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Research Article

On the Domain of 4-Dimensional Catalan Matrix in the Space of Absolutely Summable Double Sequences

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Abstract

The primary objective of this study is to construct a novel double sequence space by utilizing the domain of a 4-dimensional (4D) matrix in the space $\mathscr{L}_u$ of the absolutely summable double sequences defined via the well-known Catalan numbers. Within this framework, various algebraic and topological properties of the newly introduced space are investigated. The study further aims to compute the α-, $\beta(bp)$-, and γ-duals of the space. Another essential part of the research involves characterizing certain classes of matrix transformations from the newly defined space to some classical double sequence spaces and vice versa. These contributions are expected to enrich the theory of sequence spaces and matrix transformations by introducing new insights based on special integer sequences.

1. Introduction

Let $\mathbb{N} = \{0,1,2,\ldots\}$ and Ξ be a non-empty set. A function $t : \mathbb{N}\times\mathbb{N} \to \Xi$ described by $(m,n) \mapsto t(m,n) = s_{mn}$ is called a double-indexed sequence, or simply a double sequence. The set of all real-valued double sequences is expressed with Ω. Equipped with coordinate-wise addition and scalar multiplication, Ω forms a linear space. Any subspace of Ω is referred to as a double sequence space.
The set of all bounded double sequences is given by

$$\mathscr{M}_u = \left\{ s = (s_{mn}) \in \Omega : \|s\|_\infty = \sup_{m,n\in\mathbb{N}} |s_{mn}| < \infty \right\},$$

which becomes a Banach space under the norm $\|s\|_\infty$.
Various types of convergence have been defined for double sequences. One of the most prominent is Pringsheim convergence. A double sequence $s = (s_{mn})$ is said to be Pringsheim convergent to a complex number A if, for every $\varepsilon > 0$, there exists a natural number l_ε such that

$$|s_{mn} - A| < \varepsilon \quad \text{whenever } m,n > l_\varepsilon.$$

In this case, we write $p\text{-}\lim_{m,n\to\infty} s_{mn} = A$. The set of all Pringsheim convergent double sequences is denoted by $\mathscr{C}_p$, and its elements are called p-convergent double sequences. Unlike the case of single-indexed sequences, a p-convergent double sequence need not be bounded. For instance, consider the double sequence defined by

$$s_{mn} = \begin{pmatrix} 1^2 & 2^2 & 3^2 & \cdots & n^2 & \cdots \\ 2^2 & 0 & 0 & \cdots & 0 & \cdots \\ 3^2 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \\ m^2 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \end{pmatrix}.$$

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We observe that $p\text{-}\lim s_{mn} = 0$, but $\|s\|_\infty = \infty$.
The set of all double sequences that are both p-convergent and bounded is denoted by $\mathscr{C}_{bp}$, which is formally defined as the intersection $\mathscr{C}_{bp} = \mathscr{C}_p \cap \mathscr{M}_u$. Furthermore, double sequences for which both row-wise and column-wise convergence occurs are called regularly convergent double sequences, and the set of such sequences is denoted by $\mathscr{C}_r$. Moricz [1] proved that both $\mathscr{C}_{bp}$ and $\mathscr{C}_r$ are Banach spaces under the norm $\|\cdot\|_\infty$.
Now, let s be a given double sequence, and define a sequence $T = (t_{jk})$ by

$$t_{jk} := \sum_{m=0}^{j}\sum_{n=0}^{k} u_{mn}, \quad (j,k \in \mathbb{N}).$$

The pair $((u_{jk}),(t_{jk}))$ is called a double series, and the sequence $T = (t_{jk})$ is called the sequence of partial sums. For a given convergence method ϑ, if

$$\vartheta\text{-}\lim_{j,k\to\infty}\sum_{m=0}^{j}\sum_{n=0}^{k} u_{mn} = L,$$

then the series is called ϑ-convergent, and the value L is referred to as the ϑ-sum of the series.
Consider that Ψ and Λ in Ω and let $B = (b_{jkmn})$ be a 4D infinite matrix. The B-transform of a double sequence u is defined by

$$Bu = \{(Bu)_{jk}\}_{j,k\in\mathbb{N}} := \vartheta\text{-}\sum_{m,n} b_{jkmn}u_{mn}. \tag{1.1}$$

