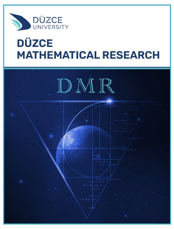

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Research Article

New vector fields and planes of framed curves in Euclidean 4-space

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Abstract. In this study, we define new Darboux vectors for curves with singular points in Euclidean 4-space. By using these vectors, we construct new planes and determine curves lying in these planes. Subsequently, we give characterizations and corollaries related to these curves.

Keywords. Darboux vector, framed curves, singular points

1. Introduction

In differential geometry, vector fields have long played a fundamental role in the study of curves and surfaces not only in Euclidean 3-space but also in higher-dimensional spaces. Among the most commonly studied are natural vector fields in space, Frenet vector fields along curves, the Darboux vector field of a curve in three-space, as well as the normal and tangent vector fields of surfaces and hypersurfaces. These vector fields are instrumental in characterizing the intrinsic geometric properties of the associated objects.

In particular, the Frenet frame, which forms an orthonormal basis along a curve, encapsulates complete information about the curve's local geometry. These formulas can also be expressed using vector products via the Darboux vector field, which identifies the instantaneous axis of rotation of the Frenet frame. Furthermore, the Darboux vector field allows for the construction of special ruled surfaces, such as the rectifying developable surface, where the base curve remains a geodesic at all points [1]. Due to these properties, the Darboux vector field plays a significant role in the study of curves in three-dimensional Euclidean space $\mathbb{R}^3$. Although generalized formulations of the Darboux vector in $\mathbb{R}^n$, and in particular $\mathbb{R}^4$, have been proposed in the literature [2], these formulations do not fully meet the specific requirements. With reference to the significance of the Darboux vector, Düldül defined new vector fields along space curves in $\mathbb{R}^4$ with nonvanishing curvatures. These fields allow the Frenet formulas to be rewritten using vector products. Using these new vector fields, Dldl subsequently defined special planes, curves, and ruled hypersurfaces in $\mathbb{R}^4$ [3].

Numerous studies in the literature highlight the significant role of the Darboux vector, which is obtained via the Frenet frame of regular curves. However, in recent years, curves with singular points, for which the Frenet frame cannot be constructed, have frequently appeared in various applications. Honda et al. introduced the concept of framed curves to explore the geometric properties of curves with singular points. Framed curves are a natural generalization of Frenet curves. Following this, they presented the idea of framed immersion. For a detailed account of framed curves and framed surfaces, see [4–14].

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Honda extended the concept of the Darboux vector to curves with singular points, in a manner analogous to that for regular curves and constructed the rectifying developable [15].

In this study, we introduce new Darboux vectors for singular curves in $\mathbb{R}^4$. These vectors allow us to define new planes and identify curves lying on these planes. Subsequently, we give characterizations and corollaries related to these curves.

2. Preliminaries

Let $\mathbb{R}^4$ be four-dimensional Euclidean space equipped with the standard inner product:

$$<x,y>=\sum_{i=1}^{4} x_i y_i,$$

where $x=(x_1,x_2,x_3,x_4)\,,y=(y_1,y_2,y_3,y_4)\in\mathbb{R}^4$. The norm of a vector $x\in\mathbb{R}^4$ is given by $\|x\|=\sqrt{<x,x>}$. The vector product in $\mathbb{R}^4$ is defined by

$$x\times y\times z=\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

where $x=(x_1,x_2,x_3,x_4)\,,y=(y_1,y_2,y_3,y_4)\,,z=(z_1,z_2,z_3,z_4)\in\mathbb{R}^4$ and e_1,e_2,e_3,e_4 are the canonical basis vectors of $\mathbb{R}^4$ [16].

