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OD_s-CHARACTERIZATION OF SOME LOW-DIMENSIONAL FINITE CLASSICAL GROUPS

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Dedicated to the memory of Professor John Clark

Abstract. The solvable graph of a finite group G, which is denoted by $\Gamma_s(G)$, is a simple graph whose vertex set is comprised of the prime divisors of $|G|$ and two distinct primes p and q are joined by an edge if and only if there exists a solvable subgroup of G such that its order is divisible by pq. Let $p_1 < p_2 < \cdots < p_k$ be all prime divisors of $|G|$ and let $D_s(G) = (d_s(p_1), d_s(p_2), \ldots, d_s(p_k))$, where $d_s(p)$ signifies the degree of the vertex p in $\Gamma_s(G)$. We will simply call $D_s(G)$ the degree pattern of solvable graph of G. A finite group H is said to be OD_s-characterizable if $H \cong G$ for every finite group G such that $|G| = |H|$ and $D_s(G) = D_s(H)$. In this paper, we study the solvable graph of some subgroups and some extensions of a finite group. Furthermore, we prove that the linear groups $L_3(q)$ with certain properties, are OD_s-characterizable.

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1. Introduction

All groups appearing here are supposed to be finite. For a natural number n, we denote by $\pi(n)$ the set of prime divisors of n and set $\pi(G) = \pi(|G|)$. The set of orders of all elements in a finite group G is denoted by $\omega(G)$ and called the *spectrum* of G. This set is closed and partially ordered by the divisibility relation; therefore, it is determined uniquely from the subset $\mu(G)$ of all maximal elements of $\omega(G)$ with respect to divisibility. Recently, many new ways is discovered to characterize a finite simple group. For more details, there are a lot of ways to associate a quantitative property to the finite group G. One of the most important is to consider some properties of the graphs associated with it. In fact, one of these graphs is the *solvable graph* of G which is introduced by Abe and Iiyori in [2]. This graph is denoted by $\Gamma_{\rm s}(G)$ and is a simple and undirected graph constructed as follows. The vertex set is $\pi(G)$ and two distinct prime p and q are adjacent (we write $p \approx q$) if and only if G has a solvable subgroup whose order is divisible by pq. If this condition is replaced by "G has a cyclic subgroup of order pq", then we call this graph the *prime graph* of G denoted by GK(G). In fact, the prime graph of G is a graph whose vertex set is $\pi(G)$ and two vertices p and q are joined by an edge if and if $pq \in \omega(G)$. Therefore, the solvable graph associated with a group is a generalization of its prime graph.

The *degree* $d_{\rm s}(p)$ (resp. $d(p)$) of a vertex $p \in \pi(G)$ is the number of adjacent vertices to p in $\Gamma_{\rm s}(G)$ (resp. GK(G)). Clearly, $d(p) \leqslant d_{\rm s}(p)$ for every vertex $p \in \pi(G)$.

In the case when $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \cdots < p_k$, we define

$$\mathrm{D_s}(G) = \Big(d_{\rm s}(p_1), d_{\rm s}(p_2), \ldots, d_{\rm s}(p_k)\Big),$$

which is called the *degree pattern of the solvable graph of* G. For every non-negative integer $m \in \{0, 1, 2, \ldots, k-1\}$, we put

$$\Delta_m(G) := \{p \in \pi(G) |\ d_{\rm s}(p) = m\}.$$

It is obvious that

$$\pi(G) = \bigcup_{m=0}^{k-1} \Delta_m(G).$$

When $\Delta_{k-1}(G) \neq \emptyset$, the prime p with $d_{\rm s}(p) = k-1$ is called a *complete prime.*

One of the purpose of this paper is to consider the solvable graphs of some groups. For more details, we examine the solvable graphs of some subgroups of a group named local subgroups which introduced in section 3 completely. We also investigate the solvable graph of a certain extension of groups.

Given a finite group G, denote by $h_{\mathrm{OD_s}}(G)$ the number of isomorphism classes of finite groups H such that $|H| = |G|$ and $\mathrm{D_s}(H) = \mathrm{D_s}(G)$. In terms of the function $h_{\mathrm{OD_s}}(\cdot)$, we have the following definition.

Definition 1.1. A finite group G is said to be k-fold $\mathrm{OD_s}$-characterizable if $h_{\mathrm{OD_s}}(G) = k$. The group G is $\mathrm{OD_s}$-characterizable if $h_{\mathrm{OD_s}}(G) = 1$.

In this paper, we are going to characterize some simple groups by order and degree pattern of solvable graph. In [3], it was shown that the following groups are $\mathrm{OD_s}$-characterizable.

(1) All sporadic simple groups;

(2) Projective special linear groups $\mathrm{L}_2(q)$ with one of the following conditions:

(a) $p = 2$, $|\pi(q+1)| = 1$ or $|\pi(q-1)| = 1$,
(b) $q \equiv 1 \pmod 4$, $|\pi(q+1)| = 2$ or $|\pi(q-1)| \leqslant 2$,
(c) $q \equiv -1 \pmod 4$.

(3) A finite group H such that $H \notin \{\mathrm{B}_n(q), \mathrm{C}_n(q)\}$ ($n \geqslant 3$ and q is odd), $|\pi(H)| = k \geqslant 3$ and $\Delta_{k-1}(H) = \emptyset$.

We will show that the projective special linear groups $\mathrm{L}_3(q)$ with certain properties, are $\mathrm{OD_s}$-characterizable. In fact, we prove the following Corollary.

Corollary A. *The simple groups* $\mathrm{L}_3(q)$ *with one of the following conditions are* $\mathrm{OD_s}$- *characterizable:*

(1) q *is odd and* $9 \nmid q-1$*;*
(2) q *is even and* $3 \parallel q-1$*;*
(3) $9 \mid q-1$ *and* $|\pi(\frac{q^2+q+1}{3})| = 1$*;*
(4) q *is even,* $3 \mid q+1$ *and* $|\pi(q^2+q+1)| = 1$.