If for every $u \in \Psi$, the transform Bu exists and belongs to Λ, then B is said to define a ϑ-type matrix transformation from Ψ into Λ, expressed by $B : \Psi \mapsto \Lambda$. The collection of all such matrices is denoted by $(\Psi : \Lambda)$.
A necessary and sufficient conditions for a matrix B to belong to $(\Psi : \Lambda)$ are that the series on the right-hand side of equation (1.1) is ϑ-convergent for each $j,k \in \mathbb{N}$ (i.e., $B_{jk} = (b_{jkmn})_{m,n\in\mathbb{N}} \in \Psi^{\beta(\vartheta)}$), and that $Bu \in \Lambda$ for all $u \in \Psi$.
Given a 4D matrix $B = (b_{jkmn})$, the ϑ-domain of B in the space Λ is defined as

$$\Lambda_B^{(\vartheta)} = \left\{ u \in \Omega : (Bu)_{jk} = \left(\vartheta\text{-}\sum_{m,n} b_{jkmn}u_{mn}\right)_{j,k\in\mathbb{N}} \in \Lambda \right\}.$$

If for all $j,k,m,n \in \mathbb{N}$, we have $b_{jkmn} = 0$ whenever $m > j$ or $n > k$ (or both), then the matrix $B = (b_{jkmn})$ is called a triangular matrix. If in addition $b_{jkjk} \neq 0$ for all $j,k \in \mathbb{N}$, then B is referred to as a triangle matrix. Cooke [2] proved that every triangular matrix is invertible and that its inverse is also a triangle matrix.
Zeltser [3], in his doctoral dissertation, investigated both the topological structure and summability theory of double sequences. Later, Altay and Başar [4] introduced the double series spaces $\mathscr{BS}$, $\mathscr{BS}(t)$, $\mathscr{CS}_\vartheta$, and $\mathscr{BV}$, where $\vartheta \in \{p, bp, r\}$, by considering the partial sums in the classical double sequence spaces $\mathscr{M}_u$, $\mathscr{M}_u(t)$, $\mathscr{C}_\vartheta$, and $\mathscr{L}_u$, respectively. They studied some algebraic and topological properties of these spaces and computed the α-duals of the spaces $\mathscr{BS}$, $\mathscr{CS}_{bp}$, and $\mathscr{BV}$, as well as the $\beta(\vartheta)$-duals of the spaces $\mathscr{CS}_p$, $\mathscr{CS}_{bp}$, and $\mathscr{CS}_r$. Moreover, they characterized certain classes of matrix transformations from these series spaces into classical sequence spaces.
Başar and Sever [5] constructed and analyzed in detail the Banach space $\mathscr{L}_q$ of absolutely q-summable double sequences for $1 \leq q < \infty$. They studied several topological features of this space and calculated its α-, $\beta(\vartheta)$-, and γ-duals. It is important to note that the space $\mathscr{L}_u$ introduced by Zeltser [6] is a special case of the space $\mathscr{L}_q$ when $q = 1$.
The study of domains of double sequences in the context of summability theory is a relatively recent area of exploration. In 2014, Mursaleen and Başar [7] defined new double sequence spaces whose first-order Cesàro means lie in the classical spaces $\mathscr{M}_u$, $\mathscr{C}_p$, $\mathscr{C}_{p0}$, $\mathscr{C}_{bp}$, $\mathscr{C}_r$, and $\mathscr{L}_q$, and investigated various properties of these spaces. This study is particularly significant as it represents the first known application of the domain of a 4D infinite matrix within the theory of double sequences.
For more comprehensive information on double sequences and summability methods, readers are referred to the works and references in [1, 8–15].
Among the many remarkable integer sequences in mathematics, the Catalan number sequence holds a prominent place due to its frequent occurrence in a wide variety of mathematical problems. The sequence begins as follows:

