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SUPPLEMENTS IN COATOMIC MODULES HAVING THE COMPLETE MAX-PROPERTY

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Abstract. Let R be a ring with identity. A right R-module M has *the complete max-property* if the maximal submodules of M are *completely coindependent* (i.e., every maximal submodule of M does not contain the intersection of the other maximal submodules of M). A right R-module is said to be a *good* module provided every proper submodule of M containing $\mathrm{Rad}(M)$ is an intersection of maximal submodules of M. We obtain a new characterization of good modules. Also, we study good modules which have the complete max-property. The second part of this paper is devoted to investigate supplements in a coatomic module which has the complete max-property.

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1. Introduction

Let R be a unitary ring and M a right R-module. A submodule N of M is called *small* in M (written $N \ll M$) if for every proper submodule L of M, $N + L \neq M$. A submodule L of M is called *coclosed in* M if L/K is not small in M/K for any proper submodule K of L. We denote by $\mathrm{Rad}(M)$ the radical of M. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule, that is, $\mathrm{Rad}(M/N) \neq 0$ for every proper submodule $N \leq M$. Let L be a submodule of M. A submodule K of M is called a *supplement* of L in M if K is minimal with respect to the property $M = L + K$; equivalently, $M = L + K$ and $K \cap L \ll K$. A submodule P of M is called a *supplement submodule* if P is a supplement of some submodule of M. The module M is called *supplemented* if every submodule of M has a supplement in M. A module M is called *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. A module M is called *cosemisimple* (or a *V-module*) if every simple R-module is M-injective, or equivalently, every proper submodule of M is an intersection of maximal submodules (see [7, 23.1]). A module M is called a *good* module if $M/\operatorname{Rad}(M)$ is a cosemisimple module (see [7, 23.3]). A non-empty family of submodules N_i $(i \in I)$ of a module M is called *coindependent* if, for any $j \in I$ and any finite subset J of $I \setminus \{j\}$, $N_j + \bigcap_{i\in J} N_i = M$. The family N_i $(i \in I)$ is called *completely coindependent* if, for every $j \in I$, $N_j + \bigcap_{i\neq j} N_i = M$ (see [4, p. 8]). Following [6, p. 74], a module M is said to have the *complete max-property* if the maximal submodules of M form a completely coindependent set of submodules of M. In this paper, we adopt the convention that the intersection of an empty set of submodules of a module M is M itself.

In Section 2, we provide some new characterizations of good modules (Theorem 2.3). Also, we investigate the interplay between the complete max-property and each one of the properties coatomic and good.

The investigations in Section 3 focus on supplements in a coatomic module which has the complete max-property. After characterizing them, we show that for a coatomic module M, if M has the complete max-property, then any supplement submodule in M has also the complete max-property. In addition, we prove that if M is a coatomic module which has the complete max-property and F is a supplement of a submodule K in M, then $\Delta_F(M) = K + \operatorname{Rad}(F) = K + \operatorname{Rad}(M)$ where $\Delta_F(M)$ denotes the intersection of the maximal submodules of M not containing F.

Throughout this paper, R will denote an associative ring with identity and all modules are unitary right R-modules. By $\mathbb{Q}$ and $\mathbb{Z}$ we denote the ring of rational and integer numbers, respectively.

2. Good modules having the complete max-property

Recall that a module M is said to be a *good module* if for any module N and any homomorphism $f : M \to N$, $f(\operatorname{Rad}(M)) = \operatorname{Rad}(f(M))$. In this section, we obtain a new characterization of good modules. Moreover, we shed some light on good modules which have the complete max-property.

Let F be a submodule of a module M. We follow the notation of [3]. So the intersection of all maximal submodules of M containing F will be denoted by $\operatorname{Rad}_F(M)$. It is easily seen that $F + \operatorname{Rad} M \subseteq \operatorname{Rad}_F(M)$. On the other hand, we do not have equality, in general, as shown in [3, Remark 3.4]. In the same vein, we exhibit the following examples.

Example 2.1. (i) *Consider the submodule* $F = p^k\mathbb{Z}$ *of* $M = \mathbb{Z}$ *for some prime integer* p *and some integer* $k \geq 2$. *We have* $\operatorname{Rad}(M) = 0$. *So* $F + \operatorname{Rad}(M) = F$, *but* $\operatorname{Rad}_F(M) = p\mathbb{Z}$.

