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RESEARCH ARTICLE

Invariant Holomorphic Statistical Submersions

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ABSTRACT

The notion of a statistical submersion is due to Abe and Hasegawa. In particular, one of the present authors defined holomorphic statistical submersions. In a joint paper with Kazan, he studied anti-invariant holomorphic statistical submersions. In the present paper, we investigate invariant statistical submersions and give their geometric properties. Two examples of such submersions are provided.

Keywords: *Statistical manifold, holomorphic statistical manifold, statistical submersion, holomorphic statistical submersion*

AMS Subject Classification (2020): *Primary: 53B05 ; Secondary: 53B12; 53C15.*

1. Introduction

In 1980s, the notion of statistical structure was introduced and began to play an important role to build a very effective branch called information geometry, which is a combination of differential geometry and statistics. In fact, information geometry joins fundamental differential geometry tools like connection, metric and curvature to statistical models, and makes it possible to describe statistical objects as geometric ones. Describing statistical spaces using geometry helps us better understand statistical behaviors. A detailed introduction to information geometry can be found in [3].

In 1985, Amari [2] studied statistical manifolds from the point of view of information geometry and he gave a new description of the statistical distributions by using the obtained geometric structures. In Amari's definition a statistical manifold is a triple (M, g, ∇) consisting of a smooth manifold M, a non-degenerate metric g on it and a torsion-free connection ∇ with the property that ∇g is symmetric [2].

Its applications could be found in various fields of science such as in statistical inference, time series, linear systems, quantum systems, image processing, physics, computer science and machine learning. In pure differential geometry, there are several papers studying statistical submanifolds of (special) statistical manifolds, as Kähler-like or Sasaki-like statistical manifolds. For example, in [4] Chen inequalities for statistical submanifolds of Kähler-like statistical manifolds were proven. In [5] the authors obtained relations between extrinsic and intrinsic invariants of statistical submanifolds in Sasaki-like statistical manifolds.

The notion of a Riemann submersion was introduced by O'Neill (1966) and Gray (1967), and since then, there have been many studies such as the case where the total space has geometric structures such as almost complex structures or almost contact structures. In addition, Abe and Hasegawa (2001) defined the statistical submersions. One of the present authors studied geometric structures similar to almost complex structures (2004) and almost contact structures (2006) in statistical manifolds, and considered statistical submersion when the total space allows geometric structures similar to almost complex structures and almost contact structures.

A statistical submersion whose total space is a holomorphic statistical manifold is said to be a holomorphic statistical submersion. Anti-invariant holomorphic statistical submersion were studied by Kazan and Takano,

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one of the present authors in [10]. Other types of statistical submersions were studied, see for example results on cosymplectic-like statistical submersions in [6].

In [18], the first author et al. dealt with the study of para-holomorphic statistical submersions, which are different in nature from holomorphic statistical submersions. Also, Vîlcu [20] studied para-Kähler-like statistical submersions. One points-out that a para-Kähler manifold is an almost product manifold, while a Kähler manifold is a complex manifold.

In this paper, we investigate invariant holomorphic statistical submersions. Two examples of invariant holomorphic statistical submersions are provided. One remarks that it is not easy to construct examples that meet the conditions. This time, we reduced the amount of calculation by making the metric a diagonal matrix, and based on the definitions, we decided the coefficients of affine connection. If the even dimension of statistical models (for example, the multinomial distribution, the negative multinomial distribution, the multivariate normal distribution, etc), then we can construct structures that are similar to Kählerian structures. These are the motivation for research on manifolds with almost complex structures.

2. Holomorphic statistical manifolds

An m-dimensional semi-Riemannian manifold is a smooth manifold M furnished with a metric tensor g, where g is a symmetric nondegenerate tensor field on M of constant index. The common value ν of the index of g on M is called the index of M $(0 \leq \nu \leq m)$ and we denote a semi-Riemannian manifold by M^m_ν. If $\nu = 0$, then M is a Riemannian manifold. For each $p \in M^m_\nu$, a tangent vector E to M^m_ν is spacelike (resp. null, timelike) if $g(E,E) > 0$ or $E = 0$ (resp. $g(E,E) = 0$ and $E \neq 0$, $g(E,E) < 0$).

Let $\mathbb{R}^m_\nu$ be an m-dimensional real vector space with an inner product of signature $(\nu, m-\nu)$ given by

$$\langle x, x\rangle = -\sum_{i=1}^{\nu} x_i^2 + \sum_{i=\nu+1}^{m} x_i^2, \tag{2.1}$$

where $x = (x_1, \ldots, x_m)$ is the natural coordinate of $\mathbb{R}^m_\nu$. $\mathbb{R}^m_\nu$ is called an m-dimensional semi-Euclidean space. If $\nu = 0$ (resp. $\nu = 1$), then $\mathbb{R}^m$ (resp. $\mathbb{R}^m_1$) is a Euclidean space (resp. a Lorentzian space).

Let M be a semi-Riemannian manifold and ∇ a torsion-free affine connection on M. The triple (M, g, ∇) is called a statistical manifold if ∇g is symmetric. For the statistical manifold (M, g, ∇), one defines another affine connection ∇^* by

$$Eg(F, G) = g(\nabla_E F, G) + g(F, \nabla^*_E G), \tag{2.2}$$

for vector fields E, F and G on M. The affine connection ∇^* is called conjugate (or dual) to ∇ with respect to g. The affine connection ∇^* is torsion-free, $\nabla^* g$ is symmetric and satisfies $(\nabla^*)^* = \nabla$. Clearly, the triple (M, g, ∇^*) is a statistical manifold. It is easy to see that $\nabla^0 = \frac{1}{2}(\nabla + \nabla^*)$ holds, where ∇^0 is the Levi-Civita connection. We denote by R and R^* the curvature tensors on M with respect to the affine connection ∇ and its conjugate ∇^*, respectively. Then one has

$$g(R(E, F)G, H) = -g(G, R^*(E, F)H), \tag{2.3}$$

for any vector fields E, F, G and H on M, where $R(E, F)G = [\nabla_E, \nabla_F]G - \nabla_{[E,F]}G$. Therefore R vanishes identically if and only if R^* vanishes identically. M is called flat if R vanishes identically. If the curvature tensor R with respect to the affine connection ∇ satisfies

$$R(E, F)G = c\,\{\, g(F, G)E - g(E, G)F \,\},$$

then the statistical manifold (M, g, ∇) is said to be a space of constant curvature c. The triple (M, g, ∇) is of constant curvature c if and only if (M, g, ∇^*) is of constant curvature c.

Let M be a smooth manifold with a tensor field J of type $(1, 1)$ on M such that

$$J^2 = -I, \tag{2.4}$$

where I stands for the identity transformation. Then M is called an almost complex manifold with almost complex structure J. An almost complex manifold is a manifold with a fixed almost complex structure. An almost complex manifold is necessarily orientable and must have an even dimension. If J preserves the metric g, i.e.,

$$g(JE, JF) = g(E, F), \tag{2.5}$$

then (M, g, J) is an almost Hermitian manifold. Moreover, if J is parallel with respect to the Levi-Civita connection ∇', namely, $(\nabla'_E J)F = 0$, then (M, g, J) is called a Kählerian manifold. For each plane π in the tangent space T_pM, the sectional curvature $K(\pi)$ is defined to be $K(\pi) = g(R(E, F)F, E)$, where $\{E, F\}$ is an orthogonal basis of π. If π is invariant by J, then $K(\pi)$ is called a holomorphic sectional curvature. The holomorphic sectional curvature $K(\pi)$ is given by

$$K(\pi) = g(R(E, JE)JE, E),$$

where E is a unit vector in π. If $K(\pi)$ is a constant for all J-invariant planes π in T_pM and for all points $p \in M$, then M is called a space of constant holomorphic sectional curvature or complex space form. It is known that

Theorem A [21] *A Kählerian manifold M is of constant holomorphic sectional curvature c if and only if*

$$R(E, F)G = \frac{c}{4}\{\, g(F, G)E - g(E, G)F + g(JF, G)JE - g(JE, G)JF + 2g(E, JF)JG \,\},$$

for any vector fields E, F, G on M.