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ \ldots.$$

It was first encountered by Leonhard Euler in 1751 in connection with the problem of counting the number of distinct triangulations of a convex polygon using non-intersecting diagonals. Although Euler initiated the study, the sequence later became known as the Catalan numbers, named after the Belgian mathematician Eugène Charles Catalan, who investigated the number of ways a sequence of $k+1$ factors can be fully parenthesized. His work revealed yet another context in which this sequence naturally appears (see [16, Example 20.3]).
Since Catalan's contributions, an extensive array of studies and combinatorial problems involving these numbers has emerged. The Catalan numbers are now regarded as one of the most pervasive and versatile sequences in mathematics, due to their occurrence in a multitude of areas, including but not limited to enumerative combinatorics, algebraic structures, number theory, probability, real and complex analysis, geometry, and topology.
Catalan numbers are known to solve numerous combinatorial enumeration problems, such as the number of binary tree structures, Dyck paths, non-crossing partitions, valid sequences of nested parentheses, and stack-sortable permutations. Their widespread utility is a testament to their combinatorial richness and algebraic depth.
Formally, the jth Catalan number C_j is given by the closed-form expression involving binomial coefficients:

$$C_j = \frac{1}{j+1}\binom{2j}{j} = \frac{(2j)!}{(j+1)!\,j!}, \quad \text{for } j \geq 0.$$

An essential property of the Catalan numbers is that they satisfy a simple recursive formula, which is frequently used in both theoretical and computational contexts:

$$C_{j+1}=\sum_{m=0}^{j} C_m C_{j-m}, \quad \text{with } C_0=1.$$

This recurrence relation reflects their recursive nature and is foundational in their combinatorial interpretations. In addition, an alternative nonlinear recurrence relation also holds:

$$C_{j+1}=\frac{2(2j+1)}{j+2}C_j, \quad \text{with } C_0=1,$$

which can be particularly useful for deriving successive terms efficiently.
Moreover, Catalan numbers have notable connections to special functions in analysis. In particular, they can be expressed in terms of the gamma function $\Gamma(z)$, a generalization of the factorial function to complex numbers. The gamma function, introduced by Adrien-Marie Legendre, is defined as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt.$$

Using this definition, the jth Catalan number can also be represented as

$$C_j=\frac{4^j\,\Gamma\left(j+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(j+2)},$$

which provides an analytical extension of Catalan numbers beyond the realm of integer arguments.
For more in-depth discussions concerning the theoretical aspects and diverse applications of Catalan numbers, the reader is referred to the comprehensive texts [16, 17].
The 2D Catalan matrix was defined by İlkhan [18], and together with this study, Kara and Kara [19] described new sequence spaces as the domains of this matrix on classical single sequence spaces and studied these spaces under the titles of completeness, inclusion relations, duals, matrix mappings, and compact operators.
Erdem and Demiriz [20, 21] and Erdem [22] utilized the Catalan numbers to define an infinite 4D Catalan matrix and introduced new Catalan double sequence spaces as the domains of this matrix in the classical double sequence spaces $\mathscr{C}_r$, $\mathscr{C}_p$, and $\mathscr{C}_{bp}$. As a continuation of the aforementioned studies, the present work defines a novel double sequence space as the domain of the 4D Catalan matrix in the space $\mathscr{L}_u$ consisting of absolutely convergent double series. Additionally, the study investigates various algebraic and topological properties of this newly introduced space, computes its α-, $\beta(bp)$-, and γ-duals, and characterizes certain classes of matrix transformations from this space into classical double sequence spaces.

2. The Double Sequence Space $\mathfrak{C}(\mathscr{L}_u)$

The 4D Catalan matrix $\mathfrak{C}=\mathfrak{c}_{jkmn}$ was introduced by Erdem and Demiriz [20] as

$$\mathfrak{c}_{jkmn}:=\begin{cases}\dfrac{C_mC_nC_{j-m}C_{k-n}}{C_{j+1}C_{k+1}} & , \quad \begin{array}{l}0\le m\le j\\ 0\le n\le k\end{array}, \\[2ex] 0 & , \quad \text{otherwise}\end{cases}$$

for all $j,k,m,n\in\mathbb{N}$. Based on this definition, the $\mathfrak{C}$-transform of a double sequence $u=(u_{mn})\in\Omega$ results in a new sequence $v=(v_{mn})$ computed as

$$v_{jk}:=(\mathfrak{C}u)_{jk}=\frac{1}{C_{j+1}C_{k+1}}\sum_{m,n=0}^{j,k}C_mC_nC_{j-m}C_{k-n}u_{mn}. \tag{2.1}$$