(ii) *Let p and q be two prime integers such that $p \neq q$. Consider the submodule $F = p^n q^m \mathbb{Z}$ of $M = \mathbb{Z}$, where n and m are natural numbers with $n \geq 2$ and $m \geq 2$. Clearly,* $\mathrm{Rad}(M) = 0$. *Then* $F + \mathrm{Rad}(M) = F$. *However,* $\mathrm{Rad}_F(M) = pq\mathbb{Z}$.

In [3], the authors provided some conditions under which $\mathrm{Rad}_F(M) = F + \mathrm{Rad}\, M$ for a submodule F of M. Among other results, it is shown in [3, Proposition 3.8] that if M is a good module, then $\mathrm{Rad}_F(M) = F + \mathrm{Rad}\, M$ for any submodule F of M. The next proposition shows that the converse of this result is true.

Proposition 2.2. *The following statements are equivalent for a module M:*

(i) *M is a good module;*
(ii) *Every proper submodule of M containing* $\mathrm{Rad}(M)$ *is an intersection of maximal submodules of M;*
(iii) $\mathrm{Rad}_F(M) = F + \mathrm{Rad}(M)$ *for every submodule F of M.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from [7, 23.1 and 23.3].

(i) $\Rightarrow$ (iii) By [3, Proposition 3.8].

(iii) $\Rightarrow$ (ii) Let L be a proper submodule of M such that $\mathrm{Rad}(M) \subseteq L$. By hypothesis, we have $\mathrm{Rad}_L(M) = L + \mathrm{Rad}(M) = L$. Hence L is an intersection of maximal submodules of M. □

Let F be a submodule of a module M. The intersection of the maximal submodules of M not containing F will be denoted by $\Delta_F(M)$.

Theorem 2.3. *The following statements are equivalent for a module M:*

(i) *M is a good module;*
(ii) $\mathrm{Rad}_F(M) = F + \mathrm{Rad}(M)$ *for every submodule F of M;*
(iii) $\mathrm{Rad}_F(M) \subseteq F + \Delta_F(M)$ *for every submodule F of M;*
(iv) *For any submodule F of M and any collection of maximal submodules N_i ($i \in I$) of M, we have $F + (\bigcap_{i\in I} N_i) = M$ or $F + (\bigcap_{i\in I} N_i)$ is an intersection of maximal submodules of M;*
(v) *For any submodule F of M, we have $F + \Delta_F(M) = M$ or $F + \Delta_F(M)$ is an intersection of maximal submodules of M.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from Proposition 2.2.

(ii) $\Leftrightarrow$ (iii) By [3, Proposition 3.5].

(i) $\Rightarrow$ (iv) This follows from Proposition 2.2.

(iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) These are obvious.

□

Remark 2.4. *From Theorem 2.3, it follows that a module M for which*

$$F + \Delta_F(M) = M \text{ for all } F \le M$$

is a good module.

Definition 2.5. A module M is said to have the *strong max-property* if for every submodule F of M, we have $F + \Delta_F(M) = M$.

We shall say that a module M has the *max-property* if the maximal submodules of M form a coindependent set of submodules of M (i.e., $M = L + \cap_{i=1}^{n} L_i$ for every positive integer n and distinct maximal submodules L, L_i $(1 \le i \le n)$ of M) (see [6]).

It is clear that the following implications hold:

Strong max-property $\Rightarrow$ complete max-property $\Rightarrow$ max-property.

The following lemma is a direct consequence of [6, Proposition 4.2 and Theorem 6.8].

