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A NOTE ON MAL'TSEV-NEUMANN PRODUCTS OF RADICAL CLASSES

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Dedicated to the memory of Professor John Clark

Abstract. A radical class $\mathcal{R}$ of rings is *elementary* if it contains precisely those rings whose singly generated subrings are in $\mathcal{R}$. Many examples of elementary radical classes are presented, and all those which are either contained in the Jacobson radical class or disjoint from it are described. There is a discussion of Mal'tsev products of radical classes in general, in which it is shown, among other things, that a product of elementary radical classes need not be a radical class, and if it is, it need not be elementary.

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1. Introduction

The *Mal'tsev-Neumann product* of classes $\mathcal{X}$ and $\mathcal{Y}$ is the class $\mathcal{X} \circ \mathcal{Y}$ of rings A having an ideal $I \in \mathcal{X}$ such that $A/I \in \mathcal{Y}$.

There are two such products, $\mathcal{X} \circ \mathcal{Y}$ and $\mathcal{Y} \circ \mathcal{X}$, and in general they are not equal. This concept was introduced by Mal'tsev [11] for general algebras and about the same time for groups by Neumann [13], so it seems appropriate to associate both names with these products. More usually they have been called Malt'sev products. For brevity, and as it introduces no ambiguity, we shall call them simply *products* in the sequel.

Mal'tsev [11] proved that for algebras with commuting congruences, if $\mathcal{X}$ and $\mathcal{Y}$ are varieties, then so is $\mathcal{X} \circ \mathcal{Y}$. (In the general version the role of ideals is played by congruence classes which are subalgebras.) The case of groups was exploited in [13].

The nature of products of radical classes is of some interest, particularly radical classes which have some resemblance to varieties. This seems not to have been directly investigated, though there are a couple of examples in the literature of radical classes which are products of radical subclasses (see the next section).

A radical class $\mathcal{R}$ is *elementary* if it contains a given ring A precisely when it contains all singly generated subrings of A. Such classes are exemplified by the nil radical class $\mathcal{N}$ and the class of boolean rings, or, more generally, a semi-simple radical class (SSR-class) [15]. The SSR-classes are varieties and elementary radical classes generally are examples of locally equational classes in the sense of Hu [8]. Further information on elementary radical classes can be obtained from [5].

We shall show that the product of two elementary radical classes may be an elementary radical class, a non-elementary radical class or not a radical class at all. Thus such a product need not be a locally equational class, so Mal'tsev's result on products of varieties does not generalize to locally equational classes. We also have a result for radicals in general. If $\mathcal{R}$ and $\mathcal{U}$ are radical classes then both $\mathcal{R} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{R}$ are radical classes if and only if they are equal.

All radical classes in this paper are radical classes of associative rings. Our notation and terminology are generally consistent with those of [6] and are identical with those of [5].

2. Results

While the Mal'tsev product of varieties is always a variety (not just for rings, but for all algebraic structures with permutable congruences; see [11, Teorema 7]) not much is known about Mal'tsev-Neumann products of radical classes. An example of a radical class of abelian groups which is a Mal'tsev-Neumann product of two such is given by Proposition 4.6 of [4] (and *via* A-radicals this gives an example for rings), while in [12] it is shown that the classes $\mathcal{T}_E \circ \mathcal{J}$, where $\mathcal{J}$ is the Jacobson radical class, $\mathcal{T}_E$ is the class of torsion rings with non-zero p-components only for $p \in E$ and E is finite, are radical classes. As elementary radical classes are in a sense generalized varieties (see Introduction) it is interesting that the Mal'tsev-Neumann product of two elementary radical classes need not be elementary, even when it is a radical class, as we shall see shortly. We begin, though, with some results about Mal'tsev-Neumann products of radical classes in general.

Lemma 2.1. *Let $\mathcal{R}$ and $\mathcal{U}$ be radical classes. If $\mathcal{R} \circ \mathcal{U}$ is a radical class, then $\mathcal{U} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{U}$.*

Proof. As radical classes are closed under extensions $\mathcal{R} \circ \mathcal{U}$ is the lower radical class defined by $\mathcal{R} \cup \mathcal{U}$, and for the same reason it contains $\mathcal{U} \circ \mathcal{R}$. $\square$

The product of two radical classes satisfies "most of" the requirements for a radical class.