We define a $6-$dimensional smooth manifold as:

$$\Delta_3=\left\{\mu=(\mu_1,\mu_2,\mu_3)\in\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4 \middle| <\mu_i,\mu_j>=\delta_{ij}, i,j=1,2,3\right\}.$$

A unit vector ν is then defined by $\nu=\mu_1\times\mu_2\times\mu_3$, satisfying $\det(\nu,\mu_1,\mu_2,\mu_3)=1$ and $(\nu,\mu)\in\Delta_4$.

Definition 2.1. *A map $(\gamma,\mu):I\to\mathbb{R}^4\times\Delta_3$ is called a framed curve if $<\gamma'(s),\mu_i(s)>=0$ for all $s\in I$ and $i=1,2,3$. $\gamma:I\to\mathbb{R}^4$ is called as a framed base curve if there exists $\mu:I\to\Delta_3$ such that (γ,μ) is a framed curve, [4].*

By following similar way as the curvatures of regular curve, we can define smooth functions for framed curve. Given a framed base curve (γ,μ) with an associated moving frame $\{\nu(s),\mu(s)\}$, the Frenet-Serret type formula is given by

$$\begin{bmatrix} \mu_1'(s) \\ \mu_2'(s) \\ \mu_3'(s) \\ \nu'(s) \end{bmatrix}=\begin{bmatrix} 0 & f(s) & g(s) & h(s) \\ -f(s) & 0 & j(s) & k(s) \\ -g(s) & -j(s) & 0 & l(s) \\ -h(s) & -k(s) & -l(s) & 0 \end{bmatrix}\begin{bmatrix} \mu_1(s) \\ \mu_2(s) \\ \mu_3(s) \\ \nu(s) \end{bmatrix}$$

where $f(s),g(s),h(s),j(s),k(s),l(s)$ are smooth curvature functions. Moreover, there exists a smooth function $\alpha(s)$ such that $\gamma'(s)=\alpha(s)\nu(s)$. (f,g,h,j,k,l,α) are called the framed curvatures of γ. A point $s_0\in I$ is a singular point if $\alpha(s_0)=0$.

Theorem 2.2. *Let (f,g,h,j,k,l,α) be smooth functions. There exists a framed curve (γ,μ) with these framed curvatures [4].*

Theorem 2.3. *Given framed curves (γ,μ) and $(\widetilde{\gamma},\eta)$, they are congruent as framed curves, if their curvatures coincide [4].*

By using smooth Euler angles θ,φ,ψ, one can define a new orthonormal frame $\eta=(\eta_1,\eta_2,\eta_3)$ by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}=A(\theta,\varphi,\psi)\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

where $A(\theta,\varphi,\psi)$ is a rotation matrix depending on the Euler angles. It follows that the new vector $\widetilde{\nu}=\eta_1\times\eta_2\times\eta_3=\nu$, so the pair (γ,η) is still a framed curve.

Assume that the following relations exist between the Euler angles and the framed curvatures;

$$\frac{\tan\psi}{\cos\theta}=l\sin\varphi-k\cos\varphi$$

and

$$h=\cot\theta\,(l\cos\varphi+k\sin\varphi),$$

then, the adapted frame $\{\nu(s),\eta_1(s),\eta_2(s),\eta_3(s)\}$ along $\gamma(s)$ is given by

$$\begin{bmatrix}\nu'(s)\\ \eta_1'(s)\\ \eta_2'(s)\\ \eta_3'(s)\end{bmatrix}=\begin{bmatrix}0 & p(s) & 0 & 0\\ -p(s) & 0 & q(s) & 0\\ 0 & -q(s) & 0 & r(s)\\ 0 & 0 & -r(s) & 0\end{bmatrix}\begin{bmatrix}\nu(s)\\ \eta_1(s)\\ \eta_2(s)\\ \eta_3(s)\end{bmatrix} \tag{2.1}$$

where $(p(s),q(s),r(s),\alpha(s))$ are framed curvatures of $\gamma(s)$, which relate to the original ones as follows:

$$\begin{aligned}
p&=-h\sec\theta\sec\psi,\\
q&=-\left(j-\varphi'\right)\sin\theta-\psi',\\
r&=\frac{\cos\theta}{\cos\psi}\left(j-\varphi'\right),\\
f&=-\sin\varphi\left(\theta'-r\sin\psi\right),\\
g&=-\cos\varphi\left(\theta'-r\sin\psi\right),\\
j&=r\frac{\cos\psi}{\cos\theta}+\theta'.
\end{aligned}$$

The vectors ν,η_1,η_2,η_3 are referred to as the generalized tangent, generalized principal normal, generalized first binormal, and generalized second binormal vectors of framed curve, respectively.

