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Stability conditions for non-autonomous linear differential equations in a Hilbert space via commutators

Michael Gil'[a*]

[a]*Department of Mathematics Ben Gurion University of the Negev P.0. Box 653, Beer-Sheva 84105, Israel*
[*]*Corresponding author E-mail: gilmi@bezeqint.net*

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Abstract

In a Hilbert space $\mathscr{H}$ we consider the equation $dx(t)/dt = (A+B(t))x(t)$ $(t \geq 0)$, where A is a constant bounded operator, and $B(t)$ is a piece-wise continuous function defined on $[0,\infty)$ whose values are bounded operators in $\mathscr{H}$. Conditions for the exponential stability are derived in terms of the commutator $AB(t)-B(t)A$. Applications to integro-differential equations are also discussed. Our results are new even in the finite dimensional case.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space with a scalar product $\langle .,. \rangle$, the norm $\|.\| = \sqrt{\langle .,. \rangle}$ and unit operator I. In addition, $\mathscr{B}(\mathscr{H})$ denotes the algebra of bounded linear operators in $\mathscr{H}$. For an $A \in \mathscr{B}(\mathscr{H})$, A^* is the adjoint operator, $\sigma(A)$ is the spectrum of A, $\Re A := (A+A^*)/2$, $\Im A := (A-A^*)/2i$, $\|A\|$ denotes the operator norm of A.
We consider the equation

$$\frac{du(t)}{dt} = (A+B(t))u(t) \quad (t \geq 0), \tag{1.1}$$

where A is a constant bounded operator and $B(t) : [0,\infty) \to \mathscr{B}(\mathscr{H})$ is a strongly piece-wise continuous function. A solution of (1.1) is a function $u(t)$, defined on $[0,\infty)$ with values in $\mathscr{H}$, absolutely continuous in t and satisfying the given initial condition and (1.1) almost everywhere on $[0,\infty)$. The existence of solutions follows from the a priory estimates proved below. We will say that equation (1.1) is exponentially stable, if there are positive constants M and ε, such that any solution $u(t)$ of (1.1) satisfies $\|u(t)\| \leq Me^{-\varepsilon t}\|u(0)\|$ $(t \geq 0)$.
Equation (1.1) can be considered as the equation

$$\frac{dx(t)}{dt} = C(t)x(t), \tag{1.2}$$

with a variable linear operator $C(t)$. This identification which is a common device in the theory of concrete differential or integro-differential equations when passing from a given equation to an abstract evolution equation turns out to be useful also here. Observe that $C(t)$ in the considered case has a special form: it is the sum of operators A and $B(t)$. This fact allows us to use the information about the coefficients more completely than the theory of differential equations (1.2) containing an arbitrary operator $C(t)$.
The basic method for the stability analysis of (1.2) is the direct Lyapunov method, cf. [2]. By that method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties.
For a selfadjoint operator S put $\Lambda(S) = \sup\, \sigma(S)$ and $\lambda(S) = \inf\, \sigma(S)$. So $\Lambda(\Re C(s)) = \sup\, \sigma(\Re C(s))$ and $\lambda(\Re C(s)) = \inf\, \sigma(\Re C(s))$. The important tool of the stability analysis is the Wintner inequalities [7, Theorem III.4.7]:

$$\exp[\int_s^t \lambda(\Re C(s_1))ds_1] \leq \frac{\|u(t)\|}{\|u(s)\|} \leq \exp[\int_s^t \Lambda(\Re C(s_1))ds_1] \quad (t \geq s \geq 0), \tag{1.3}$$

for any solution $x(t)$ of equation (1.2). If $C(t)$ is not dissipative, i.e. if $C(t)+C^*(t)$ is not negative definite for sufficiently large t, then the just mentioned inequalities do not give us stability conditions even in the case of a constant operator. In addition, in [14] the stability test

Email addresses: **gilmi@bezeqint.net (M. Gil')**

for (1.2) has been derived for equations whose operator coefficients have ”small” derivatives. The approach in [14] is the extension of the freezing method for ordinary differential equations. In this paper, we suggest a stability test via the commutator $K(t) = AB(t) - B(t)A$, which in the appropriate situations improves the published results. To the best of our knowledge, our results are new even in finite dimensional case, cf. [20].
As an illustrative example we consider a class of the so called Barbashin integro-differential equations, which play an essential role in numerous applications, in particular, in kinetic theory [5], transport theory [18], continuous mechanics [1], radiation theory [4], the dynamics of populations [21], etc.