Proposition 2.2. *Let $\mathcal{R}$,$\mathcal{U}$ be radical classes. Then $\mathcal{R} \circ \mathcal{U}$ is*

(i) *homomorphically closed and*
(ii) *closed under unions of ascending chains of ideals.*

Proof. (i) If $I \triangleleft A \in \mathcal{R} \circ \mathcal{U}$, then

$$\mathcal{R} \ni \mathcal{R}(A)/\mathcal{R}(A) \cap I \cong (\mathcal{R}(A) + I)/I \triangleleft A/I$$

and

$$(A/I)/((\mathcal{R}(A) + I)/I) \cong A/(\mathcal{R}(A) + I) \cong (A/\mathcal{R}(A)/((\mathcal{R}(A) + I)/\mathcal{R}(A)) \in \mathcal{U}$$

as $A/\mathcal{R}(A) \in \mathcal{U}$ (Proposition 2.14).

(ii) Let $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots$ be a chain of ideals of a ring A with each $I_\alpha \in \mathcal{R} \circ \mathcal{U}$, and let $I = \bigcup_\alpha I_\alpha$. If $I_\alpha \subseteq I_\beta$ then $I_\alpha \triangleleft I_\beta$ so $\mathcal{R}(I_\alpha) \subseteq \mathcal{R}(I_\beta)$. Let $J = \bigcup_\alpha \mathcal{R}(I_\alpha)$. Then each $\mathcal{R}(I_\alpha) \in \mathcal{R}$ and each is an ideal of A, so $J \in \mathcal{R}$. Also if $I_\alpha \subseteq I_\beta$, then $I_\alpha + J \subseteq I_\beta + J$ so the $I_\alpha + J$ form a chain of ideals of A and hence the $(I_\alpha + J)/J$ form a chain of ideals of A/J. But $(I_\alpha + J)/J \cong I_\alpha/(I_\alpha \cap J)$ for all α, while $\mathcal{R}(I_\alpha) \subseteq I_\alpha \cap J$, so each $(I_\alpha + J)/J$, as a homomorphic image of $I_\alpha/\mathcal{R}(I_\alpha)$, is in $\mathcal{U}$. Hence $\mathcal{U}$ also contains

$$\bigcup_\alpha((I_\alpha + J)/J) = (\bigcup_\alpha I_\alpha + J)/J = \bigcup_\alpha I_\alpha/J,$$

whence $\bigcup_\alpha I_\alpha \in \mathcal{R} \circ \mathcal{U}$. ☐

Theorem 2.3. *Let $\mathcal{R}$, $\mathcal{U}$ be radical classes. Then $\mathcal{R} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{R}$ are both radicals if and only if $\mathcal{R} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{R}$.*

Proof. By Proposition 2.2, both $\mathcal{R} \circ \mathcal{U}$ and $\mathcal{U} \circ \mathcal{R}$ are homomorphically closed and closed under unions of chains of ideals. Suppose they are equal. Let J be an ideal of a ring A with J and $A/J \in \mathcal{R} \circ \mathcal{U}$. Let $\mathcal{R}(A/J) = W/J$. Note that $\mathcal{R}(J) \triangleleft A$ as $J \triangleleft A$. Also $W \triangleleft A$. For the series

$$0 \subseteq \mathcal{R}(J) \subseteq J \subseteq W \subseteq A$$

we have $\mathcal{R}(J) \triangleleft A$, $\mathcal{R}(J) \in \mathcal{R}$, $J/\mathcal{R}(J) \in \mathcal{U}$, $W/J \in \mathcal{R}$ and $A/W \in \mathcal{U}$ by Proposition 2.14 of [5]. Now $J/\mathcal{R}(J) \in \mathcal{U}$ and $(W/\mathcal{R}(J))/(J/\mathcal{R}(J)) \cong W/J \in \mathcal{R}$, so $W/\mathcal{R}(J) \in \mathcal{U} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{U}$. Let $\mathcal{R}(W/\mathcal{R}(J)) = M/\mathcal{R}(J)$. Then

$$W/M \cong (W/\mathcal{R}(J))/(M/\mathcal{R}(J)) = (W/\mathcal{R}(J))/\mathcal{R}(W/\mathcal{R}(J)) \in \mathcal{U}.$$