Notation and Terminology. Let Γ be a graph and V be the vertex set of Γ. The *complementary graph* Γ^c of Γ is a graph whose vertex set is V and two vertices of Γ^c are joined if and only if they are not joined in Γ. Let U be a subset of the vertex set V. The graph $\Gamma - U$ is defined to be a graph whose vertex set is $V - U$ and two vertices are joined if they are joined in Γ. A *spanning subgraph* of Γ is a subgraph of Γ whose vertex set is V. A graph in which every pair of distinct vertices are adjacent is called a *complete graph*. A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. Moreover, if every two vertices from X and Y are adjacent, then it is called a *complete bipartite graph* and denoted by $K_{|X|,|Y|}$. A *star graph* is a complete bipartite graph of the form $K_{1,n}$ which consists of one central vertex having edges to other vertices in it.

2. Preliminary results

In this section, we first state some obtained results on solvable graph of finite groups, and then we find the solvable graphs of the projective special linear groups $\mathrm{L}_3(q)$. Finally, we consider the solvable graph of the automorphism groups of some simple groups.

Lemma 2.1. ([2, Corollary 2]) *The solvable graph of a finite group is a connected graph.*

Lemma 2.2. ([2, Lemma 1, Theorem 2]) *Let* G *be a finite group. Then the following statements hold:*

(1) *If G is a solvable group, then $\Gamma_{\mathrm{s}}(G)$ is complete.*
(2) *If G is a non-abelian simple group, then $\Gamma_{\mathrm{s}}(G)$ is not complete.*

Lemma 2.3. ([1, Lemma 3]) *Let G be a finite group with $|\pi(G)| = k$. If $\Delta_{k-1}(G) = \emptyset$, then G is a non-abelian simple group.*

We continue this argument with the following lemma which considers the solvable graphs of subgroups and quotient groups of a finite group.

Lemma 2.4. ([2, Lemma 2]) *Let G be a group, H a subgroup of G and N a normal subgroup of G.*

(1) *If p and q are joined in $\Gamma_{\mathrm{s}}(H)$ for $p, q \in \pi(H)$, then p and q are joined in $\Gamma_{\mathrm{s}}(G)$, that is, $\Gamma_{\mathrm{s}}(H)$ is a subgraph of $\Gamma_{\mathrm{s}}(G)$.*
(2) *If p and q are joined in $\Gamma_{\mathrm{s}}(G/N)$ for $p, q \in \pi(G/N)$, then p and q are joined in $\Gamma_{\mathrm{s}}(G)$, that is, $\Gamma_{\mathrm{s}}(G/N)$ is a subgraph of $\Gamma_{\mathrm{s}}(G)$.*
(3) *For $p \in \pi(N)$ and $q \in \pi(G) \setminus \pi(N)$, p and q are joined in $\Gamma_{\mathrm{s}}(G)$.*

Lemma 2.5. ([3, Corollary 1]) *Let N be a normal subgroup of a finite group G. Then for two primes $\{p, q\} \subseteq \pi(G) \setminus \pi(N)$, $p \approx q$ in $\Gamma_{\mathrm{s}}(G/N)$ if and only if $p \approx q$ in $\Gamma_{\mathrm{s}}(G)$.*

Lemma 2.6. ([2, Theorem 3]) *Let G be a finite group and $\{p, q\} \subseteq \pi(G)$. Then p and q are not joined in $\Gamma_{\mathrm{s}}(G)$ if and only if there exists a series of normal subgroups of G, say*

$$1 \trianglelefteq M \triangleleft N \trianglelefteq G,$$

such that M and G/N are $\{p, q\}'$-groups and N/M is a non-abelian simple group such that p and q are not joined in $\Gamma_{\mathrm{s}}(N/M)$.

Using the notation taken from [1] and [2], such a series as in Lemma 2.6 is called a *GKS-series of G* and we will say *p and q are expressed to be disjoint by this GKS-series.*

Lemma 2.7. ([1, Lemma 4]) *Let G be a finite group with $|\pi(G)| = k$. If the number of connected components of*

$$\widetilde{\Gamma}(G) = (\Gamma_{\mathrm{s}}(G) - \Delta_{k-1}(G))^c$$

equals n, then at most n GKS-series of G is necessary to express any pair of vertices of $\Gamma_{\mathrm{s}}(G)$ to be disjoined.

As a direct result of Lemma 2.7, we can point out the following Lemma (see [3]).

Lemma 2.8. [3, Lemma 6] *Let G be a finite group with $|\pi(G)| = k \geqslant 4$ and $\widetilde{\Gamma}(G) := (\Gamma_{\rm s}(G) - \Delta_{k-1}(G))^c$. If one of the following conditions holds, then any disjoined pair of vertices of $\Gamma_{\rm s}(G)$ can be expressed by only one GKS-series.*

(1) $\Delta_{k-1}(G) \neq \emptyset$ *and* $\Delta_1(G) \neq \emptyset$;
(2) $\Delta_{k-1}(G) \neq \emptyset$ *and* $\Delta_2(G) \neq \emptyset$.

The following lemma is due to K. Zsigmondy (see [11]).

Lemma 2.9. [Zsigmondy Theorem] *Let q and f be integers greater than 1. There exists a prime divisor r of $q^f - 1$ such that r does not divide $q^e - 1$ for all $0 < e < f$, except in the following cases:*
(a) $f = 6$ *and* $q = 2$;
(b) $f = 2$ *and* $q = 2^l - 1$ *for some natural number* l.

Such a prime r is called a primitive prime divisor of $q^f - 1$.