Furthermore, the inverse matrix $(\mathfrak{C})^{-1}=(\mathfrak{c}^{-1}_{jkmn})$ is given by

$$\mathfrak{c}^{-1}_{jkmn}:=\begin{cases}(-1)^{j+k-(m+n)}\dfrac{C_{m+1}C_{n+1}}{C_jC_k}H_{j-m}H_{k-n} & , \quad \begin{array}{l}0\le m\le j\\ 0\le n\le k\end{array}, \\[2ex] 0 & , \quad \text{otherwise,}\end{cases} \tag{2.2}$$

where $H_0=1$ and H_j defined by

$$H_j=\begin{vmatrix} C_1 & C_0 & 0 & 0 & 0 & \dots & 0\\ C_2 & C_1 & C_0 & 0 & 0 & \dots & 0\\ C_3 & C_2 & C_1 & C_0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ C_j & C_{j-1} & C_{j-2} & C_{j-3} & C_{j-4} & \dots & C_1\end{vmatrix}.$$

Consequently, the original sequence $u=(u_{jk})$ can be recovered from $\nu=(\mathfrak{C}u)$ via the inverse operation:

$$u_{jk}=\sum_{m,n=0}^{j,k}(-1)^{j+k-(m+n)}\frac{C_{m+1}C_{n+1}}{C_jC_k}H_{j-m}H_{k-n}\nu_{mn}. \tag{2.3}$$

We now define the set $\mathfrak{C}(\mathscr{L}_u)$ as follows:

$$\mathfrak{C}(\mathscr{L}_u)=\left\{u=(u_{mn})\in\Omega:\sum_{j,k}\left|\frac{1}{C_{j+1}C_{k+1}}\sum_{m,n=0}^{j,k}C_mC_nC_{j-m}C_{k-n}u_{mn}\right|<\infty\right\}.$$

Theorem 2.1. *$\mathfrak{C}(\mathscr{L}_u)$ forms a linear space under the coordinatewise addition and scalar multiplication of double sequences.*

Theorem 2.2. *The space $\mathfrak{C}(\mathscr{L}_u)$ equipped with the norm*

$$\|u\|_{\mathfrak{C}(\mathscr{L}_u)}=\sum_{j,k}\left|\frac{1}{C_{j+1}C_{k+1}}\sum_{m,n=0}^{j,k}C_mC_nC_{j-m}C_{k-n}u_{mn}\right|$$

is a Banach space.

Theorem 2.3. *$\mathfrak{C}(\mathscr{L}_u)$ is linearly norm isomorphic to $\mathscr{L}_u$.*

Proof. To establish the isomorphism, we define a linear transformation

$$\mathscr{T}:\mathfrak{C}(\mathscr{L}_u)\to\mathscr{L}_u,\quad u\mapsto\mathscr{T}u=\left\{(\mathfrak{C}u)_{jk}\right\}.$$

For any $u,\tilde{u}\in\mathfrak{C}(\mathscr{L}_u)$ and scalars $\lambda,\mu\in\mathbb{C}$, we have:

$$\mathscr{T}(\lambda u+\mu\tilde{u})=\mathfrak{C}(\lambda u+\mu\tilde{u})=\lambda\mathfrak{C}u+\mu\mathfrak{C}\tilde{u}=\lambda\mathscr{T}(u)+\mu\mathscr{T}(\tilde{u}),$$

which shows that $\mathscr{T}$ is a linear operator.
Additionally, if $\mathscr{T}u=0$, then each entry of the transformed sequence satisfies:

$$\frac{1}{C_{j+1}C_{k+1}}\sum_{m,n=0}^{j,k}C_mC_nC_{j-m}C_{k-n}u_{mn}=0\quad\text{for all } j,k.$$

By the structure of the Catalan transformation, this implies that all $u_{mn}=0$, hence u is the zero sequence, and so $\mathscr{T}$ is injective.
Now, for any $\nu=(\nu_{jk})\in\mathscr{L}_u$, define $u=(u_{jk})$ by the inverse transformation:

$$u_{jk}=\sum_{m,n=0}^{j,k}(-1)^{j+k-(m+n)}\frac{C_{m+1}C_{n+1}}{C_jC_k}H_{j-m}H_{k-n}\nu_{mn}.$$