Here $M \triangleleft W \triangleleft A$. Also $M/\mathcal{R}(J) \in \mathcal{R}$, so $M \in \mathcal{R}$ and thus $M \subseteq \mathcal{R}(W) \subseteq \mathcal{R}(A)$. Let M^* be the ideal of A generated by M. Then $M^* \subseteq \mathcal{R}(A)$ and $M^* \subseteq W$.

Now $W/M \in \mathcal{U}$, so $W/M^* \in \mathcal{U}$. Also $(A/M^*)/(W/M^*) \cong A/W \in \mathcal{U}$ so $A/M^* \in \mathcal{U}$. But then, as $M^* \subseteq \mathcal{R}(A)$, we have $A/\mathcal{R}(A) \in \mathcal{U}$, i.e. $A \in \mathcal{R} \circ \mathcal{U}$. Thus $\mathcal{R} \circ \mathcal{U}$ $(= \mathcal{U} \circ \mathcal{R})$ is closed under extensions and is therefore a radical class.

The converse follows from Lemma 2.1. □

We shall call a set $\mathcal{E}$ of subrings of a ring A a *local family* if $\mathcal{E}$ is up-directed by inclusion and $A = \bigcup\{B : B \in \mathcal{E}\}$. The set of all finitely generated subrings is always a local family, though the set of one-generator subrings generally is not.

Freidman [2] considered a hereditary supernilpotent radical class $\mathcal{R}$ with the following extra property.

Local Radical Condition. *If a ring A has a local family of $\mathcal{R}$-subrings, then A is itself in $\mathcal{R}$.*

Radical classes satisfying this condition include the locally nilpotent, Jacobson and Brown-McCoy radical classes and $\mathcal{N}$.

Freidman showed that if $\mathcal{R}$ satisfies the Local Radical Condition, then $\mathcal{R} \circ \mathcal{C}$ is also a radical class satisfying this condition, where $\mathcal{C}$ is the class of commutative rings.

We shall now prove that if $\mathcal{V}$ is an SSR-class then $\mathcal{N} \circ \mathcal{V}$ is a radical class. Although the overall plan of the proof mimics that of Freidman, the details are quite different, so we shall present the proof in a fair amount of detail.

Lemma 2.4. *Let $\mathcal{U}$ be an elementary radical class, A a ring with a local family $\mathcal{E}$ of $\mathcal{U}$-subrings. Then $A \in \mathcal{U}$.*

Proof. If $a \in A$ then there is a subring $B \in \mathcal{E}$ with $a \in B$. But then $< a > \in \mathcal{U}$ and it follows that $A \in \mathcal{U}$. □

Corollary 2.5. *Under the conditions of Lemma 2.4, if A is the union of an ascending chain of $\mathcal{U}$-subrings, then $A \in \mathcal{U}$.*

Lemma 2.6. *Let A be a ring with a series*

$$0 = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_\alpha \subseteq B_{\alpha+1} \subseteq \ldots \subseteq B_\mu = A$$

labelled by ordinals, such that $B_\alpha \triangleleft B_{\alpha+1}$ for all α and $B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha$ if γ is a limit ordinal. If $\alpha < \beta \leq \mu$, then $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\beta)$.

Proof. Clearly $0 = \mathcal{N}(B_0) \subseteq \mathcal{N}(B_1)$. If for all $\beta < \delta$ we have

$$\alpha < \beta \Rightarrow \mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\beta),$$

then we consider two cases.