On a statistical manifold (M, ∇, g), if the curvature tensor R with respect to the affine connection ∇ satisfies

$$R(E, F)G = \frac{c}{4}\{\, g(F, G)E - g(E, G)F + g(JF, G)JE - g(JE, G)JF + 2g(E, JF)JG \,\}, \tag{2.6}$$

then (M, ∇, g, J) is called a space of constant holomorphic sectional curvature c [8]. Then we have

Lemma 2.1. *(M, ∇, g, J) is of constant holomorphic sectional curvature c if and only if (M, ∇^*, g, J) is of constant holomorphic sectional curvature c.*

Remark 2.1. If (M, ∇, g, J) is of constant holomorphic sectional curvature c, then

(1) $R(E, F)JG = JR^*(E, F)G$ holds.

(2) (M, ∇, g, J) is Einstein.

Let (M, g, J) be a Kählerian manifold and ∇ an affine connection of M. We put $\omega(F, G) = g(F, JG)$ and $(\nabla_E \omega)(F, G) = E\omega(F, G) - \omega(\nabla_E F, G) - \omega(F, \nabla_E G)$. If (∇, g) is a statistical structure and the 2-form ω is parallel with respect to ∇ on M, then (M, ∇, g, J) is called a holomorphic statistical manifold (see [8]).

It is known that

Lemma B [8] *On a holomorpic statistical manifold (M, ∇, g, J) we have*

$$\nabla_E(JF) = J(\nabla^*_E F),$$

$$R(E, F)JG = JR^*(E, F)G.$$

Lemma 2.2. *The following conditions are equivalent:*

(1) (M, ∇, g, J) is a holomorphic statistical manifold.

(2) $\nabla^*_E(JF) = J(\nabla_E F)$ holds.

(3) $\nabla_E(JF) = J(\nabla^*_E F)$ holds.

(4) (M, ∇^*, g, J) is a holomorphic statistical manifold.

Proof. We have $(\nabla_E \omega)(G, F) = g(G, \nabla^*_E(JF) - J(\nabla_E F))$. Thus ω is parallel with respect to ∇ if and only if $\nabla^*_E(JF) = J(\nabla_E F)$ holds. It is easy to see from $(\nabla^*)^* = \nabla$ that the condition (2) is equivalent to the condition (3). $\square$

If M is of constant curvature c, then we get from Lemma B

$$c\{g(F, JG)E - g(E, JG)F\} = c\{g(F, G)JE - g(E, G)JF\},$$

which yields that $(m - 2)cg(F, JG) = 0$. Hence we have

Theorem 2.1. *Let (M^m, ∇, g, J) be a holomorphic statistical manifold. If M^m $(m \geq 4)$ is of constant curvature, then M is flat.*

We give an example of a holomorphic statistical manifold.

Example 2.1. Let M^4 be a smooth manifold with local coordinate system (x_1, x_2, x_3, x_4), which admits the following almost complex structure J:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The triple (M, g, J) is an almost Hermitian manifold with

$$g = \begin{pmatrix} e^{x_1} & 0 & 0 & 0 \\ 0 & \varepsilon e^{x_2} & 0 & 0 \\ 0 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & \varepsilon e^{x_2} \end{pmatrix},$$

where $\varepsilon = 1$ or $\varepsilon = -1$. Moreover, (M, g, J) is a Kählerian manifold. If $\varepsilon = 1$ (resp. $\varepsilon = -1$), then $M = \mathbb{R}^4$ (resp. $M = \mathbb{R}^4_2$). We put

$$\begin{aligned}
&\nabla_{\partial_1}\partial_1 = -\nabla_{\partial_3}\partial_3 = \varepsilon e^{-x_2}\partial_2,\\
&\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = -\nabla_{\partial_3}\partial_4 = -\nabla_{\partial_4}\partial_3 = e^{-x_1}\partial_1,\\
&\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = \partial_3 - \varepsilon e^{-x_2}\partial_4,\\
&\nabla_{\partial_1}\partial_4 = \nabla_{\partial_4}\partial_1 = \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = -e^{-x_1}\partial_3,\\
&\nabla_{\partial_2}\partial_2 = -\nabla_{\partial_4}\partial_4 = 0,\\
&\nabla_{\partial_2}\partial_4 = \nabla_{\partial_4}\partial_2 = \partial_4,
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^*_{\partial_1}\partial_1 = -\nabla^*_{\partial_3}\partial_3 = \partial_1 - \varepsilon e^{-x_2}\partial_2,\\
&\nabla^*_{\partial_1}\partial_2 = \nabla^*_{\partial_2}\partial_1 = -\nabla^*_{\partial_3}\partial_4 = -\nabla^*_{\partial_4}\partial_3 = -e^{-x_1}\partial_1,\\
&\nabla^*_{\partial_1}\partial_3 = \nabla^*_{\partial_3}\partial_1 = \varepsilon e^{-x_2}\partial_4,\\
&\nabla^*_{\partial_1}\partial_4 = \nabla^*_{\partial_4}\partial_1 = \nabla^*_{\partial_2}\partial_3 = \nabla^*_{\partial_3}\partial_2 = e^{-x_1}\partial_3,\\
&\nabla^*_{\partial_2}\partial_2 = -\nabla^*_{\partial_4}\partial_4 = \partial_2,\\
&\nabla^*_{\partial_2}\partial_4 = \nabla^*_{\partial_4}\partial_2 = 0.
\end{aligned}$$

Then (M, ∇, g, J) and (M, ∇^*, g, J) are holomorphic statistical manifolds.

3. Statistical submersions

Let M and B be semi-Riemannian manifolds. A surjective mapping $\pi : M \to B$ is called a semi-Riemannian submersion if π has maximal rank and π_* preserves lengths of horizontal vectors. Let $\pi : M \to B$ be a semi-Riemannian submersion. We put $\dim M = m$ and $\dim B = n$. For each point $x \in B$, the semi-Riemannian submanifold $\pi^{-1}(x)$ with the induced metric $\overline{g}$, which is obtained by restricting the metric of the total space to $\pi^{-1}(x)$, is called a fiber and denoted by $\overline{M}_x$ or simply $\overline{M}$. We notice that the dimension of each fiber is always $m - n\,(= s)$. A vector field on M is vertical if it is always tangent to fibers and horizontal if always orthogonal to fibers. We denote the vertical and horizontal subspace in the tangent space T_pM of the total space M by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$ for each point $p \in M$, and the vertical and horizontal distributions in the tangent bundle TM of M by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then TM is the direct sum of $\mathcal{V}(M)$ and $\mathcal{H}(M)$. The projection mappings are denoted $\mathcal{V}: TM \to \mathcal{V}(M)$ and $\mathcal{H}: TM \to \mathcal{H}(M)$ respectively. We call a vector field X on M projectable if there exists a vector field X_* on B such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$, and say that X and X_* are π-related. Also, a vector field X on M is called basic if it is projectable and horizontal. Then we have ([7], [9], [14]):