We define a function η on $\mathbb{N}$ which will be used in the proof of Theorem 4.1, as follows:

$$\eta(m) = \begin{cases} m & m \equiv 1 \pmod 2, \\ \frac{m}{2} & m \equiv 0 \pmod 2. \end{cases}$$

It is sometimes convenient to represent the graph $\Gamma_{\rm s}(G)$ in a compact form. By the compact form we mean a graph whose vertices are displayed with disjoint subsets of $\pi(G)$. Actually, a vertex labeled U represents the complete subgraph of $\Gamma_{\rm s}(G)$ on U. An edge connecting U and W represents the set of edges of $\Gamma_{\rm s}(G)$ that connect each vertex in U with each vertex in W. Figures $1 - 7$, for instance, depicts the compact form of the solvable graph of the projective special linear groups $\mathrm{L}_3(q)$ in all cases.

To construct the solvable graph of this group, we need to state the following facts:

- The prime graph of a group is the subgraph of its solvable graph. Therefore, it is good to note that the set of maximal elements in the spectrum of $\mathrm{L}_3(q)$ is as follows:

$$\mu(\mathrm{L}_3(q)) = \begin{cases} \{q-1, \frac{p(q-1)}{3}, \frac{q^2-1}{3}, \frac{q^2+q+1}{3}\} & \text{if } d = 3; \\ \\ \{p(q-1), q^2-1, q^2+q+1\} & \text{if } d = 1, \end{cases}$$

where $q=p^n$ is odd and $d=(3,q-1)$, and

$$\mu(\mathrm{L}_3(2^n))=\begin{cases}\{4,2^n-1,\frac{2(2^n-1)}{3},\frac{2^{2n}-1}{3},\frac{2^{2n}+2^n+1}{3}\} & \text{if } d=3;\\ \\ \{4,2(2^n-1),2^{2n}-1,2^{2n}+2^n+1\} & \text{if } d=1,\end{cases}$$

where $d=(3,2^n-1)$, except $n\in\{1,2\}$.

- Considering Lemma 2.4 the solvable graph of

$$\mathrm{L}_3(q)\cong\frac{\mathrm{SL}_3(q)}{Z(\mathrm{SL}_3(q))},$$

where $|Z(\mathrm{SL}_3(q))|=(3,q-1)$, is a subgraph of $\mathrm{SL}_3(q)$. We also found from Lemma 2.5 that

$$\Gamma_{\mathrm{s}}(\mathrm{L}_3(q))-\{3\}=\Gamma_{\mathrm{s}}(\mathrm{SL}_3(q))-\{3\}.$$

On the other hand, if $3\approx p$ in $\Gamma_{\mathrm{s}}(\mathrm{SL}_3(q))$, then there exists a solvable subgroup H of $\mathrm{SL}_3(q)$ such that $3p$ divides $|H|$. Obviously,

$$\frac{HZ}{Z}\cong\frac{H}{H\cap Z}$$

where $Z=Z(\mathrm{SL}_3(q))$, is a solvable subgroup of $\mathrm{L}_3(q)$. If either $H\cap Z=1$, or $Z\leqslant H$ and $9\mid |H|$, then $3p$ divides $|\frac{HZ}{Z}|$ and so $3\approx p$ in $\Gamma_{\mathrm{s}}(\mathrm{L}_3(q))$.

In general, let G be a finite group possessing a normal cyclic subgroup $\langle x\rangle$ where $o(x)=p$ for some prime $p\in\pi(G)$. Then we can conclude from Lemma 2.5 that for every prime $q,r\in\pi(G)\setminus\{p\}$, $q\approx r$ in $\Gamma_{\mathrm{s}}(G/\langle x\rangle)$ if and only if $q\approx r$ in $\Gamma_{\mathrm{s}}(G)$. It follows that

$$\Gamma_{\mathrm{s}}(G/\langle x\rangle)-\{p\}=\Gamma_{\mathrm{s}}(G)-\{p\}.$$

- The maximal subgroups of $\mathrm{SL}_3(q)$ which is collected in [4] (Table 8. 3) are listed as follows.

Subgroup	Conditions	Subgroup	Conditions
${\mathrm{E}_q}^3:\mathrm{GL}_2(q)$		$\mathrm{SL}_3(q_0).\big(\frac{q-1}{q_0-1},3\big)$	$q={q_0}^r, r$ is a prime
${\mathrm{E}_q}^{1+2}:(q-1)^2$		${3_+}^{1+2}:\mathrm{Q}_8.\frac{(q-1,9)}{3}$	$p=q\equiv 1 \pmod 3$
$\mathrm{GL}_2(q)$		$d\times\mathrm{SO}_3(q)$	q is odd
$(q-1)^2:\mathbb{S}_3$	$q\geqslant 5$	$(q_0-1,3)\times\mathrm{SU}_3(q_0)$	$q={q_0}^2$
$(q^2+q+1):3$			

According to the notation of [4], $d = |Z(\mathrm{SL}_3(q))| = (3, q-1)$. The cyclic group of order n is denoted by n. An elementary abelian group of order p^n is denoted by E_{p^n} or just by p^n. By $[n]$ we denote a group of order n, of unspecified structure. For a prime p, p_+^{1+2n} or p_-^{1+2n} is used for the particular case of an extraspecial group. For each prime number p and positive n, there are just two types of extraspecial group, which are central products of n non-abelian groups of order p^3. For an odd prime p, the subscript is $+$ or $-$ according as the group has exponent p or p^2. For elementary abelian groups A we write A^{m+n} to mean a group with an elementary abelian normal subgroup A^m such that the quotient is isomorphic to A^n. For two groups A and B, a split extension (resp. a non-split extension) is denoted by $A : B$ (resp. $A.B$). Moreover, $A \times B$ denotes the direct product of A and B. (See [6])

- The subgroups of $\mathrm{L}_3(q)$ when q is odd and the maximal subgroups of $\mathrm{L}_3(q)$ when q is even, are as follows (see [7]):

(1) If q is odd:

Subgroup	Conditions	Subgroup	Conditions
$\mathrm{L}_3(q_0)$	q is a power of q_0	$[q^3(q+1)(q-1)^2/d]$	
$\mathrm{PGL}_3(q_0)$	q is a power of ${q_0}^3, 3 \mid q_0 - 1$	$[6(q-1)^2/d]$	
$\mathrm{PSU}_3({q_0}^2)$	q is a power of ${q_0}^2$	$[3(q^2+q+1)/d]$	
$\mathrm{PU}_3({q_0}^2)$	q is a power of ${q_0}^6, 3 \mid q_0 + 1$	$[q(q^2-1)]$	
$[720], [2520]$	q is an even power of 5	$[216]$	$9 \mid q-1$
$[168]$	-7 is square in $GF(q)$	$[36], [72]$	$3 \mid q-1$
$[360]$	5 is square in $GF(q)$, there is a nontrivial cube root of unitary		

(2) If q is even:

Subgroup	Conditions	Subgroup	Conditions
$\mathrm{L}_3(q_0)$	q is a power of q_0	$[q^3(q+1)(q-1)^2/d]$	
$\mathrm{PGL}_3(q_0)$	$q = {q_0}^3, q_0$ is square	$[6(q-1)^2/d]$	
$\mathrm{PSU}_3({q_0}^2)$	q is square	$[360]$	$q = 4$
$\mathrm{PU}_3({q_0}^2)$	$q = {q_0}^6, q$ is not square		

Using the information above, the compact form of $\Gamma_{\rm s}(\mathrm{L}_3(q))$ is found in Figures $1-7$. Note that in Figures $1-3$, $q=p^k$ where $p\neq 2,3$.

Lemma 2.10. *Let G be a simple group with $|\mathrm{Aut}(G):G|=2$. Then we have:*

$$\Gamma_{\rm s}(\mathrm{Aut}(G))-\{2\}=\Gamma_{\rm s}(G)-\{2\}.$$

In particular, if $r\in\pi(G)-\{2\}$, then $d_{\mathrm{s}_G}(r)\leqslant d_{\mathrm{s}_{\mathrm{Aut}(G)}}(r)\leqslant d_{\mathrm{s}_G}(r)+1$, and moreover; if 2 is a complete prime in $\Gamma_{\rm s}(G)$, then $d_{\mathrm{s}_{\mathrm{Aut}(G)}}(r)=d_{\mathrm{s}_G}(r)$.

Proof. We first claim that every subgroup of $\mathrm{Aut}(G)$ of odd order is a subgroup of G. Suppose that H is a subgroup of $\mathrm{Aut}(G)$ of odd order. Since $|H:H\cap G|=|HG:G|$ which divides 2, we have $HG=G$. Hence $H\leqslant G$.

Note that $\pi(\mathrm{Aut}(G))=\pi(G)$. In what follows, we will show that, if p and q are two odd primes such that $p\approx q$ in $\Gamma_{\rm s}(\mathrm{Aut}(G))$, then $p\approx q$ in $\Gamma_{\rm s}(G)$. Assume that $p\approx q$ in $\Gamma_{\rm s}(\mathrm{Aut}(G))$. Hence, there is a solvable subgroup $L\leqslant \mathrm{Aut}(G)$ such that $pq\mid |L|$. We consider $\{p,q\}$-Hall subgroup H of L. Now from the previous paragraph of the proof, H is a subgroup of G and so $p\approx q$ in $\Gamma_{\rm s}(G)$. □

Remark 2.11. *It is easy to see that in general, if G is a finite group and $G.2$ is an extension, then $\Gamma_{\rm s}(G.2)-\{2\}=\Gamma_{\rm s}(G)-\{2\}$. An example is provided by $G=\mathrm{L}_2(16)$. We can see from* [6] *that:*

- *the maximal subgroups of G are as follows: $2^4:15, \mathbb{A}_5$, the dihedral groups D_{30} and D_{34};*
- *the maximal subgroups of $G.2$ are as follows: $2^4:(3\times \mathrm{D}_{10})$, $\mathbb{A}_5\times 2$, $17:4$ and $\mathrm{D}_{10}\times\mathbb{S}_3$.*

It is seen that $\Gamma_{\rm s}(G.2)=\Gamma_{\rm s}(G)$ is as follow:

$$7\approx 2\approx 3\approx 5\approx 2.$$

It implies that

$$\Gamma_{\rm s}(G.2)-\{2\}=\Gamma_{\rm s}(G)-\{2\}.$$

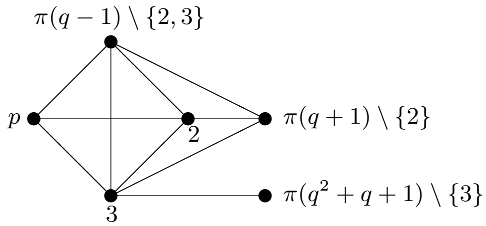

Fig. 1. $\Gamma_s(L_3(q)),\ 9 \mid q-1$.

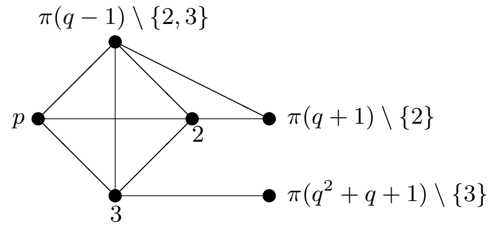

Fig. 2. $\Gamma_s(L_3(q)),\ 3 \mid q-1 (9 \nmid q-1)$.

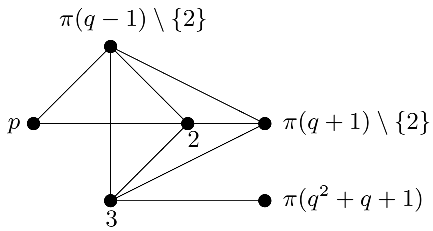

Fig. 3. $\Gamma_s(L_3(q)),\ 3 \mid q+1$.