Lemma 2.6. *Let M be an R-module which has the complete max-property such that $M/\operatorname{Rad}(M)$ is coatomic. Then M is a semilocal module.*

Proposition 2.7. *Any module which has the strong max-property is semilocal.*

Proof. Let M be a module with the strong max-property. By Theorem 2.3, M is a good module. Thus $M/\operatorname{Rad}(M)$ is a cosemisimple module. Hence $M/\operatorname{Rad}(M)$ is a coatomic module. Note that M has the complete max-property. Applying Lemma 2.6, we conclude that M is semilocal. □

Theorem 2.8. *The following statements are equivalent for a module M:*

(i) *M is a good module and M has the complete max-property;*
(ii) *M has the strong max-property.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $F + \Delta_F(M) \neq M$ for some submodule F of M. Then $F + \Delta_F(M)$ is an intersection of maximal submodules of M by Theorem 2.3. Therefore $\operatorname{Rad}_F(M) \subseteq F + \Delta_F(M)$ and hence $\operatorname{Rad}_F(M) + \Delta_F(M) = F + \Delta_F(M)$. But $\operatorname{Rad}_F(M) + \Delta_F(M) = M$ by [6, Proposition 6.1]. So $F + \Delta_F(M) = M$, a contradiction. This shows that M has the strong max-property.

(ii) $\Rightarrow$ (i) This is immediate. □

In the next example we present a coatomic good module which is not semilocal.

Example 2.9. *Let R be a right cosemisimple ring (i.e., R is a right V-ring) which is not semisimple (e.g., we take a field F and $R = \prod_{i \ge 1} F_i$ where $F_i = F$ for all $i \ge 1$). Then the R-module R_R is coatomic, but R_R is not semilocal since $Rad(R_R) = 0$. Moreover, it is clear that R_R is a good module.*

From Lemma 2.6, we get the following proposition which provides a sufficient condition for a coatomic module to be semilocal.

Proposition 2.10. *Let M be a coatomic module which has the complete max-property. Then M is semilocal. In particular, M is a good module.*

Combining Theorem 2.8 and Proposition 2.10, we obtain the following result.

Corollary 2.11. *Let M be a coatomic module. Then the following statements are equivalent:*

(i) *M has the complete max-property;*
(ii) *M has the strong max-property.*

The next example shows that, in general, a good module need not be coatomic.

Example 2.12. (i) *Let p be a prime integer and consider the $\mathbb{Z}$-module $M = \oplus_{n\geq 1}\mathbb{Z}/p^n\mathbb{Z}$. Since $\frac{\mathbb{Z}/p^n\mathbb{Z}}{\mathrm{Rad}(\mathbb{Z}/p^n\mathbb{Z})}$ is a semisimple module for all $n \geq 1$, $\mathbb{Z}/p^n\mathbb{Z}$ is a good module for all $n \geq 1$. Thus M is a good module by* [7, 23.4]*. However, M is not coatomic by* [8, Lemma 1.2].

(ii) *Let M be a module such that* $\mathrm{Rad}(M) = M$. *Then M is a good module as* $M/\mathrm{Rad}(M) = 0$ *is semisimple. On the other hand, M is not coatomic.*

In the next example, we exhibit a coatomic module which is not a good module.

Example 2.13. *Let R be a ring which is not a right V-ring such that* $\mathrm{Rad}(R) = 0$ *(e.g., we can take $R = \mathbb{Z}$). Clearly, the R-module $M = R_R$ is coatomic, but M is not a good module.*

Note that the class of semilocal modules is a proper subclass of the class of good modules (see Example 2.9). From [4, 2.8(8)], it follows that any semilocal module with a small radical is coatomic. This result can be extended to good modules as shown below.

Proposition 2.14. *Let M be a good module with a small radical. Then M is coatomic.*

Proof. Let N be a proper submodule of M. Then $N+\mathrm{Rad}(M) \neq M$ as $\mathrm{Rad}(M) \ll M$. Since M is a good module, $N + \mathrm{Rad}(M)$ is an intersection of maximal submodules of M. The result follows. □

3. Applications to supplement submodules

Our goal in this section is to characterize supplement submodules in a coatomic module which has the complete max-property. We begin with the following result on coclosed submodules of a coatomic good module.

Proposition 3.1. *Let M be a coatomic good module and let F be a submodule of M such that* $\mathrm{Rad}(M) \subseteq F$. *Then the following assertions are equivalent:*

(i) *F is coclosed in M;*

(ii) *F is coatomic and* $\mathrm{Rad}(F) = \mathrm{Rad}(M)$.

Proof. (i) $\Rightarrow$ (ii) From [2, Lemma 4.1], it follows that F is coatomic. Moreover, we have $\mathrm{Rad}(F) = F \cap \mathrm{Rad}(M)$ by [4, 3.7]. As $\mathrm{Rad}(M) \subseteq F$, we obtain $\mathrm{Rad}(F) = \mathrm{Rad}(M)$.