Lemma C. *If X and Y are basic vector fields on M which are π-related to X_* and Y_* on B, then*
(1) $g(X,Y) = g_B(X_*,Y_*)\circ\pi$, *where g is the metric on M and g_B the metric on B,*
(2) $\mathcal{H}[X,Y]$ *is basic and is π-related to* $[X_*,Y_*]$,
(3) $\mathcal{H}\nabla^0_X Y$ *is basic and π-related to $\widehat{\nabla}^0_{X_*}Y_*$, where ∇^0 and $\widehat{\nabla}^0$ are the Levi-Civita connections of M and B, respectively.*

Let (M,g,∇) be a statistical manifold and $\pi: M \to B$ be a semi-Riemannian submersion. We denote the affine connections of $\overline{M}$ be $\overline{\nabla}$ and $\overline{\nabla}^*$. Notice that $\overline{\nabla}_U V$ and $\overline{\nabla}^*_U V$ are well-defined vertical vector fields on M for vertical vector fields U and V on M, more precisely $\overline{\nabla}_U V = \mathcal{V}\nabla_U V$ and $\overline{\nabla}^*_U V = \mathcal{V}\nabla^*_U V$. Moreover, $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion-free and conjugate to each other with respect to $\overline{g}$. The triple $(\overline{M},\overline{g},\overline{\nabla})$ is a statistical manifold and so is $(\overline{M},\overline{g},\overline{\nabla}^*)$.

We call $\pi: (M,g,\nabla) \to (B,g_B,\widehat{\nabla})$ a statistical submersion if $\pi: M \to B$ satisfies

$$\pi_*(\nabla_X Y)_p = (\widehat{\nabla}_{X_*}Y_*)_{\pi(p)},$$

for basic vector fields X, Y and $p \in M$. The tensor fields T and A of type (1,2) defined by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \qquad A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$

for any vector fields E and F on M. Changing ∇ to ∇^* in the above equations, we set T^* and A^*, respectively. Then we find $T^{**} = T$ and $A^{**} = A$. For vertical vector fields, T and T^* have the symmetry property. Similarly, the tensor fields for the Levi-Civita connection ∇^0 are denoted by T^0 and A^0. For $X,Y \in \mathcal{H}(M)$ and $U,V \in \mathcal{V}(M)$, we obtain

$$g(T_U V, X) = -g(V, T^*_U X), \qquad g(A_X Y, U) = -g(Y, A^*_X U). \tag{3.1}$$

Thus $T_U V$ (resp. $T_U X$) vanishes identically if and only if $T^*_U X$ (resp. $T^*_U V$) vanishes identically. If $T_U V$ (resp. $T^*_U V$) vanishes identically, then π is called with isometric fiber with respect to ∇ (resp. ∇^*). We put $S_E F = \nabla_E F - \nabla^*_E F$ for $E, F \in TM$. Then $S_E F = S_F E$ and $g(S_E F, G) = g(F, S_E G)$ hold. It is known that

Theorem D [1] *Let $\pi: M \to B$ be a semi-Riemannian submersion. Then (M,g,∇) is a statistical manifold if and only if the following conditions hold*:
(1) $\mathcal{H}S_V X = A_X V - A^*_X V$,
(2) $\mathcal{V}S_X V = T_V X - T^*_V X$,
(3) $(\overline{M},\overline{g},\overline{\nabla})$ *is a statistical manifold for each* $x \in B$,
(4) $(B,g_B,\widehat{\nabla})$ *is a statistical manifold.*

For a statistical submersion $\pi: (M,g,\nabla) \to (B,g_B,\widehat{\nabla})$, we have the following Lemmas (see [16]):

Lemma E. *If X and Y are horizontal vector fields, then $A_X Y = -A^*_Y X$.*

From (3.1) and Lemma E, A vanishes identically if and only if A^* vanishes identically. Since A is related to the integrability of $\mathcal{H}(M)$, it is identically zero if and only if $\mathcal{H}(M)$ is integrable.

Lemma F. *For $X,Y \in \mathcal{H}(M)$ and $U,V \in \mathcal{V}(M)$ we have*

$$\begin{aligned}
\nabla_U V &= T_U V + \overline{\nabla}_U V, & \nabla^*_U V &= T^*_U V + \overline{\nabla}^*_U V,\\
\nabla_U X &= \mathcal{H}\nabla_U X + T_U X, & \nabla^*_U X &= \mathcal{H}\nabla^*_U X + T^*_U X,\\
\nabla_X U &= A_X U + \mathcal{V}\nabla_X U, & \nabla^*_X U &= A^*_X U + \mathcal{V}\nabla^*_X U,\\
\nabla_X Y &= \mathcal{H}\nabla_X Y + A_X Y, & \nabla^*_X Y &= \mathcal{H}\nabla^*_X Y + A^*_X Y.
\end{aligned}$$

Furthermore, if X is basic, then $\mathcal{H}\nabla_U X = A_X U$ and $\mathcal{H}\nabla^*_U X = A^*_X U$.

We define the covariant derivatives ∇T and ∇A by

$$\begin{aligned}
(\nabla_E T)_F G &= \nabla_E(T_F G) - T_{\nabla_E F}G - T_F(\nabla_E G),\\
(\nabla_E A)_F G &= \nabla_E(A_F G) - A_{\nabla_E F}G - A_F(\nabla_E G),
\end{aligned}$$

for any $E, F, G \in TM$. We change ∇ to ∇^* and then the covariant derivatives $\nabla^* T^*$ and $\nabla^* A^*$ are defined similarly. We consider the curvature tensor on the statistical submersion. Let $\overline{R}$ (resp. $\overline{R}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$) of each fiber. Also, let $\widehat{R}(X,Y)Z$ (resp. $\widehat{R}^*(X,Y)Z$) be horizontal vector field such that $\pi_*(\widehat{R}(X,Y)Z) = \widehat{R}(\pi_* X, \pi_* Y)\pi_* Z$ (resp. $\pi_*(\widehat{R}^*(X,Y)Z) = \widehat{R}^*(\pi_* X, \pi_* Y)\pi_* Z$) at each $p \in M$, where $\widehat{R}$ (resp. $\widehat{R}^*$) is the curvature tensor on B of the affine connection $\widehat{\nabla}$ (resp. $\widehat{\nabla}^*$). Then we have

Theorem G [16] *If $\pi : (M, g, \nabla) \to (B, g_B, \widehat{\nabla})$ is a statistical submersion, then we get for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$:*

$$\begin{aligned}
g(R(U,V)W,W') &= g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W'),\\
g(R(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X),\\
g(R(U,V)X,W) &= g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W),\\
g(R(U,V)X,Y) &= g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y) - g(T_V X, T_U^* Y)\\
&\quad - g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U),\\
g(R(X,U)V,W) &= g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W) + g(T_U^* W, A_X V),\\
g(R(X,U)V,Y) &= g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V) - g(T_U X, T_V^* Y),\\
g(R(X,U)Y,V) &= g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y) - g(A_X U, A_Y V),\\
g(R(X,U)Y,Z) &= g((\nabla_X A)_Y U, Z) - g(T_U X, A_Y^* Z) - g(T_U Y, A_X^* Z) + g(A_X Y, T_U^* Z),\\
g(R(X,Y)U,V) &= g([\mathcal{V}\nabla_X, \mathcal{V}\nabla_Y]U, V) - g(\nabla_{[X,Y]}U, V) + g(A_X U, A_Y^* V) - g(A_Y U, A_X^* V),\\
g(R(X,Y)U,Z) &= g((\nabla_X A)_Y U, Z) - g((\nabla_Y A)_X U, Z) + g(T_U^* Z, \theta_X Y),\\
g(R(X,Y)Z,U) &= g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y),\\
g(R(X,Y)Z,Z') &= g(\widehat{R}(X,Y)Z, Z') - g(A_Y Z, A_X^* Z') + g(A_X Z, A_Y^* Z') + g(\theta_X Y, A_Z^* Z'),
\end{aligned}$$

where we put $\theta_X Y = A_X Y + A_X^* Y = \mathcal{V}[X,Y]$.