2. The main result

Assume that

$$\alpha(A) := \sup \Re\sigma(A) < 0 \tag{2.1}$$

and put

$$W := 2\int_0^\infty e^{A^*t}e^{At}dt,\ \zeta(A) := 2\int_0^\infty \|e^{At}\| \int_0^t \|e^{As}\|\|e^{A(t-s)}\|ds\,dt$$

and

$$\psi(W,B(t)) := \begin{cases} \Lambda(\Re B(t))\|W\| & \text{if } \Lambda(\Re B(t)) > 0, \\ \Lambda(\Re B(t))\lambda(W) & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

Below we suggest estimates for $\|W\|$ and $\lambda(W)$. Furthermore, let $[A_1,A_2] = A_1A_2 - A_2A_1$ (the commutator of $A_1, A_2 \in \mathscr{B}(\mathscr{H})$). So $K(t) = [A, B(t)]$.
Now we are in a position to formulate our main result.

Theorem 2.1. *Let the conditions* (2.1) *and*

$$\sup_{t\ge 0}\left(\psi(W,B(t)) + \|K(t)\|\zeta(A)\right) < 1 \tag{2.2}$$

hold. Then equation (1.1) *is exponentially stable.*

This theorem is proved in the next section. If

$$\|e^{As}\| \le ce^{-\nu s}\ \ (s \ge 0; c, \nu = const > 0), \tag{2.3}$$

then

$$\langle Wv, v\rangle = 2\int_0^\infty \|e^{At}v\|^2 dt \le 2c^2 \int_0^\infty e^{-2\nu t}dt\|v\|^2\ \ (v \in \mathscr{H}).$$

Consequently,

$$\|W\| \le \frac{c^2}{\nu} \text{ and } \zeta(A) \le 2c^3\int_0^\infty e^{-\nu t}\int_0^t e^{-\nu s}e^{-\nu(t-s)}ds\,dt = 2c^3\int_0^\infty e^{-2\nu t}t\,dt = \frac{c^3}{2\nu^2}. \tag{2.4}$$

Now let us estimate $\lambda(W)$. Due to the Wintner inequalities (1.3),

$$\|e^{At}v\| \ge e^{\lambda(\Re A)t}\|v\|\ \ (v \in \mathscr{H}).$$

So in view of (2.1), $\lambda(\Re A)$ is negative. Consequently,

$$\langle Wv, v\rangle = 2\int_0^\infty \|e^{At}v\|^2 dt \ge 2\int_0^\infty e^{2\lambda(\Re A)t}\|v\|^2 dt \ge \|v\|^2/|\lambda(\Re A)|\ \ (v \in \mathscr{H}).$$

Thus

$$\lambda(W) \ge 1/|\lambda(\Re A)|. \tag{2.5}$$

If A is a normal operator: $AA^* = A^*A$, then $\|e^{At}\| = e^{\alpha(A)t}$ $(t \ge 0)$, and according to (2.4),

$$\|W\| \le \frac{1}{|\alpha(A)|}, \zeta(A) = \frac{1}{2|\alpha(A)|^2} \text{ and, in addition, } \lambda(\Re A) = \beta(A),$$

where $\beta(A) := \inf\ \Re\sigma(A)$. Consequently, $\psi(W,B(t)) = \psi_0(A,B(t))$, where

$$\psi_0(A,B(t)) = \begin{cases} \frac{\Lambda(\Re B(t))}{|\alpha(A)|} & \text{if } \Lambda(\Re B(t)) > 0, \\ \frac{\Lambda(\Re B(t))}{|\beta(A)|} & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