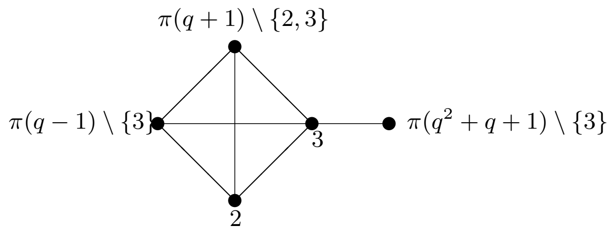

Fig. 4. $\Gamma_s(L_3(q)),\ q = 2^k,\ 9 \mid q-1$.

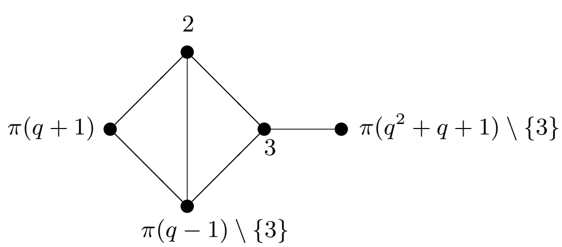

Fig. 5. $\Gamma_s(L_3(q)),\ q = 2^k,\ 3 \mid q-1 (9 \nmid q-1)$.

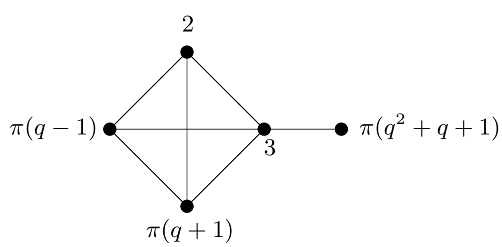

Fig. 6. $\Gamma_s(L_3(q)),\ q = 2^k,\ 3 \mid q+1$.

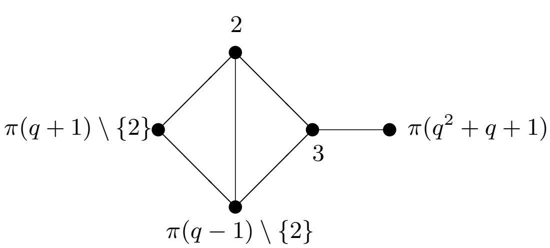

Fig. 7. $\Gamma_s(L_3(q)),\ q = 3^k$.

3. Local subgroups and their solvable graphs

A local subgroup of a group G is a subgroup H of G if there is a nontrivial solvable subgroup K of G such that $H = N_G(K)$. A subgroup of a finite group is *p-local* if it is the normalizer of some nontrivial p-subgroup. It is good to note that we denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G.

Let $P \in \mathrm{Syl}_p(G)$ for a prime $p \in \pi(G)$ and $N_G(P)$ be a p-local subgroup. If $x \in N_G(P)$ is an element, then $\langle x\rangle P$ is a solvable subgroup of G. Therefore, $d_{\mathrm{s}}(p) \geqslant |\pi(N_G(P))| - 1$. Moreover, the spanning subgraph of the solvable graph of $N_G(P)$ is a star graph with central vertex p.

In general, it is easy to see that if $N_G(H)$ is a local subgroup of G for some solvable subgroup H of G, then for all prime $p \in \pi(H)$, $d_{\mathrm{s}}(p) \geqslant |\pi(N_G(H))| - 1$.

Let G be a finite group which is not a non-abelian simple group. Then G has a normal nontrivial subgroup K_1. Suppose that $p_1 \in \pi(K_1)$. If $P_1 \in \mathrm{Syl}_{p_1}(K_1)$, then we obtain from Frattini's argument that $G = N_G(P_1)K_1$. Again, if K_1 is not a simple group, then by a similar way we have $K_1 = N_{K_1}(P_2)K_2$ where K_2 is a nontrivial normal subgroup of K_1 and $P_2 \in \mathrm{Syl}_{p_2}(K_2)$. By continuing this way, we get

$$G = N_G(P_1)N_{K_1}(P_2)N_{K_2}(P_3)\dots N_{K_{n-1}}(P_k)K_n,$$

where $K_n \lhd K_{n-1} \lhd \cdots \lhd K_1 \lhd G$ and K_n is a simple group. Furthermore, it is easily seen that for every two subgroups H and K of G such that $H \leqslant K$, $N_K(H) \leqslant N_G(H)$. Therefore, we can study the p-local subgroups of G and their influence on the structure of the solvable graph of G.

We can ask this question that if G is a non-solvable group which is not a non-abelian simple group and all of whose local subgroups are solvable, is the solvable graph of G complete? Actually, the answer is no. To explain it, we should mention that the structures of these groups are completely classified.

An *N-group* is a group that all of whose local subgroups are solvable groups. It is clear that every solvable group is an N-group. The simple N-groups were classified by John Thompson in series of papers. In fact, the simple N-groups are as follows: $\mathrm{L}_2(q)$ ($q = p^f$ where p is prime), $\mathrm{L}_3(3)$, ${}^2\mathrm{B}_2(2^{2m+1})$, $\mathrm{U}_3(3)$, $\mathbb{A}_7$, M_{11}, ${}^2\mathrm{F}_4(2)'$. More generally, Thompson showed that any non-solvable N-group is a subgroup of $\mathrm{Aut}(G)$ where G is a simple N-group.

Now, we consider the group $G = \mathrm{Aut}(\mathrm{U}_3(3))$. In fact, $G = \mathrm{U}_3(3).2$ and so by the structures of the maximal subgroups of G in [6], we can see that the solvable graph of G is as follows: $2 \approx 3 \approx 7$. Hence, $\Gamma_{\mathrm{s}}(G)$ is not complete.

4. $\mathrm{OD_s}$-characterization of some projective special linear groups

As mentioned before, it was shown in [3] that all sporadic groups and the projective special linear groups $\mathrm{L}_2(q)$ with certain properties are $\mathrm{OD_s}$-characterizable. Moreover, the following Lemma was proved.