(1) If δ is a limit ordinal, then for each $\alpha < \beta < \delta$ we have $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\beta)$ so by Corollary 2.5, $\bigcup_{\alpha<\delta} \mathcal{N}(B_\alpha) \in \mathcal{N}$. If $b \in \mathcal{N}(B_\theta)$ for some $\theta < \delta$ and $d \in B_\delta$, then $d \in B_\beta$ for some $\beta < \delta$, so for $\epsilon = max\{\theta, \beta\}$ we have $b \in \mathcal{N}(B_\epsilon)$ and $d \in B_\epsilon$, so $bd, db \in \mathcal{N}(B_\epsilon) \subseteq \bigcup_{\alpha<\delta} \mathcal{N}(B_\alpha)$. Thus $\bigcup_{\alpha<\delta} \mathcal{N}(B_\alpha) \triangleleft B_\delta$, so $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\delta)$.

(2) If $\delta = \gamma + 1$ for some γ, then for $\alpha < \delta$ we have either $\alpha = \gamma$ or $\alpha < \gamma$. In the first case $\mathcal{N}(B_\alpha) = \mathcal{N}(B_\gamma) \subseteq \mathcal{N}(B_{\gamma+1}) = \mathcal{N}(B_\delta)$ by ADS. In the second, the inductive hypothesis says that $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\gamma)$, and since by ADS $\mathcal{N}(B_\gamma) \subseteq \mathcal{N}(B_\delta)$, we have $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(B_\delta)$. □

Corollary 2.7. *Under the conditions of Lemma 2.6, $\mathcal{N}(B_\alpha) \subseteq \mathcal{N}(A)$ for all α. In particular, if $\mathcal{N}(A) = 0$ then $\mathcal{N}(B_\alpha) = 0$ for all α.*

Lemma 2.8. *Let $\mathcal{V}$ be an SSR-class, A a ring with $\mathcal{N}(A) = 0$. If A has an ideal I with $I \in \mathcal{V}$ and $A/I \in \mathcal{V} \cup \mathcal{N}$, then $A \in \mathcal{V}$ (and so $A/I \in \mathcal{V}$).*

Proof. (i) If $A/I \in \mathcal{V}$ then $A \in \mathcal{V}$, as $I \in \mathcal{V}$.
(ii) If $A/I \in \mathcal{N}$, let J be an ideal of A with $J \cap I = 0$. Then $J \cong J/J \cap I \cong (J + I)/I \triangleleft A/I \in \mathcal{N}$, so $J = 0$ (as $\mathcal{N}(A) = 0$). Thus A is an essential extension of I, so $A \in \mathcal{V}$ [10]. □

We shall call a series with the properties of the one in Lemma 2.6 a $(\mathcal{V}, \mathcal{N})$-*series* for a SSR-class $\mathcal{V}$ if all its factors are in $\mathcal{V} \cup \mathcal{N}$.

Lemma 2.9. *If*

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_\alpha \subseteq \ldots \subseteq B_\mu = A$$

is a $(\mathcal{V}, \mathcal{N})$-series for a ring A, then $A \in \mathcal{N} \circ \mathcal{V}$.

Proof. Let $\overline{A} = A/\mathcal{N}(A)$. Then $\overline{A}$ has the series

$$0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_\alpha \subseteq C_{\alpha+1} \subseteq \ldots \subseteq C_\mu = \overline{A},$$

where $C_\alpha = (B_\alpha + \mathcal{N}(A))/\mathcal{N}(A)$ for each α. We have $C_\alpha \triangleleft C_{\alpha+1}$ for every α and when δ is a limit,

$$C_\delta = (B_\delta + \mathcal{N}(A))/\mathcal{N}(A) = (\bigcup_{\alpha<\delta} B_\alpha + \mathcal{N}(A))/\mathcal{N}(A) = \bigcup_{\alpha<\delta} (B_\alpha + \mathcal{N}(A))/\mathcal{N}(A) = \bigcup_{\alpha<\beta} C_\alpha.$$

Also for each α we have
$C_{\alpha+1}/C_\alpha \cong (B_{\alpha+1} + \mathcal{N}(A))/(B_\alpha + \mathcal{N}(A)) = B_{\alpha+1} + B_\alpha + \mathcal{N}(A))/(B_\alpha + \mathcal{N})$
$\cong B_{\alpha+1}/B_{\alpha+1} \cap (B_\alpha + \mathcal{N}(A)) \in \mathcal{V} \cup \mathcal{N}$, as it is a homomorphic image of $B_{\alpha+1}/B_\alpha$.
Thus the C_α form a $(\mathcal{V}, \mathcal{N})$-series for $\overline{A}$. By Corollary 2.7, $\mathcal{N}(C_\alpha) = 0$ for each α.