So we arrive at

Corollary 2.2. *Let A be a normal operator, and the conditions* (2.1) *and*

$$\sup_{t\ge 0}\left(\psi_0(A,B(t)) + \frac{\|K(t)\|}{2|\alpha(A)|^2}\right) < 1 \tag{2.6}$$

hold. Then equation (1.1) *is exponentially stable.*

Theorem 2.1 *is sharp* in the following sense: if $B(t) = 0$, then $\psi(A, B(t)) = \|K(t)\| = 0$, and (2.2) obviously holds. But condition (2.1) is necessary in this case.
Traditionally (1.1) is considered as a perturbation of the equation $du/dt = Au$ with stable A. Besides, it is supposed that

$$\int_0^\infty \|e^{sA}\|ds \, \sup_t \|B(t)\| < 1, \tag{2.7}$$

e.g. [2, 14] and references therein. We do not assume this condition. For example, if A and $B(t)$ commute, then takes the form

$$\sup_{t\geq 0} \psi_0(A, B(t)) < 1$$

which is sharper than (2.7).
Moreover, in the contrary to the Wintner inequalities, *we do not require the dissipativity* of $A + B(t)$.

3. Proof of theorem 2.1

Lemma 3.1. *Let A, B be constant bounded operators and $K = [A, B]$. Then*

$$[e^{At}, B] = \int_0^t e^{As} K e^{A(t-s)} ds \ \ (t \geq 0). \tag{3.1}$$

Proof: For the proof see [15].
Under condition (2.1), the Lyapunov equation

$$WA + A^*W = -2I \tag{3.2}$$

has a unique solution $W \in \mathscr{B}(\mathscr{H})$ and it can be represented as in Section 2, cf. [7, Theorem I.5.1] (see also equation (4.12) from Chapter I of [7]). For two selfadjoint operators S and S_1 the inequality $S < S_1$ $(S \leq S_1)$ means $(Sh, h) < (S_1 h, h)$ $((Sh, h) \leq (S_1 h, h))$ $(h \in \mathscr{H})$. In particular, the inequality $S < 0$ $(S > 0)$ means that S is strongly negative (strongly positive) definite.

Lemma 3.2. *If condition* (2.1) *holds, then*

$$\Re(WB(t)) = \frac{1}{2}(WB(t) + (WB(t))^*) \leq (\psi(W, B(t)) + \|K(t)\|\zeta(A))I.$$

Proof. Making use of (2.1) we can write

$$\Re(WB(t)) = \frac{1}{2}(WB(t) + B^*(t)W) = \int_0^\infty (e^{A^*t_1} e^{At_1} B(t) + B^*(t) e^{A^*t_1} e^{At_1}) dt_1.$$

But

$$e^{At_1} B(t) = B(t) e^{At_1} + [e^{At_1}, B(t)], B^*(t) e^{A^*t_1} = e^{A^*t_1} B^*(t) + [B^*(t), e^{A^*t_1}].$$

So $\Re(WB(t)) = J_1 + J_2$, where

$$J_1 = \int_0^\infty e^{A^*t_1} (B(t) + B^*(t)) e^{At_1} dt \text{ and } J_2 = \int_0^\infty (e^{A^*t_1}[e^{At_1}, B(t)] + (e^{A^*t_1}[e^{At_1}, B(t)])^*) dt_1.$$

We have

$$J_1 \leq 2\Lambda(\Re B(t)) \int_0^\infty e^{A^*t_1} e^{At_1} dt_1 = \Lambda(\Re B(t)) W.$$

If $\Lambda(\Re B(t)) > 0$, then $J_1 \leq \Lambda(\Re B(t))\|W\|I$. If $\Lambda(\Re B(t)) < 0$, then $J_1 \leq \Lambda(\Re B(t))\lambda(W)I$. So $J_1 \leq \psi(W, B(t))I$.
In addition, by Lemma 3.1

$$\|J_2\| \leq 2\int_0^\infty \|e^{At_1}\|\|[e^{At_1}, B(t)]\| dt_1 \leq 2\int_0^\infty \|e^{At_1}\|\|K(t)\| \int_0^{t_1} \|e^{As}\|\|e^{A(t_1-s)}\| ds \, dt_1$$

$$= \|K(t)\|\zeta(A).$$

This proves the lemma. □

Proof of Theorem 2.1: Due to the Lyapunov equation and Lemma 3.2 we have,

$$\Re W(A + B(t)) \leq -(1 - \psi(W, B(t)) - \|K(t)\|\zeta(A))I.$$

So (2.2) implies

$$\Re W(A + B(t)) < \sup_t(-1 + \psi(W, B(t)) + \|K(t)\|\zeta(A))I < 0. \tag{3.3}$$