Lemma 4.1. *Suppose that H is a finite group and $|\pi(H)| = k \geqslant 3$. If $\Delta_{k-1}(H) = \emptyset$ and*

$$H \notin \{\mathrm{B}_n(q), \mathrm{C}_n(q) : n \geqslant 3 \text{ and } q \text{ is odd}\},$$

then H is $\mathrm{OD_s}$-characterizable.

In this section, we are going to examine $\mathrm{OD_s}$-characterizability of projective special linear groups $\mathrm{L}_3(q)$. Considering the Figures $1-7$, we can find from Lemma 4.1 that if either q is odd and $9 \nmid q-1$, or q is even and $3 \parallel q-1$, these groups are $\mathrm{OD_s}$-characterizable. So we only examine other cases whose solvable graphs are shown in Figures 1, 4 and 6.

In [1], the author introduced a new terminology. Let m be a positive integer with the following factorization into distinct prime power factors $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for some positive integers e_i and k. Then we put

$$\mathrm{mpf}(m) := \max\{p_i^{e_i} \mid 1 \leqslant i \leqslant k\}.$$

In [3], $\mathrm{mpf}(|S|)$ for sporadic simple groups and all simple groups of Lie type S were completely listed. For convenience, we tabulate $|S|$ and $\mathrm{mpf}(|S|)$ for sporadic simple groups and all simple groups of Lie type S in Tables 1 and 2.

Table 1. *The order and mpf of a sporadic simple group.*

S	$\|S\|$	mpf($\|S\|$)	S	$\|S\|$	mpf($\|S\|$)
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2^7	Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	2^{18}
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	2^4	Fi_{23}	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	3^{13}
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2^6	Co_1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$	2^{21}
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2^7	Ru	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$	2^{14}
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2^9	Fi'_{24}	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot$	3^{16}
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	3^6		$23 \cdot 29$	
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2^{13}	$O'N$	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$	2^9
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	2^{17}	Th	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$	3^{10}
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2^{10}	J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot$	2^{21}
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	19		$31 \cdot 37 \cdot 43$	
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	3^5	B	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot$	2^{41}
HN	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$	2^{14}		$19 \cdot 23 \cdot 31 \cdot 47$	
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	2^7	Ly	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$	5^6
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	2^{10}	M	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot$	2^{46}
Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	3^7		$19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	

Table 2. *The order and* mpf *of a simple group of Lie type.*

S	Restrictions on S	$\lvert S\rvert$	$\mathrm{mpf}(\lvert S\rvert)$
$\mathrm{L}_{n+1}(q)$	$n \geqslant 2$	$(n+1, q-1)^{-1}q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-1)$	$q^{n(n+1)/2}$
$\mathrm{L}_2(q)$	$\lvert\pi(q+1)\rvert = 1$	$(2, q-1)^{-1}q(q-1)(q+1)$	$q+1$
$\mathrm{L}_2(q)$	$\lvert\pi(q+1)\rvert \geqslant 2$	$(2, q-1)^{-1}q(q-1)(q+1)$	q
$\mathrm{B}_n(q)$	$n \geqslant 2$	$(2, q-1)^{-1}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	q^{n^2}
$\mathrm{C}_n(q)$	$n \geqslant 3$	$(2, q-1)^{-1}q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$	q^{n^2}
$\mathrm{D}_n(q)$	$n \geqslant 4$	$(4, q^n-1)^{-1}q^{n(n-1)}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$q^{n(n-1)}$
$\mathrm{G}_2(q)$		$q^6(q^6-1)(q^2-1)$	q^6
$\mathrm{F}_4(q)$		$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	q^{24}
$\mathrm{E}_6(q)$		$(3, q-1)^{-1}q^{12}(q^9-1)(q^5-1)\lvert\mathrm{F}_4(q)\rvert$	q^{36}
$\mathrm{E}_7(q)$		$(2, q-1)^{-1}q^{39}(q^{18}-1)(q^{14}-1)(q^{10}-1)\lvert\mathrm{F}_4(q)\rvert$	q^{63}
$\mathrm{E}_8(q)$		$q^{96}(q^{30}-1)(q^{12}+1)(q^{20}-1)(q^{18}-1)(q^{14}-1)$	q^{120}
		$(q^6+1)\lvert\mathrm{F}_4(q)\rvert$	
$\mathrm{U}_{n+1}(q)$	$(n, q) \neq (2, 3), (3, 2)$	$(n+1, q+1)^{-1}q^{n(n+1)/2}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$q^{n(n+1)/2}$
	$n \geqslant 2$		
$\mathrm{U}_4(2)$		$2^6 \cdot 3^4 \cdot 5$	3^4
$\mathrm{U}_3(3)$		$2^5 \cdot 3^3 \cdot 7$	2^5
${}^2\mathrm{B}_2(q)$	$q = 2^{2m+1}$	$q^2(q^2+1)(q-1)$	q^2
	$\lvert\pi(q^2+1)\rvert \geqslant 2$		
${}^2\mathrm{B}_2(q)$	$q = 2^{2m+1}$	$q^2(q^2+1)(q-1)$	q^2+1
	$\lvert\pi(q^2+1)\rvert = 1$		
${}^2\mathrm{D}_n(q)$	$n \geqslant 4$	$(4, q^n+1)^{-1}q^{n(n-1)}(q^n+1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$q^{n(n-1)}$
${}^3\mathrm{D}_4(q)$		$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	q^{12}
${}^2\mathrm{G}_2(q)$	$q = 3^{2m+1}$	$q^3(q^3+1)(q-1)$	q^3
${}^2\mathrm{F}_4(q)$	$q = 2^{2m+1}$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	q^{12}
${}^2\mathrm{E}_6(q)$		$(3, q+1)^{-1}q^{12}(q^9+1)(q^5+1)\lvert\mathrm{F}_4(q)\rvert$	q^{36}

Theorem 4.2. *Let G be a finite group satisfying $|G| = |\mathrm{L}_3(q)|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(\mathrm{L}_3(q))$, where $q = p^f$ is odd. In addition, assume $9 \mid q-1$ and $|\pi(\frac{q^2+q+1}{3})| = 1$. Then $G \cong \mathrm{L}_3(q)$.*

Proof. The solvable graph of $\mathrm{L}_3(q)$ with given conditions are shown in Figures $1-7$.