(ii) $\Rightarrow$ (i) Let $L \leq F$ such that $F/L \ll M/L$. Then $F/L \subseteq \mathrm{Rad}(M/L)$. Since M is a good module, we have

$$\mathrm{Rad}(M/L) = (L + \mathrm{Rad}(M))/L = (L + \mathrm{Rad}(F))/L.$$

Therefore $\mathrm{Rad}(M/L) \subseteq \mathrm{Rad}(F/L)$ by [4, 2.8 (1)]. So $F/L \subseteq \mathrm{Rad}(F/L)$. Hence, $F/L = \mathrm{Rad}(F/L)$. As F is coatomic, it follows that $F/L = 0$; that is, $L = F$. This completes the proof. □

It was shown in [5, Theorem 2.1] that if F is a supplement of a submodule K in a module M, then it is possible to define a bijective map between maximal submodules of F and maximal submodules of M which contain K. In the next result, we use this fact to characterize supplement submodules in a coatomic module.

Proposition 3.2. *Let F and K be submodules of a coatomic module M. Then the following statements are equivalent:*

(i) *F is a supplement of K in M;*

(ii) (1) *F is coatomic, and*

(2) *for any submodule N of F, N is a maximal submodule of F if and only if $N = F \cap L$ for some maximal submodule L of M with $K \subseteq L$.*

Proof. (i) $\Rightarrow$ (ii) This follows from [2, Lemma 4.1] and [5, Theorem 2.1].

(ii) $\Rightarrow$ (i) Suppose that $K + F \neq M$. Since M is coatomic, there exists a maximal submodule X of M such that $K + F \subseteq X$. By (2), $F \cap X = F$ is a maximal submodule of F, a contradiction. So $K + F = M$. Now let H be a proper submodule of F. Since F is coatomic, $H \subseteq Y$ for some maximal submodule Y of F. By hypothesis, there exists a maximal submodule Z of M such that $K \subseteq Z$ and $Y = F \cap Z$. Therefore $H + K \subseteq Y + K = (F \cap Z) + K \subseteq Z$. It follows that $H + K \neq M$. This proves that F is a supplement of K in M. □

Theorem 3.3. *Let M be a coatomic module which has the complete max-property. Then the following statements about a submodule F of M are equivalent:*

(i) *F is a supplement in M;*

(ii) *F is coatomic and* $F \cap \mathrm{Rad}(M) = \mathrm{Rad}(F)$*;*
(iii) $F \cap \mathrm{Rad}(M) \ll F$*;*
(iv) *F is coclosed in M;*
(v) *F is a supplement of $\Delta_F(M)$ in M;*
(vi) *F is a supplement of* $\mathrm{Rad}(M)$ *in* $\mathrm{Rad}_F(M)$*;*
(vii) $F \cap \Delta_F(M) \ll F$*;*
(viii) *F is coatomic and* $F \cap \Delta_F(M) = \mathrm{Rad}(F)$.

Proof. Note that M is a good module by Proposition 2.10. Applying Theorems 2.3 and 2.8, we conclude that $\mathrm{Rad}_N(M) = N + \mathrm{Rad}(M)$ and $N + \Delta_N(M) = M$ for every submodule N of M.

(i) $\Rightarrow$ (v) Assume that F is a supplement of a submodule U in M. Note that $\mathrm{Rad}\, M \ll M$ as M is coatomic. So F is also a supplement of $U + \mathrm{Rad}\, M$ in M by [4, 20.4 (4)]. Since $\mathrm{Rad}_U(M) = U + \mathrm{Rad}(M)$, F is a supplement of $\mathrm{Rad}_U(M)$ in M. Moreover, we have $\Delta_F(M) \subseteq \mathrm{Rad}_U(M)$ as $F + U = M$. Since $F + \Delta_F(M) = M$, it follows that F is a supplement of $\Delta_F(M)$ in M by [4, 20.4 (1)].

(v) $\Rightarrow$ (vii) This is obvious.