Then, we find:

$$\begin{aligned}\|u\|_{\mathfrak{C}(\mathscr{L}_u)}=\|\mathscr{T}u\|_{\mathscr{L}_u}&=\sum_{j,k}\left|\frac{1}{C_{j+1}C_{k+1}}\sum_{m,n=0}^{j,k}C_mC_nC_{j-m}C_{k-n}u_{mn}\right|\\&=\sum_{j,k}\left|\nu_{jk}\right|=\|\nu\|_{\mathscr{L}_u}<\infty.\end{aligned}$$

This confirms that $u\in\mathfrak{C}(\mathscr{L}_u)$ and that $\mathscr{T}$ is surjective and norm-preserving. Therefore, $\mathscr{T}$ establishes a linear norm isomorphism between $\mathfrak{C}(\mathscr{L}_u)$ and $\mathscr{L}_u$. □

3. α-, $\beta(bp)$-, and γ-duals of the Double Sequence Space $\mathfrak{C}(\mathscr{L}_u)$

Let Ψ be an arbitrary double sequence space. Then, its α-, $\beta(\vartheta)$-, and γ-duals are defined respectively as follows:

$$\Psi^{\alpha}:=\left\{y=(y_{mn})\in\Omega:\sum_{m,n}|y_{mn}u_{mn}|<\infty\quad\text{for all }(u_{mn})\in\Psi\right\},$$

$$\Psi^{\beta(\vartheta)}:=\left\{y=(y_{mn})\in\Omega:\vartheta\text{-}\sum_{m,n}y_{mn}u_{mn}\text{ exists for all }(u_{mn})\in\Psi\right\},$$

$$\Psi^{\gamma}:=\left\{y=(y_{mn})\in\Omega:\sup_{j,k\in\mathbb{N}}\left|\sum_{m=0}^{j}\sum_{n=0}^{k}y_{mn}u_{mn}\right|<\infty\quad\text{for all }(u_{mn})\in\Psi\right\}.$$

Lemma 3.1. *[14] Let $B=(b_{jkmn})$ be a 4D matrix. Then the following conditions hold:*

(i) The matrix B belongs to the class $(\mathscr{L}_u:\mathscr{M}_u)$ if and only if

$$M=\sup_{j,k,m,n\in\mathbb{N}}|b_{jkmn}|<\infty. \tag{3.1}$$

(ii) $B \in (\mathcal{L}_u : \mathcal{L}_u)$ *if and only if*

$$\sup_{m,n \in \mathbb{N}} \sum_{j,k} |b_{jkmn}| < \infty.$$

(iii) B *maps* $\mathcal{L}_u$ *into* $\mathcal{C}_{bp}$*, i.e.,* $B \in (\mathcal{L}_u : \mathcal{C}_{bp})$*, if and only if condition (3.1) holds and there exists at least one sequence* $(\delta_{mn}) \in \Omega$ *such that*

$$bp\text{-}\lim_{j,k \to \infty} b_{jkmn} = \delta_{mn}.$$

Theorem 3.2. *Define the set*

$$\xi_1 = \left\{ y = (y_{jk}) \in \Omega : \sup_{m,n \in \mathbb{N}} \sum_{j,k} \left| \frac{C_{m+1}C_{n+1}}{C_j C_k} H_{j-m} H_{k-n} y_{jk} \right| < \infty \right\}.$$

Then, the α*-dual of* $\mathfrak{C}(\mathcal{L}_u)$ *is equal to* ξ_1*, that is,*

$$(\mathfrak{C}(\mathcal{L}_u))^{\alpha} = \xi_1.$$

Proof. Consider the 4D matrix $G = (g_{jkmn})$ for all $j,k,m,n \in \mathbb{N}$ described by the rule:

$$g_{jkmn} := \begin{cases} (-1)^{j+k-(m+n)} \dfrac{C_{m+1}C_{n+1}}{C_j C_k} H_{j-m} H_{k-n} y_{jk} & , \quad \begin{array}{l} 0 \le m \le j \\ 0 \le n \le k \end{array}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Using the transformation given in equation (2.2), we express u_{jk} as

$$u_{jk} = \sum_{m,n=0}^{j,k} (-1)^{j+k-(m+n)} \frac{C_{m+1}C_{n+1}}{C_j C_k} H_{j-m} H_{k-n} v_{mn},$$

so that the product $y_{jk}u_{jk}$ becomes

$$\begin{aligned} y_{jk}u_{jk} &= y_{jk} \sum_{m,n=0}^{j,k} (-1)^{j+k-(m+n)} \frac{C_{m+1}C_{n+1}}{C_j C_k} H_{j-m} H_{k-n} v_{mn} \\ &= \sum_{m,n=0}^{j,k} \left((-1)^{j+k-(m+n)} \frac{C_{m+1}C_{n+1}}{C_j C_k} H_{j-m} H_{k-n} y_{jk} \right) v_{mn} \\ &= (Gv)_{jk}. \end{aligned} \tag{3.2}$$