Theorem 2.4. *Let $(\gamma,\eta):I\to\mathbb{R}^4\times\Delta_3$ be a framed curve with non-zero framed curvatures p,q, and r. Then γ is congruent to a framed rectifying curve if and only if the following condition is satisfied [17]:*

$$\frac{\beta(t)\,p(t)\,r(t)}{q(t)}+\left(\frac{\beta'(t)\,p(t)+\beta(t)\,p'(t)}{q^2(t)\,r(t)}\right)'=0,\qquad \beta(t)=\int\alpha(t)\,dt.$$

Theorem 2.5. *Let $(\gamma,\eta):I\to\mathbb{R}^4\times\Delta_3$ be framed curve with non-zero framed curvatures p,q, and r. Then γ is congruent to a framed osculating curve if and only if the following condition is satisfied [17]:*

$$-c\frac{p(t)\,r(t)}{q(t)}-\left(c\frac{r'(t)\,q(t)-r(t)\,q'(t)}{q^2(t)\,p(t)}\right)'=\alpha(t),\qquad c\in\mathbb{R}.$$

3. New vector fields and planes of framed curves

In this section, we define some new Darboux vector fields along a curve which has any singular point in $\mathbb{R}^4$. By considering these vector fields, we define some new special curves, and we give their characterizations. Furthermore, we investigate the relationships between these newly defined curves and rectifying curves, as well as osculating curves.

Let γ be a framed curve in $\mathbb{R}^4$ with nonzero curvatures p,q,r and let us denote $\{\nu,\eta_1,\eta_2,\eta_3\}$ its framed frame. We now define the following vector fields along γ:

$$\begin{aligned}
\mathfrak{D}_1&=\eta_3,\\
\mathfrak{D}_2&=q\nu+p\eta_2,\\
\mathfrak{D}_3&=r\eta_1+q\eta_3,\\
\mathfrak{D}_4&=\nu.
\end{aligned}$$

It is worth noting that $\mathfrak{D}_1$ and $\mathfrak{D}_4$ coincide with two of the framed vectors of the curve, and the set $\{\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4\}$ remains linearly independent along γ. Moreover, the pairs $\{\mathfrak{D}_1, \mathfrak{D}_2\}, \{\mathfrak{D}_3, \mathfrak{D}_4\}$, and $\{\mathfrak{D}_2, \mathfrak{D}_3\}$ form mutually orthogonal sets. We denote the subspaces spanned by these pairs as the $\mathfrak{D}_1\mathfrak{D}_2-$plane, $\mathfrak{D}_3\mathfrak{D}_4-$plane and $\mathfrak{D}_2\mathfrak{D}_3-$plane, respectively. Therefore, as a response to the previously posed question, Equation (framed formula) can be reformulated as follows:

$$\begin{aligned} \nu' &= \mathfrak{D}_1 \times \mathfrak{D}_2 \times \nu, \\ \eta_1' &= \mathfrak{D}_1 \times \mathfrak{D}_2 \times \eta_1, \\ \eta_2' &= \mathfrak{D}_3 \times \mathfrak{D}_4 \times \eta_2, \\ \eta_3' &= \mathfrak{D}_3 \times \mathfrak{D}_4 \times \eta_3. \end{aligned} \tag{3.1}$$

As observed from Equation (3.1), the framed vectors ν and η_1 rotate around the $\mathfrak{D}_1\mathfrak{D}_2-$plane, while the vectors η_2 and η_3 rotate around the $\mathfrak{D}_3\mathfrak{D}_4-$plane. These two planes collectively serve a role analogous to that of the Darboux vector in $\mathbb{R}^4$ [15, 18]

Definition 3.1. *Let $(\gamma, \eta) : I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p, q, r and $\{\nu, \eta_1, \eta_2, \eta_3\}$ be its Framed frame.*

A curve γ is referred to as a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve if its position vector lies in its $\mathfrak{D}_1\mathfrak{D}_2-$plane.