(vii) $\Rightarrow$ (iv) Assume that $\Delta_F(M) \cap F \ll F$. Since $F + \Delta_F(M) = M$, it follows that F is a supplement of $\Delta_F(M)$ in M. Hence F is coclosed in M by [4, 20.2].

(iv) $\Rightarrow$ (ii) From [2, Lemma 4.1], it follows that F is coatomic. Furthermore, $F \cap \mathrm{Rad}(M) = \mathrm{Rad}(F)$ by [4, 3.7 (3)].

(ii) $\Rightarrow$ (viii) Note that $F \cap \Delta_F(M) = F \cap \mathrm{Rad}_F(M) \cap \Delta_F(M) = F \cap \mathrm{Rad}(M)$. Then $F \cap \Delta_F(M) = \mathrm{Rad}(F)$ by (ii).

(viii) $\Rightarrow$ (iii) Since F is coatomic, we have $\mathrm{Rad}(F) \ll F$. Thus $F \cap \Delta_F(M) \ll F$. But $F \cap \mathrm{Rad}(M) \subseteq F \cap \Delta_F(M)$. So $F \cap \mathrm{Rad}(M) \ll F$.

(iii) $\Rightarrow$ (vi) This follows from the fact that $F + \mathrm{Rad}(M) = \mathrm{Rad}_F(M)$.

(vi) $\Rightarrow$ (i) Note that $F + \Delta_F(M) = M$. In addition, we have $F \cap \Delta_F(M) \subseteq F \cap \mathrm{Rad}_F(M) \cap \Delta_F(M) \subseteq F \cap \mathrm{Rad}(M) \ll F$ by (vi). Therefore F is a supplement of $\Delta_F(M)$ in M. $\square$

The next example shows that the conditions in the hypothesis of Theorem 3.3 are not superfluous.

Example 3.4. (i) *Let p be a prime integer and consider the $\mathbb{Z}$-module $M = M_1 \oplus M_2$ where $M_1 = \mathbb{Z}/p^2\mathbb{Z} \oplus 0$ is a maximal submodule of M and $M_2 = 0 \oplus \mathbb{Z}/p\mathbb{Z}$ is simple. It is clear that M is a coatomic module. However, the module M does not have the complete max-property as $M/\mathrm{Rad}(M) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (see* [6, Theorems 2.3 and 6.8] *or* [6, Corollary 6.11]*). Let $N = (\bar{1}, \tilde{1})\mathbb{Z} \leq M$. It is easily seen that $N \oplus M_2 = M$. So N is a maximal submodule of M. Note that M_2 is a supplement* *in M. Moreover, $M_2 \nsubseteq M_1$ and $M_2 \nsubseteq N$. Hence $\Delta_{M_2}(M) \subseteq M_1 \cap N \subseteq p\mathbb{Z}/p^2\mathbb{Z} \oplus 0$. Thus $M_2 + \Delta_{M_2}(M) \subseteq (p\mathbb{Z}/p^2\mathbb{Z} \oplus 0) \oplus M_2$. It follows that $M_2 + \Delta_{M_2}(M) \neq M$. This implies that M_2 is not a supplement of $\Delta_{M_2}(M)$ in M.*

(ii) *Let M be a nonzero module with $\mathrm{Rad}(M) = M$. Then M is a supplement in M, but $M = M \cap \mathrm{Rad}(M)$ is not small in M. Note that M has the complete max-property but M is not coatomic.*

Following [2], a module M is called an *ms-module* if every maximal submodule of M is a supplement in M. As an application of Theorem 3.3, we get the following corollaries.

Corollary 3.5. *Let M be a coatomic module which has the complete max-property. Then M is an ms-module if and only if $\mathrm{Rad}(M) \ll K$ for every maximal submodule K of M.*

Corollary 3.6. *Let M be a coatomic module which has the complete max-property. Let L and F be submodules of M such that $F \subseteq L$ and $F \cap \mathrm{Rad}(M) = L \cap \mathrm{Rad}(M)$. If F is a supplement in M, then so is L.*

Corollary 3.7. *Let M be a coatomic module which has the complete max-property. Let L and F be submodules of M such that $\mathrm{Rad}(M) \subseteq F \subseteq L$. If F is a supplement in M, then so is L.*

Corollary 3.8. *Let M be a coatomic module which has the complete max-property and let N be a maximal submodule of M. If N and $\Delta_N(M)$ are supplements in M, then M is an ms-module.*