Hence, from (3.2), $yu = (y_{jk}u_{jk}) \in \mathcal{L}_u$ while $u \in \mathfrak{C}(\mathcal{L}_u)$ iff $Gv \in \mathcal{L}_u$ while $v \in \mathcal{L}_u$. This observation implies that $y = (y_{jk})$ belongs to the α-dual of $\mathfrak{C}(\mathcal{L}_u)$ if and only if $G \in (\mathcal{L}_u : \mathcal{L}_u)$.
By invoking Lemma 3.1(ii), which gives necessary and sufficient conditions for such mappings, we deduce that

$$(\mathfrak{C}(\mathcal{L}_u))^{\alpha} = \xi_1.$$

□

Theorem 3.3. *Let the sets* ξ_2 *and* ξ_3 *be defined respectively as*

$$\xi_2 = \left\{ y = (y_{jk}) \in \Omega : \sup_{j,k,m,n \in \mathbb{N}} |\sigma(j,k,i,l,m,n)| < \infty \right\},$$

$$\xi_3 = \left\{ y = (y_{jk}) \in \Omega : bp\text{-}\lim_{j,k \to \infty} \sigma(j,k,i,l,m,n) \text{ exists} \right\},$$

where $\sigma(j,k,i,l,m,n) = \sum_{i=m}^{j} \sum_{l=n}^{k} (-1)^{i+l-(m+n)} \frac{C_{m+1}C_{n+1}}{C_i C_l} H_{i-m} H_{l-n} y_{il}$.
Then, the $\beta(bp)$*-dual of the sequence space* $\mathfrak{C}(\mathcal{L}_u)$ *is given by the intersection:*

$$(\mathfrak{C}(\mathcal{L}_u))^{\beta(bp)} = \xi_2 \cap \xi_3.$$

Proof. Let $y = (y_{jk}) \in \Omega$ and $u = (u_{jk}) \in \mathfrak{C}(\mathcal{L}_u)$. From the transformation given in (2.3), there exists a double sequence $v = (v_{jk}) \in \mathcal{L}_u$.
Define an infinite 4D matrix $A = (a_{jkmn})$ for all $j,k,m,n \in \mathbb{N}$ by:

$$a_{jkmn} := \begin{cases} \displaystyle\sum_{i=m}^{j} \sum_{l=n}^{k} (-1)^{i+l-(m+n)} \frac{C_{m+1}C_{n+1}}{C_i C_l} H_{i-m} H_{l-n} y_{il} & , \quad \begin{array}{l} 0 \le m \le j \\ 0 \le n \le k \end{array}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Using equation (2.3), we compute:

$$\begin{aligned} z_{jk} &= \sum_{m,n=0}^{j,k} y_{mn} u_{mn} \\ &= \sum_{m,n=0}^{j,k} y_{mn} \left(\sum_{i,l=0}^{m,n} (-1)^{m+n-(i+l)} \frac{C_{i+1}C_{l+1}}{C_m C_n} H_{m-i} H_{n-j} v_{il} \right) \\ &= \sum_{m,n=0}^{j,k} \left(\sum_{i=m}^{j} \sum_{l=n}^{k} (-1)^{i+l-(m+n)} \frac{C_{m+1}C_{n+1}}{C_i C_l} H_{i-m} H_{j-n} y_{il} \right) v_{mn} \\ &= (Ay)_{jk}. \end{aligned}$$