Now $C_1 \cong C_1/C_0 \in \mathcal{V} \cup \mathcal{N}$ and $\mathcal{N}(C_1) = 0$ so ($C_1 = 0$ or) $C_1 \in \mathcal{V}$. Suppose $C_\alpha \in \mathcal{V}$ for all $\alpha < \delta$. If δ is a limit, then $C_\delta = \bigcup_{\alpha<\delta} C_\alpha \in \mathcal{V}$ by Corollary 2.5. If $\delta = \gamma + 1$ for some γ, then $C_\gamma \in \mathcal{V}, C_\gamma \triangleleft C_\delta, \mathcal{N}(C_\delta) = 0$ and $C_\delta/C_\gamma \in \mathcal{V} \cup \mathcal{N}$, so by Lemma 4.8, $C_\delta \in \mathcal{V}$. By induction all $C_\alpha \in \mathcal{V}$, so in particular $A/\mathcal{N}(A) = \overline{A} = C_\mu \in \mathcal{V}$. $\square$

Lemma 2.10. *Let $\mathcal{W}$ be any variety, not necessarily a SSR-class. For each ring R, let $R(\mathcal{W}) = \bigcap\{I \triangleleft R : R/I \in \mathcal{W}\}$. If $\{B_\lambda : \lambda \in \Lambda\}$ is a local family of subrings of R, then $\{B_\lambda(\mathcal{W}) : \lambda \in \Lambda\}$ is a local family for $R(\mathcal{W})$.*

Proof. Every homomorphism $f : R \to W \in \mathcal{W}$ induces a homomorphism from B_λ to W for each λ, so each $B_\lambda(\mathcal{W}) \subseteq Ker(f)$. Therefore each $B_\lambda(\mathcal{W})$ is a subring of $R(\mathcal{W})$. In the same way, if $B_\lambda \subseteq B_\theta$ then $B_\lambda(\mathcal{W}) \subseteq B_\theta(\mathcal{W})$. It follows that the set of $B_\lambda(\mathcal{W})$ is up-directed.

Let $J = \bigcup_{\lambda\in\Lambda} B_\lambda(\mathcal{W})$. If $a \in J$ and $r \in R$, let a be in $B_\nu(\mathcal{W})$, $r \in B_\tau$. If $B_\nu, B_\tau \subseteq B_\sigma$ then $a \in B_\sigma(\mathcal{W})$ and $r \in B_\sigma$ so $ra, ar \in B_\sigma(\mathcal{W}) \subseteq J$. Thus $J \triangleleft R$. Let$f(x_1, x_2, \ldots, x_n) = 0$ be an equation satisfied by $\mathcal{W}$. If $a_1, a_2, \ldots, a_n \in R$, then there is a $\rho \in \Lambda$ such that $a_1, a_2, \ldots, a_n \in B_\rho$. Hence $a_1 + J, a_2 + J, \ldots, a_n + J \in (B_\rho + J)/J \cong B_\rho/B_\rho \cap J$. But $B_\rho(\mathcal{W}) \subseteq B_\rho \cap J$ and $B_\rho/B_\rho(\mathcal{W}) \in \mathcal{W}$, so $(B_\rho + J)/J \in \mathcal{W}$. Hence $f(a_1+J, a_2+J, \ldots, a_n+J) = 0$. This means that $R/J \in \mathcal{W}$ and so $R(\mathcal{W}) \subseteq J$. The reverse inclusion was established at the beginning of the proof. $\square$

Proposition 2.11. *Let I be an ideal of a ring A. Then A has a $(\mathcal{V}, \mathcal{N})$-series if and only if I and A/I have such series.*

