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The new UP-isomorphism theorems for UP-algebras in the meaning of the congruence determined by a UP-homomorphism

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Abstract

The aim of this paper is to construct the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also give an application of the theorem to the first, second, and third UP-isomorphism theorems in UP-algebras.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [7], BCI-algebras [8], BCH-algebras [4], KU-algebras [15], SU-algebras [10], UP-algebras [6] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [8] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [7, 8] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.
The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches such as: In 1998, Jun, Hong, Xin and Roh [9] proved isomorphism theorems by using Chinese Remainder Theorem in BCI-algebras. In 2001, Park, Shim and Roh [14] proved isomorphism theorems of IS-algebras. In 2004, Hao and Li [3] introduced the concept of ideals of an ideal in a BCI-algebra and some isomorphism theorems are obtained by using this concept. They obtained several isomorphism theorems of BG-algebras and related properties. In 2006, Kim [12] introduced the notion of KS-semigroups. He characterized ideals of a KS-semigroup and proved the first isomorphism theorem for KS-semigroups. In 2007, Dar and Akram [2] introduced the notion of K-homomorphism of K-algebras. In 2008, Kim and Kim [11] introduced the notion of BG-algebras which is a generalization of B-algebras. They obtained several isomorphism theorems of BG-algebras and related properties. In 2009, Paradero-Vilela and Cawi [13] characterized KS-semigroup homomorphisms and proved the isomorphism theorems for KS-semigroups. In 2011, Keawrahun and Leerawat [10] introduced the notion of SU-semigroups and proved the isomorphism theorems for SU-semigroups. In 2012, Asawasamrit [1] introduced the notion of KK-algebras and studied isomorphism theorems of KK-algebras. In 2015, Iampan [5] studied UP-isomorphism theorems of UP-algebras.
In this paper, we construct the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also give an application of the theorem to the first, second, and third UP-isomorphism theorems in UP-algebras.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.1. *[6] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,*

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2) $0 \cdot x = x$,
(UP-3) $x \cdot 0 = 0$,
(UP-4) $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

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Example 1.2. *[6] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A\cdot B = B\cap A'$ and $A*B = B\cup A'$ for all $A,B\in\mathscr{P}(X)$. Then $(\mathscr{P}(X),\cdot,\emptyset)$ and $(\mathscr{P}(X),*,X)$ are UP-algebras and we shall call it the power UP-algebra of type 1 and the power UP-algebra of type 2, respectively.*

Example 1.3. *[6] Let $A = \{0,a,b,c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

(1.1)

Then $(A,\cdot,0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [6] *In a UP-algebra A, the following properties hold: for any $x,y,z\in A$,*

(1) $x\cdot x = 0$,
(2) $x\cdot y = 0$ and $y\cdot z = 0$ implies $x\cdot z = 0$,
(3) $x\cdot y = 0$ implies $(z\cdot x)\cdot(z\cdot y) = 0$,
(4) $x\cdot y = 0$ implies $(y\cdot z)\cdot(x\cdot z) = 0$,
(5) $x\cdot(y\cdot x) = 0$,
(6) $(y\cdot x)\cdot x = 0$ if and only if $x = y\cdot x$, and
(7) $x\cdot(y\cdot y) = 0$.

Definition 1.5. *[6] Let A be a UP-algebra. A nonempty subset B of A is called a UP-ideal of A if it satisfies the following properties:*

(1) the constant 0 of A is in B, and
(2) for any $x,y,z\in A, x\cdot(y\cdot z)\in B$ and $y\in B$ implies $x\cdot z\in B$.

Definition 1.6. *[6] Let $A = (A,\cdot,0)$ be a UP-algebra. A subset S of A is called a UP-subalgebra of A if the constant 0 of A is in S, and $(S,\cdot,0)$ itself forms a UP-algebra.*

Proposition 1.7. [6] *A nonempty subset S of a UP-algebra $A = (A,\cdot,0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.8. *[6] Let A be a UP-algebra. An equivalence relation ρ on A is called a congruence if for any $x,y,z\in A$,*

$$x\rho y \text{ implies } x\cdot z\rho y\cdot z \text{ and } z\cdot x\rho z\cdot y.$$

Lemma 1.9. [6] *An equivalence relation ρ on A is a congruence if and only if for any $x,y,u,v\in A$, $x\rho y$ and $u\rho v$ imply $x\cdot u\rho y\cdot v$.*

Definition 1.10. *[6] Let A be a UP-algebra and B a UP-ideal of A. Define the binary relation $\sim_B$ on A as follows: for all $x,y\in A$,*

$$x\sim_B y \text{ if and only if } x\cdot y\in B \text{ and } y\cdot x\in B. \tag{1.2}$$

Proposition 1.11. [6] *Let A be a UP-algebra and B a UP-ideal of A with a binary relation $\sim_B$ defined by (1.2). Then $\sim_B$ is a congruence on A.*

Let A be a UP-algebra and ρ a congruence on A. If $x\in A$, then the ρ-class of x is the $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y\in A \mid y\rho x\}.$$

Then the set of all ρ-classes is called the *quotient set of A by ρ*, and is denoted by A/ρ. That is,

$$A/\rho = \{(x)_\rho \mid x\in A\}.$$

Theorem 1.12. [6] *Let A be a UP-algebra and B a UP-ideal of A. Then $(A/\sim_B,*,(0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B}*(y)_{\sim_B} = (x\cdot y)_{\sim_B}$ for all $x,y\in A$, called the quotient UP-algebra of A induced by the congruence $\sim_B$.*