Applying the right-hand Wintner inequality (1.3) with the scalar product $(.,.)_W$ defined by $(h, g)_W = \langle Wh, g\rangle$ $(h, g \in \mathscr{H})$, we can assert that equation (1.1) is exponentially stable, as claimed. □

4. Equations with finite dimensional operators

In this section $\mathscr{H} = \mathbb{C}^n$-the n-dimension complex Euclidean space, A and $B(t)$ are $n \times n$ matrices. Put

$$g(A) = [N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2\,]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues of A, counted with their multiplicities; $N_2(A) = (\text{trace } AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt) norm of A. The following relations are checked in [12, Section 2.1]: $g^2(A) \le N_2^2(A) - |\text{trace } A^2|$,

$$g(e^{i\tau}A + zI) = g(A)\ (\tau \in \mathbb{R}, z \in \mathbb{C},) \text{ and } g^2(A) \le \frac{N_2^2(A - A^*)}{2}.$$

If A is a normal matrix, then $g(A) = 0$.
It is shown in [12, Example 2.7.3], that

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}}\ \ (t \ge 0).$$

So

$$\|W\| \le 2\int_0^\infty \|e^{At}\|^2 dt \le 2\int_0^\infty e^{2\alpha(A)t} \left(\sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}}\right)^2 dt = \chi_n(A),$$

where

$$\chi_n(A) = \sum_{j,k=0}^{n-1} \frac{g^{j+k}(A)(k+j)!}{2^{j+k}|\alpha(A)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Put

$$p_n(A,t) = \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}}\ \ (t \ge 0).$$

Then $\|e^{At}\| \le e^{\alpha(A)t} p_n(A,t)$ and $\zeta(A) \le \zeta_n(A)$, where

$$\zeta_n(A) := 2\int_0^\infty e^{2\alpha(A)t} p_n(A,t) \int_0^t p_n(A,t-s)p(A,s)ds\,dt.$$

Moreover, according to (2.5), $\psi(W, B(t)) \le \hat{\psi}_n(A, B(t))$, where

$$\hat{\psi}_n(A,B(t)) := \begin{cases} \chi_n(A)\Lambda(\Re B(t)) & \text{if } \Lambda(\Re B(t)) > 0, \\ \frac{\Lambda(\Re B(t))}{|\lambda(\Re A)|} & \text{if } \Lambda(\Re B(t)) \le 0. \end{cases}$$

Now Theorem 2.1 and (2.5) imply

Corollary 4.1. *Let $\mathscr{H} = \mathbb{C}^n$, A be a Hurwitzian matrix (i.e. condition (2.1) holds), and*

$$\sup_{t \ge 0} (\hat{\psi}_n(A, B(t)) + \|K(t)\|\zeta_n(A)) < 1.$$

Then (1.1) is exponentially stable.

5. Equations with infinite dimensional operators

In this section we consider equation (1.1) in the infinite dimensional space assuming that

$$\Im A \text{ is a Hilbert-Schmidt operator.} \tag{5.1}$$

i.e. $N_2(\Im A) = (\text{trace } (\Im A)^2)^{1/2} < \infty$. Put

$$\hat{u}(A) = [2N_2^2(\Im A) - 2\sum_{k=1}^{\infty} |\Im \hat{\lambda}_k(A)|^2\,]^{1/2},$$

where $\hat{\lambda}_k(A), k = 1, 2, ...,$ are nonreal eigenvalues of A, enumerated with their multiplicities in the decreasing order of the absolute values of their imaginary parts. Recall the classical Weyl inequality

$$N_2^2(\Im A) \ge \sum_{k=1}^{\infty} |\Im \hat{\lambda}_k(A)|^2,$$

cf. [12, p. 98]. So $\hat{u}(A) \le \sqrt{2}N_2(\Im A)$. If A is a normal operator, then $\hat{u}(A) = 0$, cf. [12, Section 7.7]. As is shown in [12, Example 7.10.3],

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{\infty} \frac{t^k \hat{u}^k(A)}{(k!)^{3/2}}\ \ (t \ge 0),$$