Proof. Let

$$0 = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_\mu = A$$

be a $(\mathcal{V}, \mathcal{N})$-series. Consider the series

$$0 = B_0 \cap I \subseteq B_1 \cap I \subseteq \ldots \subseteq B_\mu \cap I = I.$$

For each α we have $B_\alpha \cap I \triangleleft B_{\alpha+1} \cap I$ and

$$B_{\alpha+1} \cap I/B_\alpha \cap I = B_{\alpha+1} \cap I/B_{\alpha+1} \cap I \cap B_\alpha \cong (B_{\alpha+1} \cap I + B_\alpha)/B_\alpha \triangleleft B_{\alpha+1}/B_\alpha \in \mathcal{V} \cup \mathcal{N}.$$

If δ is a limit ordinal, then $B_\delta \cap I = (\bigcup_{\alpha<\delta} B_\alpha) \cap I = \bigcup_{\alpha<\delta}(B_\alpha \cap I)$, so this is a $(\mathcal{V}, \mathcal{N})$-series for I. As in the proof of Lemma 2.9,

$$0 = (B_0 + I)/I \subseteq (B_1 + I)/I \subseteq \ldots \subseteq (B_\mu + I)/I = A/I$$

is a $(\mathcal{V}, \mathcal{N})$-series for A/I.

Conversely, if I and A/I have $(\mathcal{V},\mathcal{N})$-series

$$0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_\nu = I$$

and

$$0 = D_0/I \subseteq D_1/I \subseteq D_2/I \subseteq \ldots \subseteq D_\lambda/I = A/I$$

respectively, we consider the series

$$0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_\nu = I = D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_\lambda = A.$$

The terms of this series are well ordered with order type $\nu + \lambda$. We can therefore label its terms by the ordinals in an initial segment by re-naming each D_β as $C_{\nu+\beta}$. the series behaves as a $(\mathcal{V},\mathcal{N})$-series as far as C_ν. For each relevant β we have

$$C_{\nu+\beta+1}/C_{\nu+\beta} = D_{\beta+1}/D_\beta \cong (D_{\beta+1}/I)/(D_\beta/I) \in \mathcal{V} \cup \mathcal{N}.$$

If δ is a limit, then so is $\nu + \delta$ and $C_{\nu+\delta} = D_\delta$, while $D_\delta/I = \bigcup_{\beta<\delta}(D_\beta/I) = (\bigcup_{\beta<\delta} D_\beta)/I$, so

$$C_{\nu+\delta} = \bigcup_{\beta<\delta} C_{\nu+\beta} = \bigcup_{\alpha<\nu+\delta} C_\alpha.$$

We therefore have a $(\mathcal{V},\mathcal{N})$-series for A. □

Lemma 2.12. *If a ring A has a local system $\mathcal{E}$ of subrings, each of which has a $(\mathcal{V},\mathcal{N})$-series, then also A has such a series.*

Proof. By Lemma 2.9, $B/\mathcal{N}(B) \in \mathcal{V}$ for all $B \in \mathcal{E}$, so $B(\mathcal{V}) \subseteq \mathcal{N}(B)$, and hence $B(\mathcal{V}) \triangleleft \mathcal{N}(B)$ and finally $B(\mathcal{V})$ is nil for every $B \in \mathcal{E}$. By Lemma 2.10 $A(\mathcal{V}) = \bigcup\{B(\mathcal{V}) : B \in \mathcal{E}\} \in \mathcal{N}$ (since $\mathcal{N}$ satisfies the Local Radical Condition). Hence $A(\mathcal{V}) \subseteq \mathcal{N}(A)$, so $A/\mathcal{N}(A) \in \mathcal{V}$. But then

$$0 \subseteq \mathcal{N}(A) \subseteq A$$

is a $(\mathcal{V},\mathcal{N})$-series for A. □

Corollary 2.13. *If*

$$0 = I_0 \subseteq I_1 \subseteq \ldots \subseteq I_j \subseteq \ldots$$

is a chain of ideals in some ring and each I_j has a $(\mathcal{V},\mathcal{N})$-series, then so does $\bigcup_j I_j$.

Now we have all the ingredients of the principal result of this section.