By the hypothesis

$$|G| = |\mathrm{L}_3(q)| = \frac{1}{3}q^3(q^2-1)(q^3-1).$$

Moreover, $\mathrm{D_s}(G) = \mathrm{D_s}(\mathrm{L}_3(q))$ which implies from Figure 1 that

- $d_s(3) = |\pi(G)| - 1$,
- $d_s(t) = |\pi(q^2+q+1)| - 1$ for every prime $t \in \pi(\frac{q^2+q+1}{3})$.

Since $|\pi(\frac{q^2+q+1}{3})| = 1$ and $9 \nmid q^2+q+1$, so there exists a prime p' such that $\pi(\frac{q^2+q+1}{3}) = \{p'\}$. We can see from Lemma 2.8 that $\tilde{\Gamma}(G) = (\Gamma_s(G) - \{3\})^c$ is connected and any disjoint pair of vertices of $\Gamma_s(G)$ can be expressed by only one GKS-series, say $1 \trianglelefteq M \lhd N \trianglelefteq G$, such that M and G/N are 3-groups. Furthermore, using the structure of the degree pattern of the solvable graph of G, we can get that 3 is adjacent to p'. It is also seen from Lemma 2.6 that $p' \in \pi(N/M)$. Let $|M| = 3^m$ and $|G/N| = 3^n$. Thus we can conclude that

$$|N/M| = 3^{-m-n-1}q^3(q^2-1)(q^3-1).$$

On the other hand, N/M is a non-abelian simple group and so according to the classification of finite simple groups, the possibilities for N/M are: an alternating group $\mathbb{A}_l$ on $l \geqslant 5$ letters, one of the 26 sporadic simple groups, and a simple group of Lie type. If $N/M \cong \mathrm{L}_3(q)$, then $M = 1$, $N = G$ and thus $G \cong \mathrm{L}_3(q)$, as required. Therefore, we assume that N/M is isomorphic to the non-abelian simple group $S \ncong \mathrm{L}_3(q)$ and we will try to get a contradiction. To this aim, we use the following facts.

First, the solvable graph of N/M is a subgraph of the solvable graph of G. It yields that $d_s(p') \leqslant 1$ in the solvable graph of N/M. Second,

$$\mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|).$$

Therefore, we need to compute the value $\mathrm{mpf}(3^{-m-n-1}q^3(q^2-1)(q^3-1))$.

It is easily seen that

$$q-1 < q^2-1 < q^2+q+1 < q^3,$$

because $q > 3$. So we can conclude that

$$\mathrm{mpf}(3^{-m-n-1}q^3(q^2-1)(q^3-1)) = \mathrm{mpf}(3^{-m-n-1}q^3(q-1)(q^2-1)(q^2+q+1)) = q^3.$$

Hence, $\mathrm{mpf}(|S|) = q^3$.

(1) S is not isomorphic to an alternating group $\mathbb{A}_l$, $l \geqslant 5$.

Suppose that S is isomorphic to an alternating group $\mathbb{A}_l$, $l \geqslant 5$. Thus p' divides $|\mathbb{A}_l|$ which yields that $p' \leqslant l$. If $l \geqslant p'+4$, then we can obtain from [10] that $d(p') \geqslant 2$ and hence $d_s(p') \geqslant 2$ that is impossible. So we may assume that $p' \leqslant l \leqslant p'+3$.

It is good to mention that $\pi(q^2+q+1) = \{3, p'\}$ and $3 \parallel q^2+q+1$. Then according to the order of $\mathbb{A}_l$, we deduce that $q^2+q+1 = 3p'$. On the other hand, we have

$$\frac{l!}{2} = |\mathbb{A}_l| = |S| = |N/M| = 3^{-m-n-1}q^3(q-1)(q^2-1)(q^2+q+1),$$

which follows that $p'-1 = (q^2+q-2)/3$ divides $3^{-m-n-1}q^3(q-1)(q^2-1)$, a contradiction.

(2) S is not isomorphic to one of the 26 sporadic simple groups.

Suppose that S is isomorphic to one of the 26 sporadic simple groups. As mentioned above, $\mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|)$. It implies that $\mathrm{mpf}(|S|) = q^3$. Hence, we obtain from Table 1 that S is one of the following groups:

$$\mathrm{M}_{12}, \mathrm{HS}, \mathrm{M^cL}, \mathrm{Co}_2, \mathrm{Co}_1, \mathrm{O'N}, \mathrm{J}_4, \mathrm{Ly}, \mathrm{M}.$$

On the other hand, $p \neq 2$ which forces that $S \cong Ly$. So we can conclude that $q = 25$. It follows that $q^2+q+1 = 651$ which is a contradiction.

(3) S is not isomorphic to a simple group of Lie type, except $\mathrm{L}_3(q)$.

We only examine the cases when S is isomorphic to the groups $\mathrm{L}_{n+1}(q_0)$, $\mathrm{C}_n(q_0)$, $\mathrm{D}_n(q_0)$, ${}^2\mathrm{E}_6(q_0)$. We omit other cases because they are similar.