A curve γ is referred to as a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve if its position vector lies in its $\mathfrak{D}_3\mathfrak{D}_4-$plane.

A curve γ is referred to as a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve if its position vector lies in its $\mathfrak{D}_2\mathfrak{D}_3-$plane.

Theorem 3.2. *Let $(\gamma, \eta) : I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p, q, r. Then γ is a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve if and only if its curvatures satisfy the following differential equation:*

$$\left(\left(\frac{1}{r(t)} \left(\frac{p(t)}{q(t)} \left(\int \alpha(t) dt \right) \right) \right)' \right)' + \frac{p(t) r(t)}{q(t)} \left(\int \alpha(t) dt \right) = 0. \tag{3.2}$$

Proof. Let γ be a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve with nonvanishing curvatures p, q, r. By the definition of a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve, the position vector of γ can be expressed as

$$\gamma(t) = \lambda(t) \mathfrak{D}_1(t) + \mu(t) \mathfrak{D}_2(t) \tag{3.3}$$

for some smooth functions $\lambda(t)$ and $\mu(t)$. Differentiating Equation (3.3) with respect to t, and applying the framed formulas, yields the following system of differential equations:

$$\begin{aligned} &\alpha(t) = \mu'(t) q(t) + \mu(t) q'(t), \\ &-\lambda(t) r(t) + \mu'(t) p(t) + \mu(t) p'(t) = 0, \\ &\lambda'(t) + \mu(t) p(t) r(t) = 0. \end{aligned} \tag{3.4}$$

Solving the first two equations in system (3.4) yields the following expressions:

$$\mu(t) = \frac{1}{q(t)} \left(\int \alpha(t) dt \right), \ \lambda(t) = \frac{1}{r(t)} \left(\frac{p(t)}{q(t)} \left(\int \alpha(t) dt \right) \right)'$$

where c is a constant of integration. Substituting these expressions into the third equation of system (3.4) leads directly to the desired result.

Conversely, assume that the equation given in (3.2) holds. Consider the vector field

$$X(t) = \gamma(t) - \frac{1}{r(t)} \left(\frac{p(t)}{q(t)} \left(\int \alpha(t) dt \right) \right)' \mathfrak{D}_1(t) - \frac{1}{q(t)} \left(\int \alpha(t) dt \right) \mathfrak{D}_2(t).$$

Differentiating $X(t)$ with respect to t yields the zero vector, implying that $X(t)$ is constant along the curve. Hence, $\gamma(t)$ differs from a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve by a constant vector, and is therefore congruent to a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve. □

By considering Theorem 2.4, we obtain the following corollary.

Corollary 3.3. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p,q,r. Then γ is a framed $\mathfrak{D}_1\mathfrak{D}_2-$curve if and only if it is a framed rectifying curve.*

Theorem 3.4. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p,q,r. Then γ is a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve if and only if its curvatures satisfy the following differential equation:*

$$c\left(\frac{1}{p(t)}\left(\frac{r(t)}{q(t)}\right)'\right)' + \frac{cp(t)r(t)}{q(t)} + \alpha = 0, \tag{3.5}$$

where $c \in \mathbb{R}$.