For each $p \in M$, we denote by $\{E_1, \dots, E_m\}$, $\{X_1, \dots, X_n\}$ and $\{U_1, \dots, U_s\}$ local orthonormal bases of $T_p M$, $\mathcal{H}_p(M)$ and $\mathcal{V}_p(M)$ such that $E_i = X_i$ $(i = 1, \dots, n)$ and $E_{n+\alpha} = U_\alpha$ $(\alpha = 1, \dots, s)$. The mean curvature vector of ∇ (resp. ∇^*) is given by the horizontal vector field $N = \sum_\alpha \varepsilon_\alpha T_{U_\alpha} U_\alpha$ (resp. $N^* = \sum_\alpha \varepsilon_\alpha T^*_{U_\alpha} U_\alpha$). The statistical submersion π is called minimal with respect to ∇ (resp. ∇^*) if $N = 0$ (resp. $N^* = 0$). If $T_U V = \frac{1}{s} g(U,V)N$ (resp. $T_U^* V = \frac{1}{s} g(U,V)N^*$) holds, then π is said to be with conformal fiber with respect to ∇ (resp. ∇^*). We put $\sigma = \sum_i \varepsilon_i A_{X_i} X_i$.

4. Holomorphic statistical submersions

Let (M, ∇, g, J) be a holomorphic statistical manifold and $(B, \widehat{\nabla}, g_B)$ be a statistical manifold. The statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is called a holomorphic statistical submersion. For $X \in \mathcal{H}(M)$ and $U \in \mathcal{V}(M)$ we put

$$JX = PX + FX, \qquad JU = tU + fU,$$

where PX, $tU \in \mathcal{H}(M)$ and FX, $fU \in \mathcal{V}(M)$. From $J^2 = -I$, we get

$$P^2 = -I - tF, \quad FP + fF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft.$$

Because of $g(JE, G) + g(E, JG) = 0$ for E, $G \in TM$, we find

$$\begin{aligned}
g(PY, Z) + g(Y, PZ) &= 0,\\
g(FY, V) + g(Y, tV) &= 0,\\
g(fV, W) + g(V, fW) &= 0.
\end{aligned}$$

Thus t vanishes identically if and only if F vanishes identically. If $J(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (resp. $J(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$), for each $p \in M$, that is, $t = 0$ (resp. $f = 0$), then $\overline{M}$ is said to be an invariant (resp. anti-invariant) submanifold of M. We put

$$(\mathcal{H}\nabla_X P)Y = \mathcal{H}\nabla_X(PY) - P(\mathcal{H}\nabla_X Y), \qquad (\mathcal{H}\nabla_U P)Y = \mathcal{H}\nabla_U(PY) - P(\mathcal{H}\nabla_U Y),$$
$$(\mathcal{V}\nabla_X F)Y = \mathcal{V}\nabla_X(FY) - F(\mathcal{H}\nabla_X Y), \qquad (\mathcal{V}\nabla_U F)Y = \mathcal{V}\nabla_U(FY) - F(\mathcal{H}\nabla_U Y),$$
$$(\mathcal{H}\nabla_X t)V = \mathcal{H}\nabla_X(tV) - t(\mathcal{V}\nabla_X V), \qquad (\mathcal{H}\nabla_U t)V = \mathcal{H}\nabla_U(tV) - t(\overline{\nabla}_U V),$$
$$(\mathcal{V}\nabla_X f)V = \mathcal{V}\nabla_X(fV) - f(\mathcal{V}\nabla_X V), \qquad (\overline{\nabla}_U f)V = \overline{\nabla}_U(fV) - f(\overline{\nabla}_U V).$$

Similarly, we set $(\mathcal{H}\nabla^*_X P)Y$ and $(\mathcal{H}\nabla^0_X P)Y$, etc. with respect to the conjugate connection ∇^* and the Levi-Civita connection ∇^0. We find

$$(\nabla^0_X J)Y = (\mathcal{H}\nabla^0_X P)Y + A^0_X(FY) - t(A^0_X Y) + (\mathcal{V}\nabla^0_X F)Y + A^0_X(PY) - f(A^0_X Y),$$
$$(\nabla^0_X J)V = (\mathcal{H}\nabla^0_X t)V + A^0_X(fV) - P(A^0_X V) + (\mathcal{V}\nabla^0_X f)V + A^0_X(tV) - F(A^0_X V),$$
$$(\nabla^0_U J)Y = (\mathcal{H}\nabla^0_U P)Y + T^0_U(FY) - t(T^0_U Y) + (\mathcal{V}\nabla^0_U F)Y + T^0_U(PY) - f(T^0_U Y),$$
$$(\nabla^0_U J)V = (\mathcal{H}\nabla^0_U t)V + T^0_U(fV) - P(T^0_U V) + (\overline{\nabla}^0_U f)V + T^0_U(tV) - F(T^0_U V).$$

Since (M, g, J) is a Kählerian manifold, we have

Lemma 4.1. *Let M be a Kählerian manifold and $\pi : M \to B$ be a semi-Riemanian submersion. One has*

$$(\mathcal{H}\nabla^0_X P)Y + A^0_X(FY) - t(A^0_X Y) = 0, \qquad (\mathcal{V}\nabla^0_X F)Y + A^0_X(PY) - f(A^0_X Y) = 0,$$
$$(\mathcal{H}\nabla^0_X t)V + A^0_X(fV) - P(A^0_X V) = 0, \qquad (\mathcal{V}\nabla^0_X f)V + A^0_X(tV) - F(A^0_X V) = 0,$$
$$(\mathcal{H}\nabla^0_U P)Y + T^0_U(FY) - t(T^0_U Y) = 0, \qquad (\mathcal{V}\nabla^0_U F)Y + T^0_U(PY) - f(T^0_U Y) = 0,$$
$$(\mathcal{H}\nabla^0_U t)V + T^0_U(fV) - P(T^0_U V) = 0, \qquad (\overline{\nabla}^0_U f)V + T^0_U(tV) - F(T^0_U V) = 0.$$

For a holomorphic statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$, we have

Lemma 4.2. *Let $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ be a holomorphic statistical submersion. We get*

$$g((\mathcal{H}\nabla_X P)Y, Z) + g(Y, (\mathcal{H}\nabla^*_X P)Z) = 0, \qquad g((\mathcal{H}\nabla_U P)Y, Z) + g(Y, (\mathcal{H}\nabla^*_U P)Z) = 0,$$
$$g((\mathcal{V}\nabla_X F)Y, V) + g(Y, (\mathcal{H}\nabla^*_X t)V) = 0, \qquad g((\mathcal{V}\nabla_U F)Y, V) + g(Y, (\mathcal{H}\nabla^*_U t)V) = 0,$$
$$g((\mathcal{H}\nabla_X t)V, Y) + g(V, (\mathcal{V}\nabla^*_X F)Y) = 0, \qquad g((\mathcal{H}\nabla_U t)V, Y) + g(V, (\mathcal{V}\nabla^*_U F)Y) = 0,$$
$$g((\mathcal{V}\nabla_X f)V, W) + g(V, (\mathcal{V}\nabla^*_X f)W) = 0, \qquad g((\overline{\nabla}_U f)V, W) + g(V, (\overline{\nabla}^*_U f)W) = 0.$$

Corollary 4.1. *If $\pi :$ is a holomorphic statistical submersion, then we find*
(1) $\mathcal{H}\nabla P = 0$ is equivalent to $\mathcal{H}\nabla^* P = 0$.
(2) $\mathcal{V}\nabla F = 0$ is equivalent to $\mathcal{H}\nabla^* t = 0$.
(3) $\mathcal{V}\nabla f = 0$ is equivalent to $\mathcal{V}\nabla^* f = 0$.