Proof. Let K be a maximal submodule of M such that $K \neq N$. Then $\mathrm{Rad}(M) \subseteq \Delta_N(M) \subseteq K$. By Corollary 3.7, it follows that K is a supplement in M. Since N is a supplement in M, M is an ms-module. □

Corollary 3.9. *Let R be a right noetherian ring and let M be a finitely generated R-module which has the complete max-property. Then the following statements about a submodule F of M are equivalent:*

(i) *F is a supplement in M;*

(ii) *$F \cap \mathrm{Rad}(M) = \mathrm{Rad}(F)$.*

Proof. Since R is right noetherian and M is finitely generated, every submodule of M is finitely generated. So every submodule of M is coatomic. The result follows from Theorem 3.3. □

It is shown in [8, Lemma 1.1] that over a commutative noetherian ring, every submodule of a coatomic module is coatomic. Combining this fact and Theorem 3.3, we obtain the following result.

Corollary 3.10. *Let R be a commutative noetherian ring and let M be a coatomic R-module which has the complete max-property. Then the following statements about a submodule F of M are equivalent:*

(i) *F is a supplement in M;*
(ii) *$F \cap \operatorname{Rad}(M) = \operatorname{Rad}(F)$.*

As noted in [6, p. 80], the class of modules which have the complete max-property is not closed under submodules. For example, the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$ has the complete max-property, however the submodule $\mathbb{Z}$ does not have the complete max-property. Next, we will show that for a coatomic module M, if M has the complete max-property, then any supplement submodule in M inherits the property.

Proposition 3.11. *Let M be a coatomic module. If M has the complete max-property, then every supplement submodule of M has the complete max-property.*

Proof. Assume that the module M has the complete max-property. Then M is a good module by Proposition 2.10. Let F be a supplement submodule in M. Then $M/\Delta_F(M)$ has the complete max-property by [6, Lemma 3.4]. Moreover, from Corollary 2.11 and Theorem 3.3, it follows that

$$F/\operatorname{Rad}(F) = F/F \cap \Delta_F(M) \cong (F + \Delta_F(M))/\Delta_F(M) = M/\Delta_F(M).$$

So $F/\operatorname{Rad}(F)$ has the complete max-property. Using again [6, Lemma 3.4], it follows that F has the complete max-property. □

Proposition 3.12. *Let M be a module. Assume that* $\operatorname{Rad}(M)$ *has a supplement F in M such that F has the complete max-property. Then M has the complete max-property.*

Proof. By hypothesis, we have $\operatorname{Rad}(M) + F = M$. Then

$$M/\operatorname{Rad}(M) = (\operatorname{Rad}(M) + F)/\operatorname{Rad}(M) \cong F/(F \cap \operatorname{Rad}(M)).$$

Since F has the complete max-property, $F/(F \cap \operatorname{Rad}(M))$ has also the complete max-property by [6, Lemma 3.4]. Therefore $M/\operatorname{Rad}(M)$ has the complete max-property. Again by [6, Lemma 3.4], it follows that M has the complete max-property. □

Proposition 3.13. *Let $M = M_1 + M_2$ be a good module such that every maximal submodule of M contains M_1 or M_2. Assume that M_1 and M_2 are mutual supplements in M and they both have the complete max-property. Then M has the complete max-property.*

Proof. Let N be a maximal submodule of M. Without loss of generality we can assume that $M_1 \subseteq N$. Since M_2 is a supplement of M_1, the maximal submodules of M_2 are $\{N_i \cap M_2 \mid i \in I\}$ where $\{N_i \mid i \in I\}$ are the maximal submodules of M containing M_1 by [5, Theorem 2.1]. So $N = N_{i_0}$ for some $i_0 \in I$. Since M_2 has the complete max-property, we have

$$(N_{i_0} \cap M_2) + \bigcap_{i \neq i_0} (N_i \cap M_2) = M_2. \qquad (*)$$

Let $\{N_j \mid j \in J\}$ be the set of the maximal submodules of M containing M_2. Hence

$$N_{i_0} + \Delta_{N_{i_0}}(M) = N_{i_0} + \left(\bigcap_{i \neq i_0} N_i \right) \bigcap \left(\bigcap_{j \in J} N_j \right).$$