Theorem 2.14. (i) *A ring A has a $(\mathcal{V},\mathcal{N})$-series if and only if $A \in \mathcal{N} \circ \mathcal{V}$.*
(ii) *$\mathcal{N} \circ \mathcal{V}$ is a strongly hereditary radical class satisfying $(*)$.*

Proof. (i) If $A \in \mathcal{N} \circ \mathcal{V}$ then $A/\mathcal{N}(A) \in \mathcal{V}$ ([5, Proposition 2.14]), so

$$0 \subseteq \mathcal{N}(A) \subseteq A$$

is a $(\mathcal{V}, \mathcal{N})$-series. The converse is given by Lemma 2.9 .
(ii) By Proposition 2.11, $\mathcal{N} \circ \mathcal{V}$ is closed under extensions. The other requirements for a radical class are given by Proposition 2.2 or by Proposition 2.11 and Corollary 2.13. If B is a subring of a ring $A \in \mathcal{N} \circ \mathcal{V}$, then $B \cap \mathcal{N}(A) \in \mathcal{N}$ and $B/B \cap \mathcal{N}(A) \cong (B + \mathcal{N}(A))/\mathcal{N}(A) \subseteq A/\mathcal{N}(A) \in \mathcal{V}$ so $B \in \mathcal{N} \circ \mathcal{V}$. The condition involving $(*)$ is just Lemma 2.12. $\square$

Condition $(*)$ is weaker than the elementary property: for instance the locally nilpotent radical class $\mathcal{L}$ satisfies $(*)$, but $\mathcal{N}$ is the smallest elementary radical class containing $\mathcal{L}$. Moreover, $\mathcal{N} \circ \mathcal{V}$ need not be elementary.

Example 2.15. *$\mathcal{N} \circ VAR(K(4))$ is not an elementary radical class. We'll show this by examining $M_2(K(2))$. It is a routine matter to show that all one-generator subrings of $M_2(K(2))$ except those generated by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are in $\mathcal{N}, \mathfrak{B}$ or $\mathcal{N} \circ \mathfrak{B}$. Also $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ are inverse to each other. It follows that the subring generated by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (or $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$) is a four-element field. Thus all singly-generated subrings of $M_2(K(2))$ are in $\mathcal{N} \circ VAR(K(4))$. But as $M_2(K(2))$ is simple and neither nil nor commutative, it is not in $\mathcal{N} \circ VAR(K(4))$.*

On the other hand, the radical class $\mathcal{N} \circ \mathfrak{B}$ *is* elementary. This follows from our next theorem, which makes use of a ring theoretic property we now introduce.

A ring A is *(strongly) nil-clean* [1] if for each $a \in A$ there exist an idempotent e and a nilpotent element z such that

$$a = e + z \text{ (and } ez = ze).$$

It has recently been shown [9] that the strongly nil-clean rings are precisely the rings in $\mathcal{N} \circ \mathfrak{B}$. The (strongly) nil-clean property is a generalization of the (strongly) clean property [1]. All of these properties have been mostly studied for rings with identity; the paper of Nicholson and Zhou [14] extended cleanness to rings generally, and in subsequent papers some attention has been given to the wider setting.

Theorem 2.16. *The following conditions are equivalent for a ring A.*

(i) $< a >\in \mathcal{N} \circ \mathfrak{B}$ *for all* $a \in A$.

(ii) A *is strongly nil-clean.*

(iii) $A \in \mathcal{N} \circ \mathfrak{B}$.

Proof. (i) $\Rightarrow$ (ii) If $a \in A$ then $< a + \mathcal{N}(< a >) > = < a > /\mathcal{N}(< a >) \in \mathfrak{B}$ so $a + \mathcal{N}(< a >)$ is idempotent, possibly 0. In any case, $a + \mathcal{N}(< a >)$ can be lifted to an idempotent $e_a \in < a >$. Thus $a - e_a \in \mathcal{N}(< a >)$. Let $z_a = a - e_a$. Then z_a is nilpotent, $a = e_a + z_a$ and (as both elements are in $< a >$) $e_a z_a = z_a e_a$.
(ii) $\Rightarrow$ (iii) [9, Theorem 5.6].
(iii) $\Rightarrow$ (i) This is clear because $\mathcal{N} \circ \mathfrak{B}$ is strongly hereditary (Theorem 2.14). $\square$