Proof. Let γ be a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve with nonvanishing curvatures p,q,r. By the definition of a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve, the position vector of γ can be expressed as

$$\gamma(t) = \upsilon(t)\mathfrak{D}_3(t) + \omega(t)\mathfrak{D}_4(t) \tag{3.6}$$

for some smooth functions $\upsilon(t)$ and $\omega(t)$. Differentiating Equation (3.6) with respect to t, and applying the framed formulas, yields the following system of differential equations:

$$\begin{aligned}
&(\upsilon(t)q(t))' = 0,\\
&(\upsilon(t)r(t))' + \omega(t)p(t) = 0,\\
&\alpha(t) = -\upsilon(t)p(t)r(t) + \omega'(t).
\end{aligned} \tag{3.7}$$

Solving the first two equations in system (3.7) yields the following expressions:

$$\upsilon(t) = \frac{c}{q(t)},\ \omega(t) = -\frac{1}{p(t)}\left(\frac{cr(t)}{q(t)}\right)'$$

where c is a constant of integration. Substituting these expressions into the third equation of system (3.7) leads directly to the desired result.

Conversely, assume that the equation given in (3.5) holds. Consider the vector field

$$Y(t) = \gamma(t) - \frac{c}{q(t)}\mathfrak{D}_3(t) + \frac{1}{p(t)}\left(\frac{cr(t)}{q(t)}\right)'\mathfrak{D}_4(t).$$

Differentiating $Y(t)$ with respect to t yields the zero vector, implying that $Y(t)$ is constant along the curve. Hence, $\gamma(t)$ differs from a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve by a constant vector, and is therefore congruent to a framed $\mathfrak{D}_3\mathfrak{D}_4-$curve. □

By considering Theorem 2.5, we obtain the following corollary.

Corollary 3.5. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p,q,r. Then γ is a $\mathfrak{D}_3\mathfrak{D}_4-$curve if and only if it is a framed osculating curve.*

Theorem 3.6. *Let $(\gamma,\eta): I \to \mathbb{R}^4 \times \Delta_3$ be framed curve with non-zero curvatures p,q,r. Then γ is a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve if and only if its curvatures satisfy the following differential equation:*

$$c_1\left(\frac{q(t)}{p(t)}\right)' - c_2p(t) - \alpha = 0,\ c_2\left(\frac{q(t)}{r(t)}\right)' + c_1r(t) = 0 \tag{3.8}$$

where $c_1, c_2 \in \mathbb{R}$.

Proof. Let γ be a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve with nonvanishing curvatures p,q,r. By the definition of a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve, the position vector of γ can be expressed as

$$\gamma(t) = \rho(t)\mathfrak{D}_2(t) + \sigma(t)\mathfrak{D}_3(t) \tag{3.9}$$

for some smooth functions $\rho(t)$ and $\sigma(t)$. Differentiating Equation (3.9) with respect to t, and applying the framed formulas, yields the following system of differential equations:

$$\begin{aligned} &(\rho(t)p(t))' = 0, \\ &(\sigma(t)r(t))' = 0, \\ &(\rho(t)q(t))' - \sigma(t)r(t)p(t) - \alpha(t) = 0, \\ &(\sigma(t)q(t))' + \rho(t)p(t)r(t) = 0. \end{aligned} \tag{3.10}$$

Solving the first two equations in system (3.10) yields the following expressions:

$$\rho(t) = \frac{c_1}{p(t)}, \ \sigma(t) = \frac{c_2}{r(t)},$$

where c_1, c_2 is a constant of integration. Substituting these expressions into the third and fourth equation of system (3.10) leads directly to the desired result.

Conversely, assume that the equation given in (3.8) holds. Consider the vector field

$$Z(t) = \gamma(t) - \frac{c_1}{p(t)}\mathfrak{D}_2(t) - \frac{c_2}{r(t)}\mathfrak{D}_3(t).$$

Differentiating $Z(t)$ with respect to t yields the zero vector, implying that $Z(t)$ is constant along the curve. Hence, $\gamma(t)$ differs from a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve by a constant vector, and is therefore congruent to a framed $\mathfrak{D}_2\mathfrak{D}_3-$curve. $\square$

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