So

$$\|W\| \le 2\int_0^\infty \|e^{At}\|^2 dt \le 2\int_0^\infty e^{\alpha(A)t} \left(\sum_{k=0}^{\infty} \frac{t^k \hat{u}^k(A)}{(k!)^{3/2}}\right)^2 dt = \tilde{\chi}(A),$$

where

$$\tilde{\chi}(A)=\sum_{j,k=0}^{\infty}\frac{\hat{u}^{j+k}(A)(k+j)!}{2^{j+k}|\alpha(A)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Put

$$\tilde{p}(A,t)=\sum_{k=0}^{\infty}\frac{t^k\hat{u}^k(A)}{(k!)^{3/2}}\ \ (t\geq 0).$$

Then $\|e^{At}\|\leq e^{\alpha(A)t}\hat{p}(A,t)$ and

$$\zeta(A)\leq\tilde{\zeta}(A):=2\int_0^{\infty}e^{2\alpha(A)t}\tilde{p}(t,A)\int_0^t\tilde{p}(t-s,A)\tilde{p}(s,A)ds\,dt.$$

Moreover, $\psi(W,B(t))\leq\tilde{\psi}(A,B(t))$, where

$$\tilde{\psi}(A,B(t)):=\begin{cases}\tilde{\chi}(A)\Lambda(\Re B(t)) & \text{if } \Lambda(\Re B(t))>0,\\ \frac{\Lambda(\Re B(t))}{|\lambda(\Re A)|} & \text{if } \Lambda(\Re B(t))\leq 0.\end{cases}$$

Now Theorem 2.1 and (2.5) imply

Corollary 5.1. *If the conditions* (2.1)*,* (5.1) *and*

$$\sup_{t\geq 0}\left(\tilde{\psi}(A,B(t))+\|K(t)\|\tilde{\zeta}(A)\right)<1,$$

hold, then (1.1) *is exponentially stable.*

6. Example

Put $\Omega=[0,1]\times[0,1]$. In this section $\mathscr{H}=L^2(\Omega)$ is the Hilbert spaces of complex square integrable functions defined on Ω with the traditional scalar product and norm.
Consider the equation

$$\frac{\partial u(t,x,y)}{\partial t}=c(x)u(t,x,y)+\int_0^1 k_1(x,s)u(t,s,y)ds+\int_0^1 k_2(t,y,s)u(t,x,s)ds \tag{6.1}$$

$$(0\leq x,y\leq 1;t\geq 0),$$

where $c(\cdot):[0,1]\to\mathbb{R}$ is piece-wise continuous, $k_1(\cdot,\cdot):[0,1]^2\to\mathbb{C}$, $k_2(\cdot,\cdot,\cdot):[0,\infty)\times[0,1]^2\to\mathbb{C}$, are given functions satisfying the conditions pointed below. Equation of the type (6.1) is the Barbashin type integro-differential equation or simply the Barbashin equation, [2]. The stability of (6.1) can also be investigated by perturbations of the simple equation

$$\frac{\partial u(t,x,y)}{\partial t}=c(x)u(t,x,y),$$

cf. [2, Section 2.5], but this approach gives rather rough results if the norm of k_1 and k_2 are large enough.
Define the operators A and $B(t)$ by

$$(Aw)(x,y)=c(x)w(x,y)+\int_0^1 k_1(x,s)w(s,y)ds$$

and

$$(B(t)w)(x,y)=\int_0^1 k_2(t,x,s)w(x,s)ds\ \ (x,y\in[0,1];\ w\in L^2(\Omega)),$$

respectively. Under consideration we have $[A,B(t)]=0$ for all $t\geq 0$. Moreover, assume that

$$N_2(A-A^*)=\left(\int_0^1\int_0^1|k_1(x,s)-\bar{k}_1(s,x))|^2\,ds\,dx\right)^{1/2}<\infty$$

and k_2 provides the boundedness of $B(t)$. Various estimates for $\alpha(A)$ under considerations can be found in [13]. In particular, if $k_1(x,s)=0$ for $x\leq s$, then $\alpha(A)=\sup_x c(x)$. Furthermore, it is not hard to check that

$$\Lambda(\Re B(t))=\frac{1}{2}\sup_{v\in L^2(0,1)}\int_0^1\int_0^1(k_2(t,y,s)+\bar{k}_2(t,s,y))v(s)\,\bar{v}(y)\,ds\,dy$$

and

$$\lambda(\Re A)=\frac{1}{2}\inf_{v\in L^2(0,1)}\int_0^1\int_0^1(k_1(x,s)+\bar{k}_1(s,x))v(s)\,\bar{v}(x)\,ds\,dx.$$

Now we can directly apply Corollary 5.1.
Note that the theory of of various classes of integro-differential equations is rather rich, cf. [3, 6], [8]-[11], [16, 17, 19, 22, 23] and references therein, but the stability conditions in terms of the commutators have not been derived.