- Let S be isomorphic to $\mathrm{L}_{n+1}(q_0)$ for some integer $n \neq 2$ and a power q_0 of a prime p_0. If $n \geqslant 4$, then considering the spectrum of $\mathrm{L}_{n+1}(q_0)$ in [5], we can find that $d_s(p') \geqslant 2$ in the solvable graph of S that is impossible. Assume now that $n = 3$. By Table 2, it is seen that

$$\mathrm{mpf}(|\mathrm{L}_4(q_0)|) = q_0^6,$$

and so

$$q_0^6 = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = q^3.$$

Hence, we have $q = q_0^2$. On the other hand,

$$|S| = |\mathrm{L}_4(q_0)| = (4, q_0-1)^{-1}q_0^6(q_0^2-1)(q_0^3-1)(q_0^4-1).$$

It follows that

$$(4, q_0-1)^{-1}(q_0-1) = 3^{-m-n-1}(q_0^2-q_0+1),$$

which is a contradiction. Therefore, we may suppose that $n = 1$. It is seen from Table 2 that $\mathrm{mpf}(|\mathrm{L}_2(q_0)|) = q_0$ or $q_0 + 1$. If $\mathrm{mpf}(|\mathrm{L}_2(q_0)|) = q_0 + 1$, then by an easy computation it is found that

$$2(q^2 - 1) = 3^{m+n+1}(q^3 - 2),$$

a contradiction. In the case when $\mathrm{mpf}(|\mathrm{L}_2(q_0)|) = q_0$ by a similar way, we can get a contradiction.

- Assume that S is isomorphic to $\mathrm{C}_n(q_0)$. Then we observe that

$$q_0^{n^2} = \mathrm{mpf}(|\mathrm{C}_n(q_0)|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = q^3.$$

Note that

$$|\mathrm{C}_n(q_0)| = \frac{1}{2} q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1).$$

Let r be a prime dividing the order of S. Then by Proposition 2.4 in [9], and Propositions 3.1 and 4.3 in [8], we can easily find that in the case when $\eta(e(r, q_0)) \leqslant n - 1$, r is adjacent to at least two primes in the prime graph of G which follows that $d_{\mathrm{s}}(r) \geqslant 2$. Hence, if $d_{\mathrm{s}}(r) \leqslant 1$, then r is a primitive prime of $q_0^n - 1$ or $q_0^{2n} - 1$. Thus we have $q^2 + q + 1 = r^m$ for a natural number m. It yields that $q^2 + q + 1$ divides $q_0^n - 1$ or $q_0^n + 1$. Now using the fact that $q^3 = q_0^{n^2}$, we can obtain a contradiction.

- Suppose that S is isomorphic to $\mathrm{D}_n(q_0)$. Then we have

$$q_0^{n(n-1)} = \mathrm{mpf}(|\mathrm{D}_n(q_0)|) = \mathrm{mpf}(|S|) = \mathrm{mpf}(|N/M|) = q^3.$$

Note that

$$|\mathrm{D}_n(q_0)| = (4, q_0^n - 1)^{-1} q^{n(n-1)} (q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1).$$

Let r be a prime dividing the order of S. According to Proposition 2.5 in [9], and Propositions 4.3, 3.1 and 4.4 in [8], it is seen that in the case when $\eta(e(r, q_0)) \leqslant n - 1$, r is adjacent to at least two primes in the prime graph of G which implies that $d_{\mathrm{s}}(r) \geqslant 2$. It follows that if $d_{\mathrm{s}}(r) \leqslant 1$, then r is probably a primitive prime of $q_0^n - 1$, $q_0^{n-1} - 1$ or $q_0^{2(n-1)} - 1$. Then we have $q^2 + q + 1 = r^m$ for a natural number m. We can conclude that $q^2 + q + 1$ divides $q_0^n - 1$, $q_0^{n-1} - 1$ or $q_0^{n-1} + 1$. Now using the fact that $q^3 = q_0^{n(n-1)}$, we get a contradiction.

- Let S be isomorphic to ${}^2\mathrm{E}_6(q_0)$. We can see from Table 2 that $\mathrm{mpf}(|{}^2\mathrm{E}_6(q_0)|) = q_0^{36}$. It follows that $q = q_0^{12}$. On the other hand,

$$|S| = |{}^2\mathrm{E}_6(q_0)| = q_0^{36}(q_0^{12} - 1)(q_0^9 + 1)(q_0^8 - 1)(q_0^6 - 1)(q_0^5 + 1)(q_0^2 - 1).$$

So we deduce that $q_0^{36} = q^3$. Then we have

$$|N/M| = 3^{-m-n-1}q^3(q^2-1)(q^3-1) = 3^{-m-n-1}q_0^{36}(q_0^{24}-1)(q_0^{36}-1),$$

and it follows that a primitive prime of $q_0^{24}-1$ belongs to $\pi({}^2\mathrm{E}_6(q_0))$, a contradiction. □

Now by a similar way to the proof of Theorem 4.2, we can prove the following Theorem.

Theorem 4.3. *Let G be a finite group satisfying $|G| = |\mathrm{L}_3(2^f)|$ and $\mathrm{D_s}(G) = \mathrm{D_s}(\mathrm{L}_3(2^f))$. If one of the following conditions holds, then $G \cong \mathrm{L}_3(2^f)$.*

(1) $9 \mid 2^f - 1$ *and* $|\pi(\frac{2^{2f}+2^f+1}{3})| = 1$;

(2) $3 \mid 2^f + 1$ *and* $|\pi(2^{2f}+2^f+1)| = 1$.

Finally, considering Theorems 4.2 and 4.3, we state the following Corollary.

Corollary 4.4. *The simple groups $\mathrm{L}_3(q)$ with the following conditions are $\mathrm{OD_s}$-characterizable:*

(1) *q is odd and* $9 \nmid q-1$;

(2) *q is even and* $3 \parallel q-1$;

(3) $9 \mid q-1$ *and* $|\pi(\frac{q^2+q+1}{3})| = 1$;

(4) *q is even,* $3 \mid q+1$ *and* $|\pi(q^2+q+1)| = 1$.

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