Using $\nabla^*_E(JF) = J(\nabla_E F)$, we get

Lemma 4.3. *Let $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ be a holomorphic statistical submersion. We get*

$$(\mathcal{H}\nabla^*_U t)V = t(\mathcal{V}(S_U V)) + P(T_U V) - T^*_U(fV),$$
$$(\overline{\nabla}^*_U f)V = f(\mathcal{V}(S_U V)) + F(T_U V) - T^*_U(tV),$$
$$(\mathcal{H}\nabla^*_U P)X = P(\mathcal{H}(S_U X)) + t(T_U X) - T^*_U(FX),$$
$$(\mathcal{V}\nabla^*_U F)X = F(\mathcal{H}(S_U X)) + f(T_U X) - T^*_U(PX),$$
$$(\mathcal{H}\nabla^*_X t)U = t(\mathcal{V}(S_X U)) + P(A_X U) - A^*_X(fU),$$
$$(\mathcal{V}\nabla^*_X f)U = f(\mathcal{V}(S_X U)) + F(A_X U) - A^*_X(tU),$$
$$(\mathcal{H}\nabla^*_X P)Y = P(\mathcal{H}(S_X Y)) + t(A_X Y) - A^*_X(FY),$$
$$(\mathcal{V}\nabla^*_X F)Y = F(\mathcal{H}(S_X Y)) + f(A_X Y) - A^*_X(PY).$$

Next, we give an example of an anti-invariant holomorphic statistical submersion.

Example 4.1. Let (B, g_B) be a (semi-)Riemannian manifold with local coordinate system (x_1, x_2, x_3), where $g_B = \mathrm{diag}(e^{x_1}, \varepsilon e^{x_2}, e^{x_1})$, where $\varepsilon = 1$ or $\varepsilon = -1$. If $\varepsilon = 1$ (resp. $\varepsilon = -1$), then $B = \mathbb{R}^3$ (resp. $B = \mathbb{R}^3_1$). If we put

$$\begin{aligned}
&\widehat{\nabla}_{\partial_{1_*}}\partial_{1_*} = -\widehat{\nabla}_{\partial_{3_*}}\partial_{3_*} = \varepsilon e^{-x_2}\partial_{2_*}, && \widehat{\nabla}_{\partial_{1_*}}\partial_{2_*} = \widehat{\nabla}_{\partial_{2_*}}\partial_{1_*} = e^{-x_1}\partial_{1_*},\\
&\widehat{\nabla}_{\partial_{1_*}}\partial_{3_*} = \widehat{\nabla}_{\partial_{3_*}}\partial_{1_*} = \partial_{3_*}, && \widehat{\nabla}_{\partial_{2_*}}\partial_{2_*} = 0,\\
&\widehat{\nabla}_{\partial_{2_*}}\partial_{3_*} = \widehat{\nabla}_{\partial_{3_*}}\partial_{2_*} = -e^{-x_1}\partial_{3_*},
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\nabla}^*_{\partial_{1_*}}\partial_{1_*} = -\widehat{\nabla}^*_{\partial_{3_*}}\partial_{3_*} = \partial_{1_*} - \varepsilon e^{-x_2}\partial_{2_*}, && \widehat{\nabla}^*_{\partial_{1_*}}\partial_{2_*} = \widehat{\nabla}^*_{\partial_{2_*}}\partial_{1_*} = -e^{-x_1}\partial_{1_*},\\
&\widehat{\nabla}^*_{\partial_{1_*}}\partial_{3_*} = \widehat{\nabla}^*_{\partial_{3_*}}\partial_{1_*} = 0, && \widehat{\nabla}^*_{\partial_{2_*}}\partial_{2_*} = \partial_{2_*},\\
&\widehat{\nabla}^*_{\partial_{2_*}}\partial_{3_*} = \widehat{\nabla}^*_{\partial_{3_*}}\partial_{2_*} = e^{-x_1}\partial_{3_*},
\end{aligned}$$

then $(B, \widehat{\nabla}, g_B)$ and $(B, \widehat{\nabla}^*, g_B)$ are statistical manifolds, where $\partial_{i_*} = \partial/\partial x_i$ $(i = 1, 2, 3)$. Considering the holomorphic statistical manifold (M, ∇, g, J) given in Example 1, we define a holomorphic statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ by

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3).$$

For ∂_1, ∂_2, $\partial_3 \in \mathcal{H}(M)$ and $\partial_4 \in \mathcal{V}(M)$, we get

$$\begin{aligned}
&T_{\partial_4}\partial_4 = 0, && \overline{\nabla}_{\partial_4}\partial_4 = 0,\\
&\mathcal{H}\nabla_{\partial_4}\partial_1 = -e^{-x_1}\partial_3, && T_{\partial_4}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_4}\partial_2 = 0, && T_{\partial_4}\partial_2 = \partial_4,\\
&\mathcal{H}\nabla_{\partial_4}\partial_3 = -e^{-x_1}\partial_1, && T_{\partial_4}\partial_3 = 0,\\
&A_{\partial_1}\partial_4 = -e^{-x_1}\partial_3, && \mathcal{V}\nabla_{\partial_1}\partial_4 = 0,\\
&A_{\partial_2}\partial_4 = 0, && \mathcal{V}\nabla_{\partial_2}\partial_4 = \partial_4,\\
&A_{\partial_3}\partial_4 = -e^{-x_1}\partial_3, && \mathcal{V}\nabla_{\partial_3}\partial_4 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_1 = -\mathcal{H}\nabla_{\partial_3}\partial_3 = \varepsilon e^{-x_2}\partial_2, && A_{\partial_1}\partial_1 = -A_{\partial_3}\partial_3 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_2 = \mathcal{H}\nabla_{\partial_2}\partial_1 = e^{-x_1}\partial_1, && A_{\partial_1}\partial_2 = A_{\partial_2}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_1 = \partial_3, && A_{\partial_1}\partial_3 = A_{\partial_3}\partial_1 = -\varepsilon e^{-x_2}\partial_4,\\
&\mathcal{H}\nabla_{\partial_2}\partial_2 = 0, && A_{\partial_2}\partial_2 = 0,\\
&\mathcal{H}\nabla_{\partial_2}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_2 = -e^{-x_1}\partial_3, && A_{\partial_2}\partial_3 = A_{\partial_3}\partial_2 = 0.
\end{aligned}$$

Thus π is with isometric fiber with respect to ∇. Moreover, we find

$$P = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f = 0.$$

Therefore π is anti-invariant.