If K is a field in a non-trivial SSR-class $\mathcal{V}$ and K^0 the zeroring on its additive group, let $K^0 * K$ be the ring obtained by the adjunction to K^0 of the identity of K in the usual way:

$$(K^0 * K)^+ = K^+ \oplus K^+; (a, b)(c, d) = (ad + bc, bd).$$

Then $K^0 * K \in \mathcal{N} \circ \mathcal{V}$. But $\mathcal{V}(K^0 * K) = 0$ and $K^0 * K \notin \mathcal{N}$, so $K^0 * K \notin \mathcal{V} \circ \mathcal{N}$. By Theorem 2.3 we therefore have

Proposition 2.17. *If $\mathcal{V}$ is a non-trivial SSR-class, then $\mathcal{V} \circ \mathcal{N}$ is not a radical class.*

There do exist pairs of radical classes for which the two Mal'tsev products are equal.

Proposition 2.18. Let $\mathcal{V}$ and $\mathcal{W}$ be non-trivial SSR-classes such that there are no prime fields $K(p) \in \mathcal{V} \cap \mathcal{W}$. (This means there are no fields common to both classes.) Then

$$\mathcal{V} \circ \mathcal{W} = \{A \oplus B : A \in \mathcal{V}, B \in \mathcal{W}\}.$$

Proof. To see this, we first show that $\mathcal{V}$ and $\mathcal{W}$ are *independent* in the sense of [7]: there is a binary polynomial symbol $\pi(x, y)$ such that $\mathcal{V}$ satisfies the identity $\pi(x, y) = x$ and $\mathcal{W}$ satisfies $\pi(x, y) = y$. Since $\mathcal{V}$ and $\mathcal{W}$ are generated as varieties by disjoint finite strongly hereditary sets of finite fields, we can let m be the product of the characteristics of fields in $\mathcal{V}$ and n the corresponding integer for $\mathcal{W}$. Then m and n are relatively prime, so $rm + sn = 1$ for some $r, s \in \mathbb{Z}$. Let $\pi(x, y) = snx + rmy$. If $a, b \in A \in \mathcal{V}$ then $\pi(a, b) = sna = (1 - rm)a = a$ and there is an analogous result for $\mathcal{W}$. By Theorem 1 of [7],

$$\{A \oplus B : A \in \mathcal{V}, B \in \mathcal{W}\} = \mathcal{V} \vee \mathcal{W}$$

(the join of the two varieties). Also $\mathcal{V} \vee \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}$ as the latter is a variety containing $\mathcal{V}$ and $\mathcal{W}$. But if $R \in \mathcal{V} \circ \mathcal{W}$ then $\mathcal{V}(R) \in \mathcal{T}_{\{p:p|m\}}$ and $R/\mathcal{V}(R) \in \mathcal{T}_{\{p:p|n\}}$. But m and n are relatively prime, so for additive groups we have $Ext((R/\mathcal{V}(R))^+, \mathcal{V}(R)^+) = 0$ (see, e.g. [3], p.267, (C), (D)), whence

$$R^+ = \bigoplus_{p|m} R_p^+ \oplus \bigoplus_{p|n} R_p^+,$$

(where R_p^+ is the p-component of R^+). But the two direct sums on the RHS are (the additive groups of) ideals of R, so

$$R = \mathcal{T}_{\{p:p|m\}}(R) \oplus \mathcal{T}_{\{p:p|n\}}(R) = \mathcal{V}(R) \oplus \mathcal{W}(R).$$

We now have

$$\mathcal{V} \circ \mathcal{W} \subseteq \{A \oplus B : A \in \mathcal{V}, B \in \mathcal{W}\} = \mathcal{V} \vee \mathcal{W} \subseteq \mathcal{V} \circ \mathcal{W}.$$

□

Note that $\mathcal{V} \circ \mathcal{W}$ is again a SSR-class. Whether there are "non-splitting" examples of "commuting pairs" of radical classes we do not know.

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