References

[1] V.M. Aleksandrov, and E.V. Kovalenko, *Problems in Continuous Mechanics with Mixed Boundary Conditions*, Nauka, Moscow 1986. In Russian.
[2] J. Appel, A. Kalitvin and P. Zabreiko, *Partial Integral Operators and Integrodifferential Equations*, Marcel Dekker, New York, 2000.
[3] J.A.D. Appleby and D.W. Reynolds, On the non-exponential convergence of asymptotically stable solutions of linear scalar Volterra integro-differential equations, Journal of Integral Equations and Applications, 14, no 2 (2002), 521-543.
[4] K. M. Case, P. F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading Mass. 1967.
[5] M.C. Cercignani, *Mathematical Methods in Kinetic Theory*, Macmillian, New York, 1969.
[6] Chuhu Jin and Jiaowan Luo, Stability of an integro-differential equation, Computers and Mathematics with Applications, 57 (2009) 1080–1088.
[7] Yu L. Daleckii, and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, R. I. 1974.
[8] A. Domoshnitsky and Ya. Goltser, An approach to study bifurcations and stability of integro-differential equations. Math. Comput. Modelling, 36 (2002), 663-678.
[9] A.D. Drozdov, Explicit stability conditions for integro-differential equations with periodic coefficients, Math. Methods Appl. Sci. 21 (1998), 565-588.
[10] A.D. Drozdov and M. I. Gil', Stability of linear integro-differential equations with periodic coefficients. Quart. Appl. Math. 54 (1996), 609-624.
[11] N.T. Dung, On exponential stability of linear Levin-Nohel integro-differential equations, Journal of Mathematical Physics 56, 022702 (2015); doi: 10.1063/1.4906811
[12] M.I. Gil', *Operator Functions and Localization of Spectra*, Lecture Notes In Mathematics vol. 1830, Springer-Verlag, Berlin, 2003.
[13] M.I. Gil', Spectrum and resolvent of a partial integral operator. *Applicable Analysis*, 87, no. 5, (2008) 555–566.
[14] M.I. Gil', On stability of linear Barbashin type integro-differential equations, *Mathematical Problems in Engineering*, **2015**, Article ID 962565, (2015), 5 pages.
[15] M.I. Gil', A bound for the Hilbert-Schmidt norm of generalized commutators of nonself-adjoint operators, Operators and Matrices, 11, no. 1 (2017), 115-123
[16] Ya. Goltser and A. Domoshnitsky, Bifurcation and stability of integrodifferential equations, Nonlinear Anal. 47 (2001), 953-967.
[17] Ya. Coltser and A. Domoshnitsky, About reducing integro-differential equations with infinite limits of integration to systems of ordinary differential equations. Advances in Difference Equations 2013:187, (2013) 17 pages.
[18] H.G. Kaper, C.G. Lekkerkerker, and J. Hejtmanek, *Spectral Methods in Linear Transport Theory*, Birkhauser, Basel, 1982.
[19] B. G. Pachpatte, On a parabolic integrodifferential equation of Barbashin type, *Comment. Math.Univ. Carolin. 52*, no. 3 (2011) 391-401
[20] W.J. Rugh, *Linear System Theory*. Prentice Hall, Upper Saddle River, New Jersey, 1996.
[21] H. R. Thieme, A differential-integral equation modelling the dynamics of populations with a rank structure, *Lect. Notes Biomath. 68* (1986), 496-511
[22] J. Vanualailai and S. Nakagiri, Stability of a system of Volterra integro-differential equations J. Math. Anal. Appl. 281 (2003) 602-619
[23] B. Zhang, Construction of Liapunov functionals for linear Volterra integrodifferential equations and stability of delay systems, Electron. J. Qual. Theory Differ. Equ. 30 (2000) 1-17.