5. Holomorphic statistical submersions with invariant fibers

A holomorphic statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is called invariant if each fiber is an invariant submanifold of M, that is, $\varphi(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$. From $t = 0$ $(F = 0)$, we get $P^2 = -I$ and $f^2 = -I$. By Lemma 4.3, we get

Lemma 5.1. *If $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion, then*

$$P(T_U V) = T^*_U(fV), \qquad (\overline{\nabla}^*_U f)V = f(\mathcal{V}(S_U V)),$$
$$(\mathcal{H}\nabla^*_U P)X = P(\mathcal{H}(S_U X)), \qquad f(T_U X) = T^*_U(PX),$$
$$P(A_X U) = A^*_X(fU), \qquad (\mathcal{V}\nabla^*_X f)U = f(\mathcal{V}(S_X U)),$$
$$(\mathcal{H}\nabla^*_X P)Y = P(\mathcal{H}(S_X Y)), \qquad f(A_X Y) = A^*_X(PY).$$

It is easy to see from Lemma 4.1 that $(\mathcal{H}\nabla^0_X P)Y = 0$ and $(\overline{\nabla}^0_U f)V = 0$ hold, which implies that the base space and each fiber are Kählerian manifolds. Using Lemma 5.1, we find $\mathcal{H}\nabla^*_X(PY) = P(\mathcal{H}\nabla_X Y)$ and $\overline{\nabla}^*_U(fV) = f(\overline{\nabla}_U V)$. Hence we have from Lemma 4.1

Theorem 5.1. *If $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion, then*
(1) $(B, \widehat{\nabla}, g_B, \widehat{P})$ is a holomorphic statistical manifold, where $\widehat{P}$ is a tensor field of type $(1,1)$ such that $\pi_* P = \widehat{P}\pi_*$.
(2) $(\overline{M}, \overline{\nabla}, \overline{g}, f)$ is a holomorphic statistical manifold for each $x \in B$.

Also, we get from (3.1), Lemmas E and 5.1

$$g(T_{fU}(fV), X) = g(P(T^*_{fU}V), X) = -g(T^*_V(fU), PX) = -g(P(T_V U), PX) = -g(T_U V, X),$$

$$\begin{aligned} g(A_{PX}U, Y) &= -g(U, A^*_{PX}Y) = g(U, A_Y(PX)) = g(U, f(A^*_Y X)) = g(fU, A_X Y) \\ &= -g(A^*_X(fU), Y) = -g(P(A_X U), Y). \end{aligned}$$

Thus we have

Lemma 5.2. *Let π be an invariant holomorphic statistical submersion. We find*
(1) $T_{fU}(fV) = -T_U V$ and $A_{PX}(PY) = -A_X Y$.
(2) $T_{fU}X = -f(T_U X)$ and $A_{PX}U = -P(A_X U)$.

Lemma 5.3. *Let π be an invariant holomorphic statistical submersion. Then $\mathcal{H}\nabla_U P = 0$ (resp. $\mathcal{V}\nabla_X f = 0$) if and only if $A = A^*$ (resp. $T = T^*$).*

When π is an invariant holomorphic statistical submersion, $\{E_A, JE_A\}$ $(A = 1, \ldots, m)$, $\{X_i, PX_i\}$ $(i = 1, \ldots, n)$ and $\{U_\alpha, fU_\alpha\}$ $(\alpha = 1, \ldots, s)$ are local orthonormal bases of T_pM, $\mathcal{H}_p(M)$ and $\mathcal{V}_p(M)$, respectively such that $E_i = X_i$, $E_{n+\alpha} = U_\alpha$, $X_{n+i} = PX_i$ and $U_{s+\alpha} = fU_\alpha$. From Lemma 5.2 (1), we get

$$N = \sum_{\alpha=1}^{2s} \varepsilon_\alpha T_{U_\alpha} U_\alpha = \sum_{\alpha=1}^{s} \varepsilon_\alpha \{T_{U_\alpha} U_\alpha + T_{fU_\alpha}(fU_\alpha)\} = 0,$$

$$\sigma = \sum_{i=1}^{2n} \varepsilon_i A_{X_i} X_i = \sum_{i=1}^{n} \varepsilon_i \{A_{X_i} X_i + A_{PX_i}(PX_i)\} = 0.$$

Hence we have

Theorem 5.2. *If $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion, then each fiber is minimal with respect to ∇ and $\sigma = 0$.*

Corollary 5.1. *If $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion, then each fiber is minimal with respect to ∇^*, that is, $N^* = 0$.*

By virtue of Theorems G and 5.2, we have

Theorem 5.3. *Let $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion with conformal fiber with respect to ∇ or ∇^*. If (M, ∇, g, J) is of constant holomorphic sectional curvature c, then $(\overline{M}, \overline{\nabla}, \overline{g}, f)$ is of constant holomorphic sectional curvature c.*

We consider that the total space (M, ∇, g, J) is of constant holomorphic sectional curvature c. From the last equation of Theorem G, we find

$$\begin{aligned} &g(\widehat{R}(X,Y)Z, Z') - g(A_Y Z, A^*_X Z') + g(A_X Z, A^*_Y Z') + g(A_X Y, A^*_Z Z') + g(A^*_X Y, A^*_Z Z') \\ &= \frac{c}{4}\{g(Y,Z)g(X,Z') - g(X,Z)g(Y,Z') + g(PY,Z)g(PX,Z') \\ &\quad - g(PX,Z)g(PY,Z') + 2g(X,PY)g(PZ,Z')\}. \end{aligned} \tag{5.1}$$

Substituting Z by PZ in (5.1), we obtain

$$\begin{aligned}&g(\widehat{R}^*(X,Y)Z,PZ') - g(A^*_Y Z, A_X(PZ')) + g(A^*_X Z, A_Y(PZ'))\\ &-g(A_X Y, A^*_Z(PZ')) - g(A^*_X Y, A^*_Z(PZ'))\\ &= \frac{c}{4}\{g(Y,Z)g(X,PZ') - g(X,Z)g(Y,PZ') - g(Y,PZ)g(X,Z')\\ &\quad +g(X,PZ)g(Y,Z') + 2g(X,PY)g(Z,Z')\}.\end{aligned} \tag{5.2}$$

On the other hand, it is clear from Lemma 2.1 that (5.1) is rewitten as follows:

$$\begin{aligned}&g(\widehat{R}^*(X,Y)Z,Z') - g(A^*_Y Z, A_X Z') + g(A^*_X Z, A_Y Z') + g(A^*_X Y, A_Z Z') + g(A_X Y, A_Z Z')\\ &= \frac{c}{4}\{g(Y,Z)g(X,Z') - g(X,Z)g(Y,Z') + g(PY,Z)g(PX,Z')\\ &\quad -g(PX,Z)g(PY,Z') + 2g(X,PY)g(PZ,Z')\},\end{aligned}$$

which yields that

$$\begin{aligned}&g(\widehat{R}^*(X,Y)Z,PZ') - g(A^*_Y Z, A_X(PZ')) + g(A^*_X Z, A_Y(PZ'))\\ &+g(A^*_X Y, A_Z(PZ')) + g(A_X Y, A_Z(PZ'))\\ &= \frac{c}{4}\{g(Y,Z)g(X,PZ') - g(X,Z)g(Y,PZ') + g(PY,Z)g(X,Z')\\ &\quad -g(PX,Z)g(Y,Z') + 2g(X,PY)g(Z,Z')\}.\end{aligned} \tag{5.3}$$

By virtue of (5.2) and (5.3), we get $g(\theta_X Y, \theta_Z Z') = 0$, where $\theta_X Y = A_X Y + A^*_X Y$. Hence we have

Proposition 5.1. *Let $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ is an invariant holomorphic statistical submersion, and (M, ∇, g, J) be of constant holomorphic sectional curvature c. If g is positive definite, then $\mathcal{V}[X, Y] = 0$ holds, and A is symmetric.*

Finally, we give two examples of invariant holomorphic statistical submersions.