This implies that, for any $u \in \mathfrak{C}(\mathscr{L}_u)$ and corresponding $v \in \mathscr{L}_u$, the product $yu = (y_{jk}u_{jk})$ belongs to $\mathscr{CS}_{bp}$ if and only if $z = (z_{jk}) = (Ay)_{jk} \in \mathscr{C}_{bp}$.
Therefore, $y = (y_{jk})$ belongs to the $\beta(bp)$-dual of $\mathfrak{C}(\mathscr{L}_u)$ if and only if A defines a matrix in the class $(\mathscr{L}_u : \mathscr{C}_{bp})$.
By invoking part (iii) of Lemma 3.1, we conclude:

$$(\mathfrak{C}(\mathscr{L}_u))^{\beta(bp)} = \xi_2 \cap \xi_3.$$

□

Theorem 3.4. *The γ-dual of the sequence space $\mathfrak{C}(\mathscr{L}_u)$ is given by*

$$(\mathfrak{C}(\mathscr{L}_u))^{\gamma} = \xi_2.$$

Proof. The proof follows similar lines to that of Theorem 3.3, with the key difference being the use of Lemma 3.1(i), which characterizes the matrix class $(\mathscr{L}_u : \mathscr{M}_u)$.

□

4. Certain Matrix Transformations Related to the Sequence Space $\mathfrak{C}(\mathscr{L}_u)$

In this section, we provide necessary and sufficient conditions for an infinite 4D matrix $B = (b_{jkmn})$ to belong to the matrix transformation classes $(\mathfrak{C}(\mathscr{L}_u) : \Lambda)$ where $\Lambda \in \{\mathscr{M}_u, \mathscr{C}_{bp}, \mathscr{L}_q\}$.

Theorem 4.1. *Let $B = (b_{jkmn})$ be an 4D infinite matrix. Then B belongs to the matrix class $(\mathfrak{C}(\mathscr{L}_u) : \mathscr{M}_u)$ if and only if the following two conditions are simultaneously satisfied:*

$$B_{jk} \in (\mathfrak{C}(\mathscr{L}_u))^{\beta(bp)}, \tag{4.1}$$

and

$$\sup_{j,k,m,n\in\mathbb{N}} \left| \sum_{r=m}^{\infty} \sum_{d=n}^{\infty} (-1)^{r+d-(m+n)} \frac{C_{m+1}C_{n+1}}{C_r C_d} H_{r-m} H_{d-n} b_{jkrd} \right| < \infty. \tag{4.2}$$

Proof. Let $B \in (\mathfrak{C}(\mathscr{L}_u) : \mathscr{M}_u)$ and $u = (u_{mn})$ be any element of the sequence space $\mathfrak{C}(\mathscr{L}_u)$. According to the transformation identity (2.1), there subsists a double sequence $v = (v_{mn}) \in \mathscr{L}_u$ such that the entries of u can be expressed accordingly. Then Bu exists and in $\mathscr{M}_u$, therefore $B_{jk} \in (\mathfrak{C}(\mathscr{L}_u))^{\beta(bp)}$. Keeping in mind the transformation relation given in (2.3), the partial sums of the matrix transformation Bu over $m \le i$ and $n \le l$ for any $j,k,i,l \in \mathbb{N}$ can be expressed as:

$$\begin{aligned} (Bu)_{jk}^{[i,l]} &= \sum_{m=0}^{i} \sum_{n=0}^{l} b_{jkmn} u_{mn} \\ &= \sum_{m=0}^{i} \sum_{n=0}^{l} b_{jkmn} \left[\sum_{r,d=0}^{m,n} (-1)^{m+n-(r+d)} \frac{C_{r+1}C_{d+1}}{C_m C_n} H_{m-r} H_{n-d} v_{rd} \right] \\ &= \sum_{m=0}^{i} \sum_{n=0}^{l} \left[\sum_{r=m}^{i} \sum_{d=n}^{l} (-1)^{r+d-(m+n)} \frac{C_{m+1}C_{n+1}}{C_r C_d} H_{r-m} H_{d-n} b_{jkrd} \right] v_{mn}. \end{aligned} \tag{4.3}$$