Since M is a good module, from Theorem 2.3 we have

$$\bigcap_{j \in J} N_j = \mathrm{Rad}_{M_2}(M) = M_2 + \mathrm{Rad}(M).$$

Thus,

$$N_{i_0} + \Delta_{N_{i_0}}(M) = N_{i_0} + \left(\bigcap_{i \neq i_0} N_i \right) \bigcap (M_2 + \mathrm{Rad}(M)).$$

By modularity, we get

$$N_{i_0} + \Delta_{N_{i_0}}(M) = N_{i_0} + \mathrm{Rad}(M) + \left(\left(\bigcap_{i \neq i_0} N_i \right) \bigcap M_2 \right).$$

But $\mathrm{Rad}(M) \subseteq N_{i_0}$. Then, by using $(*)$, we have

$$\begin{aligned} N_{i_0} + \Delta_{N_{i_0}}(M) &= N_{i_0} + \bigcap_{i \neq i_0} (N_i \cap M_2) \\ &= N_{i_0} + (N_{i_0} \cap M_2) + \bigcap_{i \neq i_0} (N_i \cap M_2) \\ &= N_{i_0} + M_2 \\ &= M. \end{aligned}$$

This completes the proof. □

The next example illustrates that the assumption “every maximal submodule of M contains M_1 or M_2”in Proposition 3.13 cannot be dropped.

Example 3.14. *Let M be as in Example 3.4(i). The module M does not have the complete max-property. Since $M/\mathrm{Rad}(M)$ is semisimple, M is a good module. Also, M_1 and M_2 are mutual supplements in M. Let $N = (\bar{1}, \tilde{1})\mathbb{Z} \leq M$. It is easily seen that N is a maximal submodule of M such that neither M_1 nor M_2 is* *contained in N. Note that both of M_1 and M_2 have the complete max-property since each one of them has only one maximal submodule.*

Combining Proposition 3.13 and [6, Lemma 3.4], we obtain the following result.

Corollary 3.15. *Let $M = M_1 \oplus M_2$ be a good module such that every maximal submodule of M contains M_1 or M_2. Then M has the complete max-property if and only if M_1 and M_2 have the complete max-property.*

In the next result, we evaluate $\Delta_F(M)$ for a supplement submodule F of a coatomic module M which has the complete max-property.

Theorem 3.16. *Let M be a coatomic module which has the complete max-property and let K be a submodule of M. Let F be a supplement of K in M. Then*

$$\Delta_F(M) = K + \mathrm{Rad}(F) = K + \mathrm{Rad}(M).$$

Proof. Set $\Gamma = \{L \leq M \mid L \text{ is maximal in } M \text{ and } F \nsubseteq L\}$ and $\Lambda = \{N \leq M \mid N \text{ is maximal in } M \text{ and } K \subseteq N\}$. Clearly $\Lambda \subseteq \Gamma$. Let us show that $\Lambda = \Gamma$. Note that F is a supplement of $\Delta_F(M)$ in M by Theorem 3.3. It follows that for a maximal submodule X of M, $F \nsubseteq X$ if and only if $\Delta_F(M) \subseteq X$. Let $L \in \Gamma$. Then $\Delta_F(M) \subseteq L$. By [5, Proof of Theorem 2.1], $L \cap F$ is a maximal submodule of F and $N = (L \cap F) + K$ is a maximal submodule of M. Note that $N \cap F = ((L \cap F) + K) \cap F = (L \cap F) + (K \cap F)$. As F is a supplement of K in M, we have $K \cap F \ll F$. So $K \cap F \subseteq \mathrm{Rad}(M) \subseteq L$. Thus $K \cap F \subseteq L \cap F$. Hence $N \cap F = L \cap F$. Note that $F \nsubseteq N$. Then $\Delta_F(M) \subseteq N$. By modularity, we have

$$L = L \cap (F + \Delta_F(M)) = (L \cap F) + \Delta_F(M) = (N \cap F) + \Delta_F(M) = N \cap (F + \Delta_F(M)) = N.$$