Example 5.1. Let (B, g_B) be a Riemannian manifold with local coordinate system (x_1, x_3), where $g_B = \mathrm{diag}(e^{x_1}, e^{x_1})$. If we put

$$\widehat{\nabla}_{\partial_{1_*}}\partial_{1_*} = -\widehat{\nabla}_{\partial_{3_*}}\partial_{3_*} = 0, \qquad \widehat{\nabla}_{\partial_{1_*}}\partial_{3_*} = \widehat{\nabla}_{\partial_{3_*}}\partial_{1_*} = \partial_{3_*}$$

and

$$\widehat{\nabla}^*_{\partial_{1_*}}\partial_{1_*} = -\widehat{\nabla}^*_{\partial_{3_*}}\partial_{3_*} = \partial_{1_*}, \qquad \widehat{\nabla}^*_{\partial_{1_*}}\partial_{3_*} = \widehat{\nabla}^*_{\partial_{3_*}}\partial_{1_*} = 0,$$

then $(B, \widehat{\nabla}, g_B)$ is a statistical manifold and so is $(B, \widehat{\nabla}^*, g_B)$, where $\partial_{i_*} = \partial/\partial x_i \ (i = 1, 3)$. Considering the holomorphic statistical manifold (M, ∇, g, J) given in Example 1, we define a holomorphic statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ by

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_3).$$

For $\partial_1,\ \partial_3 \in \mathcal{H}(M)$ and $\partial_2,\ \partial_4 \in \mathcal{V}(M)$, we get

$$\begin{aligned}
&T_{\partial_2}\partial_2 = -T_{\partial_4}\partial_4 = 0, && \overline{\nabla}_{\partial_2}\partial_2 = -\overline{\nabla}_{\partial_4}\partial_4 = 0,\\
&T_{\partial_2}\partial_4 = T_{\partial_4}\partial_2 = 0, && \overline{\nabla}_{\partial_2}\partial_4 = \overline{\nabla}_{\partial_4}\partial_2 = \partial_4,\\
&\mathcal{H}\nabla_{\partial_2}\partial_1 = -\mathcal{H}\nabla_{\partial_4}\partial_3 = e^{-x_1}\partial_1, && T_{\partial_2}\partial_1 = -T_{\partial_4}\partial_3 = 0,\\
&\mathcal{H}\nabla_{\partial_2}\partial_3 = \mathcal{H}\nabla_{\partial_4}\partial_1 = -e^{-x_1}\partial_3, && T_{\partial_2}\partial_3 = T_{\partial_4}\partial_1 = 0,\\
&A_{\partial_1}\partial_2 = -A_{\partial_3}\partial_4 = e^{-x_1}\partial_1, && \mathcal{V}\nabla_{\partial_1}\partial_2 = -\mathcal{V}\nabla_{\partial_3}\partial_4 = 0,\\
&A_{\partial_1}\partial_4 = A_{\partial_3}\partial_2 = -e^{-x_1}\partial_3, && \mathcal{V}\nabla_{\partial_1}\partial_4 = \mathcal{V}\nabla_{\partial_3}\partial_2 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_1 = -\mathcal{H}\nabla_{\partial_3}\partial_3 = 0, && A_{\partial_1}\partial_1 = -A_{\partial_3}\partial_3 = \varepsilon e^{-x_2}\partial_2,\\
&\mathcal{H}\nabla_{\partial_1}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_1 = \partial_3, && A_{\partial_1}\partial_3 = A_{\partial_3}\partial_1 = -\varepsilon e^{-x_2}\partial_4,
\end{aligned}$$

and

$$
\begin{aligned}
&T^*_{\partial_2}\partial_2 = -T^*_{\partial_4}\partial_4 = 0, && \overline{\nabla}^*_{\partial_2}\partial_2 = -\overline{\nabla}^*_{\partial_4}\partial_4 = \partial_2,\\
&T^*_{\partial_2}\partial_4 = T^*_{\partial_4}\partial_2 = 0, && \overline{\nabla}^*_{\partial_2}\partial_4 = \overline{\nabla}^*_{\partial_4}\partial_2 = 0,\\
&\mathcal{H}\nabla^*_{\partial_2}\partial_1 = -\mathcal{H}\nabla^*_{\partial_4}\partial_3 = -e^{-x_1}\partial_1, && T^*_{\partial_2}\partial_1 = -T^*_{\partial_4}\partial_3 = 0,\\
&\mathcal{H}\nabla^*_{\partial_2}\partial_3 = \mathcal{H}\nabla^*_{\partial_4}\partial_1 = e^{-x_1}\partial_3, && T^*_{\partial_2}\partial_3 = T^*_{\partial_4}\partial_1 = 0,\\
&A^*_{\partial_1}\partial_2 = -A_{\partial_3}\partial_4 = -e^{-x_1}\partial_1, && \mathcal{V}\nabla^*_{\partial_1}\partial_2 = -\mathcal{V}\nabla^*_{\partial_3}\partial_4 = 0,\\
&A^*_{\partial_1}\partial_4 = A^*_{\partial_3}\partial_2 = e^{-x_1}\partial_3, && \mathcal{V}\nabla^*_{\partial_1}\partial_4 = \mathcal{V}\nabla^*_{\partial_3}\partial_2 = 0,\\
&\mathcal{H}\nabla^*_{\partial_1}\partial_1 = -\mathcal{H}\nabla^*_{\partial_3}\partial_3 = \partial_1, && A^*_{\partial_1}\partial_1 = -A^*_{\partial_3}\partial_3 = -\varepsilon e^{-x_2}\partial_2,\\
&\mathcal{H}\nabla^*_{\partial_1}\partial_3 = \mathcal{H}\nabla^*_{\partial_3}\partial_1 = 0, && A^*_{\partial_1}\partial_3 = A^*_{\partial_3}\partial_1 = \varepsilon e^{-x_2}\partial_4.
\end{aligned}
$$

Thus π is with isometric fiber with respect to ∇ and ∇^*. We find

$$
P = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad F = 0, \qquad t = 0, \qquad f = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

Therefore, π is invariant and minimal. It is easy to see from $\widehat{\nabla}^*_{\partial_i}(\widehat{P}\partial_j) = \widehat{P}(\widehat{\nabla}_{\partial_i}\partial_j)$ $(i = 1, 3)$ and $\overline{\nabla}^*_{\partial_\alpha}(f\partial_\beta) = f(\overline{\nabla}_{\partial_\alpha}\partial_\beta)$ $(\alpha, \beta = 2, 4)$ that $(B, \widehat{\nabla}, g_B, \widehat{P})$ and $(\overline{M}, \overline{\nabla}, \overline{g}, f)$ are holomorphic statistical manifolds. Moreover, $\mathcal{V}\nabla_X f = 0$ holds. We put $X_1 = e^{-x_1}\partial_1$, $X_2 = e^{-x_1}\partial_3$ and $U_1 = \varepsilon e^{-x_2}\partial_2$, $U_2 = \varepsilon e^{-x_2}\partial_4$. Then $\{X_1, X_2\}$ and $\{U_1, U_2\}$ are local orthonormal bases of $\mathcal{H}_p(M)$ and $\mathcal{V}_p(M)$ for each $p \in M$, respectively. We find $A_{X_1}X_1 = -A_{X_2}X_2 = e^{-2x_1}U_1$ and $A_{X_1}X_2 = A_{X_2}X_1 = -e^{-2x_1}U_2$, which implies that $\sigma = 0$.