Now, define a new 4D matrix $H = (h_{jkmn})$ by the rule:

$$h_{jkmn} := \begin{cases} \sum_{r=m}^{\infty} \sum_{d=n}^{\infty} (-1)^{r+d-(m+n)} \frac{C_{m+1}C_{n+1}}{C_r C_d} H_{r-m} H_{d-n} b_{jkrd} & , \quad \begin{matrix} 0 \le m \le j \\ 0 \le n \le k \end{matrix} , \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Next, by taking the limit on (4.3) as $i,l \to \infty$, we obtain that $Bu = Hv$. So, $B \in (\mathfrak{C}(\mathscr{L}_u) : \mathscr{M}_u)$ if and only if $H \in (\mathscr{L}_u : \mathscr{M}_u)$. Thus, (4.2) holds.
On the other hand, let the conditions (4.1) and (4.2) hold and take $u = (u_{mn}) \in \mathfrak{C}(\mathscr{L}_u)$ with $v \in \mathscr{L}_u$. Then, Bu exists because (4.1) holds. From (4.3), it is seen that $Bu = Hv$ as $i,l \to \infty$. By the condition (4.2), we see that $H \in (\mathscr{L}_u : \mathscr{M}_u)$ and consequently $B \in (\mathfrak{C}(\mathscr{L}_u) : \mathscr{M}_u)$.
The following theorems are presented without proof since their demonstrations follow a reasoning structure analogous to the one used in the proof of Theorem 4.1.

Theorem 4.2. *Let $B=(b_{jkmn})$ be an infinite 4D matrix. Then, a necessary and sufficient condition for B to belong to the class $(\mathfrak{C}(\mathscr{L}_u):\mathscr{C}_{bp})$ is that conditions (4.1) and (4.2) are satisfied, and in addition, there exists at least one double sequence $(\delta_{mn})\in\Omega$ such that the following limit holds:*

$$bp-\lim_{j,k\to\infty}\sum_{r=m}^{\infty}\sum_{d=n}^{\infty}(-1)^{r+d-(m+n)}\frac{C_{m+1}C_{n+1}}{C_rC_d}H_{r-m}H_{d-n}b_{jkrd}=\delta_{mn},$$

for all $m,n\in\mathbb{N}$.

Theorem 4.3. *Let $B=(b_{jkmn})$ be a 4D infinite matrix. Then B is an element of the class $(\mathfrak{C}(\mathscr{L}_u):\mathscr{L}_u)$ if and only if condition (4.1) holds together with the following summability condition:*

$$\sup_{j,k\in\mathbb{N}}\sum_{m,n}\left|\sum_{r=m}^{\infty}\sum_{d=n}^{\infty}(-1)^{r+d-(m+n)}\frac{C_{m+1}C_{n+1}}{C_rC_d}H_{r-m}H_{d-n}b_{jkrd}\right|<\infty.$$

5. Conclusion

In this study, we have introduced and thoroughly examined a novel double sequence space, denoted by $\mathfrak{C}(\mathscr{L}_u)$, which is defined as the domain of a 4D Catalan matrix in the space $\mathscr{L}_u$ of absolutely summable double sequences. The construction of this space was motivated by the rich combinatorial structure of the Catalan numbers and their well-known recurrence relations and closed-form expressions, which provided a foundation for developing the associated matrix transformation.
We have established that the space $\mathfrak{C}(\mathscr{L}_u)$ forms a complete normed linear space, i.e., a Banach space. Furthermore, it was proven that $\mathfrak{C}(\mathscr{L}_u)$ is linearly norm isomorphic to $\mathscr{L}_u$, implying that the newly defined space preserves many of the essential topological and algebraic characteristics of $\mathscr{L}_u$, while simultaneously introducing a new structural framework based on the Catalan sequence.
A significant portion of the study was devoted to the investigation of the dual spaces of $\mathfrak{C}(\mathscr{L}_u)$. In particular, we characterized its α-dual, $\beta(bp)$-dual, and γ-duals.
Additionally, we analyzed necessary and sufficient conditions under which infinite 4D matrices map $\mathfrak{C}(\mathscr{L}_u)$ into classical double sequence spaces such as $\mathscr{M}_u$, $\mathscr{C}_{bp}$, and $\mathscr{L}_u$.
The results obtained herein contribute to the ongoing development of sequence space theory by providing a new class of double sequence spaces constructed via special integer sequences, in this case, the Catalan numbers. The framework established in this study not only enriches the theoretical landscape of summability and matrix transformation theory but also opens new avenues for further exploration. Potential future research directions may include the construction of related spaces via other combinatorial sequences.
In conclusion, this work demonstrates how classical combinatorial sequences such as the Catalan numbers can serve as a fertile basis for defining and analyzing novel functional structures in the realm of double sequence spaces and operator theory.

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