It follows that $L \in \Lambda$. So $\Lambda = \Gamma$. Thus $\Delta_F(M) = \mathrm{Rad}_K(M)$. Since M is good, $\Delta_F(M) = \mathrm{Rad}_K(M) = K + \mathrm{Rad}(M)$ by Theorem 2.3. Moreover, by Theorem 3.3, we have $F \cap \Delta_F(M) = \mathrm{Rad}(F)$. So $\Delta_F(M) = (K + F) \cap \Delta_F(M) = K + (F \cap \Delta_F(M)) = K + \mathrm{Rad}(F)$. □

Remark 3.17. *Let M be a coatomic module which has the complete max-property and let F be a supplement in M. From the previous result, it follows that if F is a supplement of a submodule K in M, then*

(i) *$K \subseteq \Delta_F(M)$, and*
(ii) *every maximal submodule of M contains F or K.*

By the following example we see that the condition "M has the complete max-property"cannot be omitted from the hypothesis of Theorem 3.16.

Example 3.18. *Let M be as in Example 3.4(i). So M_2 is a supplement of both M_1 and N in M. Since M_1 and N are maximal submodules of M, we have $N + \mathrm{Rad}(M) = N$ and $M_1 + \mathrm{Rad}(M) = M_1$. Thus $N + \mathrm{Rad}(M) \neq M_1 + \mathrm{Rad}(M)$. Note that M is a coatomic module which does not have the complete max-property.*

As an application of Theorem 3.16, we obtain the following two propositions.

Recall that following [1], two submodules X and Y of a module M are said to be β^* *equivalent* (denoted as $X\beta^*Y$) if $(X+Y)/X \ll M/X$ and $(X+Y)/Y \ll M/Y$. It was shown in [1, Theorem 2.6 (ii)] that if X, Y are submodules of M such that $X\beta^*Y$, then X has a supplement C in M if and only if C is a supplement of Y in M.

Proposition 3.19. *Let M be a coatomic module which has the complete max-property and let H, K and F be submodules of M. Assume that F is a supplement of both H and K in M. Then $H\beta^*K$.*

Proof. By Theorem 3.16, we have $H + \mathrm{Rad}(M) = K + \mathrm{Rad}(M) = \Delta_F(M)$. From [1, Corollary 2.4], it follows that $H\beta^*K$. □

Following [1], a module M is called *Goldie*-supplemented* if for every submodule X of M, there exists a supplement submodule F in M such that $X\beta^*F$. It was shown in [1, Theorem 3.6 and Example 3.9 (iii)] that any Goldie*-supplemented module is supplemented but the converse is not true, in general. In the next proposition, we present some sufficient conditions for a supplemented module to be Goldie*-supplemented.

Proposition 3.20. *Let M be a coatomic module which has the complete max-property. If M is supplemented, then M is Goldie*-supplemented*

Proof. Assume that M is a supplemented module. Let X be a submodule of M. Let F be a supplement of X in M and let T be a supplement of F in M. Then F is a supplement of T in M by [4, 20.4 (9)]. Using Theorem 3.16, we get $X + \mathrm{Rad}(M) = T + \mathrm{Rad}(M) = \Delta_F(M)$. Note that $\mathrm{Rad}(M) \ll M$. Therefore M is Goldie*-supplemented by [1, Corollary 3.4]. □

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References

[1] G. F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez and A. Tercan, *Goldie*-supplemented modules*, Glasg. Math. J., 52(A) (2010), 41-52.

[2] E. Büyükasik and D. Pusat-Yilmaz, *Modules whose maximal submodules are supplements*, Hacet. J. Math. Stat., 39(4) (2010), 477-487.

[3] M. D. Cissé and D. Sow, *On generalizations of essential and small submodules*, Southeast Asian Bull. Math., 41(3) (2017), 369-383.

[4] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules, Supplements and Projectivity in Module Theory, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2006.

[5] C. Nebiyev and A. Pancar, *On supplement submodules*, Ukrainian Math. J., 65(7) (2013), 1071-1078.

[6] P. F. Smith, *Module with coindependent maximal submodules*, J. Algebra Appl., 10(1) (2011), 73-99.

[7] R. Wisbauer, Foundations of Module and Ring Theory, Algebra, Logic and Applications, 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[8] H. Zöschinger, *Koatomare moduln*, Math. Z., 170(3) (1980), 221-232.

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