Example 5.2. Let (B, g_B) be a (semi-)Riemannian manifold with local coordinate system (x_2, x_4), where $g_B = \mathrm{diag}(\varepsilon e^{x_2}, \varepsilon e^{x_2})$, where $\varepsilon = 1$ or -1. If $\varepsilon = 1$ (resp. $\varepsilon = -1$), then $B = \mathbb{R}^2$ (resp. $B = \mathbb{R}^2_2$). If we put

$$
\widehat{\nabla}_{\partial_{2_*}}\partial_{2_*} = -\widehat{\nabla}_{\partial_{4_*}}\partial_{4_*} = 0, \qquad \widehat{\nabla}_{\partial_{2_*}}\partial_{4_*} = \widehat{\nabla}_{\partial_{4_*}}\partial_{2_*} = \partial_{4_*}
$$

and

$$
\widehat{\nabla}^*_{\partial_{2_*}}\partial_{2_*} = -\widehat{\nabla}^*_{\partial_{4_*}}\partial_{4_*} = \partial_{2_*}, \qquad \widehat{\nabla}_{\partial^*_{2_*}}\partial_{4_*} = \widehat{\nabla}^*_{\partial_{4_*}}\partial_{2_*} = 0,
$$

then $(B, \widehat{\nabla}, g_B)$ is a statistical manifold and so is $(B, \widehat{\nabla}^*, g_B)$, where $\partial_{i_*} = \partial/\partial x_i$ $(i = 2, 4)$. Considering the holomorphic statistical manifold (M, ∇, g, J) given in Example 1, we define a holomorphic statistical submersion $\pi : (M, \nabla, g, J) \to (B, \widehat{\nabla}, g_B)$ by

$$
\pi(x_1, x_2, x_3, x_4) = (x_2, x_4).
$$

For ∂_2, $\partial_4 \in \mathcal{H}(M)$ and ∂_1, $\partial_3 \in \mathcal{V}(M)$, we get

$$
\begin{aligned}
&T_{\partial_1}\partial_1 = -T_{\partial_3}\partial_3 = \varepsilon e^{-x_2}\partial_2, && \overline{\nabla}_{\partial_1}\partial_1 = -\overline{\nabla}_{\partial_3}\partial_3 = 0,\\
&T_{\partial_1}\partial_3 = T_{\partial_3}\partial_1 = -\varepsilon e^{-x_2}\partial_4, && \overline{\nabla}_{\partial_1}\partial_3 = \overline{\nabla}_{\partial_3}\partial_1 = \partial_3,\\
&\mathcal{H}\nabla_{\partial_1}\partial_2 = -\mathcal{H}\nabla_{\partial_3}\partial_4 = 0, && T_{\partial_1}\partial_2 = -T_{\partial_3}\partial_4 = e^{-x_1}\partial_1,\\
&\mathcal{H}\nabla_{\partial_1}\partial_4 = \mathcal{H}\nabla_{\partial_3}\partial_2 = 0, && T_{\partial_1}\partial_4 = T_{\partial_3}\partial_2 = -e^{-x_1}\partial_3,\\
&A_{\partial_2}\partial_1 = -A_{\partial_4}\partial_3 = 0, && \mathcal{V}\nabla_{\partial_2}\partial_1 = -\mathcal{V}\nabla_{\partial_4}\partial_3 = e^{-x_1}\partial_1,\\
&A_{\partial_2}\partial_3 = A_{\partial_4}\partial_1 = 0, && \mathcal{V}\nabla_{\partial_2}\partial_3 = \mathcal{V}\nabla_{\partial_4}\partial_1 = -e^{-x_1}\partial_3,\\
&\mathcal{H}\nabla_{\partial_2}\partial_2 = -\mathcal{H}\nabla_{\partial_4}\partial_4 = 0, && A_{\partial_2}\partial_2 = -A_{\partial_4}\partial_4 = 0,\\
&\mathcal{H}\nabla_{\partial_2}\partial_4 = \mathcal{H}\nabla_{\partial_4}\partial_2 = \partial_4, && A_{\partial_2}\partial_4 = A_{\partial_4}\partial_2 = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&T^*_{\partial_1}\partial_1 = -T^*_{\partial_3}\partial_3 = -\varepsilon e^{-x_2}\partial_2, && \overline{\nabla}^*_{\partial_1}\partial_1 = -\overline{\nabla}^*_{\partial_3}\partial_3 = \partial_1,\\
&T^*_{\partial_1}\partial_3 = T^*_{\partial_3}\partial_1 = \varepsilon e^{-x_2}\partial_4, && \overline{\nabla}^*_{\partial_1}\partial_3 = \overline{\nabla}^*_{\partial_3}\partial_1 = 0,\\
&\mathcal{H}\nabla^*_{\partial_1}\partial_2 = -\mathcal{H}\nabla^*_{\partial_3}\partial_4 = 0, && T^*_{\partial_1}\partial_2 = -T^*_{\partial_3}\partial_4 = -e^{-x_1}\partial_1,\\
&\mathcal{H}\nabla^*_{\partial_1}\partial_4 = \mathcal{H}\nabla^*_{\partial_3}\partial_2 = 0, && T^*_{\partial_1}\partial_4 = T^*_{\partial_3}\partial_2 = e^{-x_1}\partial_3,\\
&A^*_{\partial_2}\partial_1 = -A^*_{\partial_4}\partial_3 = 0, && \mathcal{V}\nabla^*_{\partial_2}\partial_1 = -\mathcal{V}\nabla^*_{\partial_4}\partial_3 = -e^{-x_1}\partial_1,\\
&A^*_{\partial_2}\partial_3 = A^*_{\partial_4}\partial_1 = 0, && \mathcal{V}\nabla^*_{\partial_2}\partial_3 = \mathcal{V}\nabla^*_{\partial_4}\partial_1 = e^{-x_1}\partial_3,\\
&\mathcal{H}\nabla^*_{\partial_2}\partial_2 = -\mathcal{H}\nabla^*_{\partial_4}\partial_4 = \partial_2, && A^*_{\partial_2}\partial_2 = -A^*_{\partial_4}\partial_4 = 0,\\
&\mathcal{H}\nabla^*_{\partial_2}\partial_4 = \mathcal{H}\nabla^*_{\partial_4}\partial_2 = 0, && A^*_{\partial_2}\partial_4 = A^*_{\partial_4}\partial_2 = 0.
\end{aligned}
$$

Thus $\mathcal{H}(\mathcal{M})$ is integrable. We have

$$
P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad F = 0, \qquad t = 0, \qquad f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Therefore, π is invariant, and $(B, \widehat{\nabla}, g_B, \widehat{P})$ and $(\overline{M}, \overline{\nabla}, \overline{g}, f)$ are holomorphic statistical manifolds. Moreover, $\mathcal{H}\nabla_U P = 0$ holds. We put $U_1 = e^{-x_1}\partial_1$, $U_2 = e^{-x_1}\partial_3$ and $X_1 = \varepsilon e^{-x_2}\partial_2$, $X_2 = \varepsilon e^{-x_2}\partial_4$. Then $\{U_1, U_2\}$ and $\{X_1, X_2\}$ are local orthonormal bases of $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$ for each $p \in M$, respectively. We find $T_{U_1}U_1 = -T_{U_2}U_2 = e^{-2x_1}X_1$ and $T_{U_1}U_2 = T_{U_2}U_1 = -e^{-2x_1}X_2$, which implies that $N = 0$, namely, π is minimal with respect to ∇. Similarly, it is clear that π is minimal with respect to ∇^*.

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Author's contributions

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