_2^*(t)$

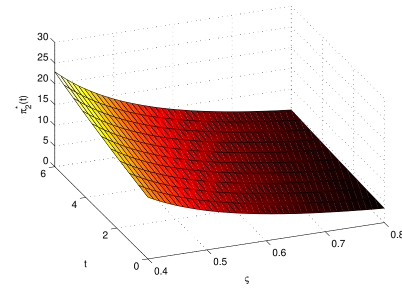

Figure 6. The impact of parameter ς, t on $\pi_2^*(t)$

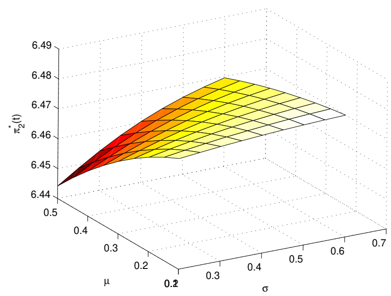

Figure 7. The impact of parameters μ, σ on $\pi_2^*(t)$

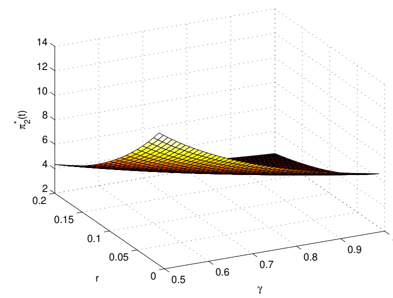

Figure 8. The impact of parameter γ, r on $\pi_2^*(t)$

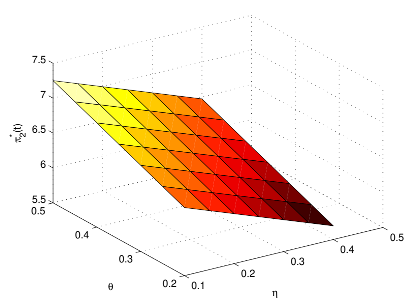

Figure 9. The impact of parameters η, θ on $\pi_2^*(t)$

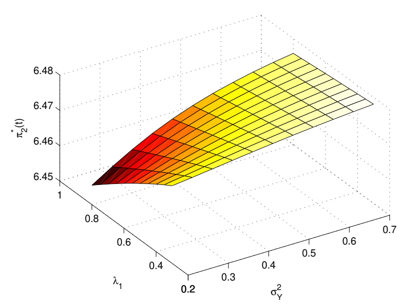

Figure 10. The impact of parameter λ_1, σ_Y^2 on $\pi_2^*(t)$

respect to the default intensity h^P. That is to say, the insurer should invest more money in the defaultable bond with higher credit spread δ and smaller default intensity h^P. Fig.

6 shows that $\pi_2^*(t)$ decreases with loss rate ς. Indeed, a higher loss rate induces a less recovery amount. Thus the insurer will reduce the investment in the defaultable bond as the loss rate ς increases. Moreover, $\pi_2^*(t)$ increases with the current time t, which implies that the insurer should seek more investment opportunities from undertaking default risk. As described in Zhao et al. (2016), the insurer can easily ensure the optimal strategy' time-consistency too.

Fig. 7 illustrates the impact of the parameter of the risky asset's price μ and σ, which shows that $\pi_2^*(t)$ increases with σ and decreases with μ. The insurer should enhance investment in defaultable bond when σ increases or μ decreases. Fig. 8 also depicts the relationship between the optimal investment strategy of the defaultable bond and the risk-free interest rate r. The larger r is, the less money is invested in defaultable bond. In Fig. 8, we find that $\pi_2^*(t)$ decreases with respect to the risk aversion coefficient γ, i.e., an insurer with a lower risk aversion coefficient will invest more in the defaultable bond.

Figs. 9 indicates that the optimal investment strategy $\pi_2^*(t)$ decreases with respect to the reinsurer's safety loading η but increases with respect to the insurer's safety loading θ. This implies that the insurer should invest more in defautable bond with a larger flow of the net premium. From Fig. 10, we also see that $\pi_2^*(t)$ decreases with λ_1 and increase with σ_Y^2. This reveals that the insurer will purchase more defaultable bonds when the intensity of the jumps of the risky asset's price λ_1 becomes smaller and the second-order moment of each jump amplitude of the risky asset's price σ_Y^2 becomes larger.

6. Conclusion

In this paper, we study a time-consistent mean-variance reinsurance-investment strategy selection problem involving a defaultable security and jumps. In our model, the insurer can purchase proportional reinsurance or acquire new business and invest a financial market consisting of a risk-free asset, a risky asset and a defaultable bond. Compared with [22], we introduce jump by assuming that the risky asset's price process evolves according to a geometric Léry process. By using a game theoretic framework, we establish the extended Hamilton-Jacobi-Bellman systems of equations for pre-default case and post-default case. We provide the corresponding verification theorem without proof and derive the closed-from expression of the optimal reinsurance-investment strategies and the corresponding value functions for both the problem before default and the problem after default. Moreover, some special cases of our model are discussed and the time-consistent strategies are obtained. Finally, numerical examples are presented to show how the time-consistent strategy we have derived changes when some model parameters vary. However, there are some limits in our paper: (1) we do not consider the correlation between the risky asset's price model and the risk model; (2) the time horizon T is pregiven; (3) we only consider a single defaultable security. We need to adopt much more sophisticated methods to solve these complicate optimal problem.

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