Definition 1.13. *[6] Let $(A,\cdot,0)$ and $(A',\cdot',0')$ be UP-algebras. A mapping f from A to A' is called a UP-homomorphism if*

$$f(x\cdot y) = f(x)\cdot' f(y) \text{ for all } x,y\in A.$$

A UP-homomorphism $f\colon A\to A'$ is called a

(1) UP-epimorphism if f is surjective,
(2) UP-monomorphism if f is injective,
(3) UP-isomorphism if f is bijective. Moreover, we say A is UP-isomorphic to A', symbolically, $A\cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the image of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the image of f, denoted by $\operatorname{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the inverse image of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the kernel of f, written by $\operatorname{Ker}(f)$. That is,

$$\operatorname{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\operatorname{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.14. [6] *Let A be a UP-algebra and B a UP-ideal of A. Then the mapping $\pi_B \colon A \to A/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_B$.*

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.3}$$

Proposition 1.15. [6] *Let A be a UP-algebra with a binary relation $\leq$ defined by (1.3). Then $(A, \leq)$ is a partially ordered set with 0 as the greatest element.*

We often call the partial ordering $\leq$ defined by (1.3) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

Theorem 1.16. [6] *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f \colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) $f(0_A) = 0_B$,
(2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$,
(3) if C is a UP-subalgebra of A, then the image $f(C)$ is a UP-subalgebra of B. In particular, $\operatorname{Im}(f)$ is a UP-subalgebra of B,
(4) if D is a UP-subalgebra of B, then the inverse image $f^{-1}(D)$ is a UP-subalgebra of A. In particular, $\operatorname{Ker}(f)$ is a UP-subalgebra of A,
(5) if C is a UP-ideal of A such that $\operatorname{Ker}(f) \subseteq C$, then the image $f(C)$ is a UP-ideal of $f(A)$,
(6) if D is a UP-ideal of B, then the inverse image $f^{-1}(D)$ is a UP-ideal of A. In particular, $\operatorname{Ker}(f)$ is a UP-ideal of A, and
(7) $\operatorname{Ker}(f) = \{0_A\}$ if and only if f is injective.

2. Main results

In this section, we introduce the congruence determined by a UP-homomorphism and prove the new fundamental theorem of UP-algebras in the meaning of the congruence determined by a UP-homomorphism. We also prove the first, second, and third UP-isomorphism theorems in UP-algebras.

Definition 2.1. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Define the binary relation $\sim_f$ on A as follows: for all $x, y \in A$,*

$$x \sim_f y \text{ if and only if } f(x) = f(y). \tag{2.1}$$

Theorem 2.2. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism with a binary relation $\sim_f$ on A defined by (2.1). Then $\sim_f$ is a congruence on A, called the congruence determined by f.*

Proof. Reflexive: For all $x \in A$, we have $f(x) = f(x)$. Thus $x \sim_f x$.
Symmetric: Let $x, y \in A$ be such that $x \sim_f y$. Then $f(x) = f(y)$, so $f(y) = f(x)$. Thus $y \sim_f x$.
Transitive: Let x, y, z be such that $x \sim_f y$ and $y \sim_f z$. Then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$. Thus $x \sim_f z$.
Therefore, $\sim_f$ is an equivalence relation on A. Finally, let $x, y, u, v \in A$ be such that $x \sim_f u$ and $y \sim_f v$. Then $f(x) = f(u)$ and $f(y) = f(v)$. Since f is a UP-homomorphism, we get

$$f(x \cdot y) = f(x) \bullet f(y) = f(u) \bullet f(v) = f(u \cdot v).$$

Thus $x \cdot y \sim_f u \cdot v$. By Lemma 1.9, we have $\sim_f$ is a congruence on A. □

Theorem 2.3. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Then $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}$ for all $x, y \in A$, called the quotient UP-algebra of A induced by the congruence $\sim_f$.*

Proof. Let $x, y, u, v \in A$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$ and $(u)_{\sim_f} = (v)_{\sim_f}$. Since $\sim_f$ is an equivalence relation on A, we get $x \sim_f y$ and $u \sim_f v$. By Lemma 1.9, we have $x \cdot u \sim_f y \cdot v$. Hence, $(x)_{\sim_f} * (u)_{\sim_f} = (x \cdot u)_{\sim_f} = (y \cdot v)_{\sim_f} = (y)_{\sim_f} * (v)_{\sim_f}$, showing $*$ is well defined.
(UP-1): Let $x, y, z \in A$. By (UP-1), we have $((y)_{\sim_f} * (z)_{\sim_f}) * (((x)_{\sim_f} * (y)_{\sim_f}) * ((x)_{\sim_f} * (z)_{\sim_f})) = ((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)))_{\sim_f} = (0_A)_{\sim_f}$.
(UP-2): Let $x \in A$. By (UP-2), we have $(0_A)_{\sim_f} * (x)_{\sim_f} = (0_A \cdot x)_{\sim_f} = (x)_{\sim_f}$.
(UP-3): Let $x \in A$. By (UP-3), we have $(x)_{\sim_f} * (0_A)_{\sim_f} = (x \cdot 0_A)_{\sim_f} = (0_A)_{\sim_f}$.
(UP-4): Let $x, y \in A$ be such that $(x)_{\sim_f} * (y)_{\sim_f} = (y)_{\sim_f} * (x)_{\sim_f} = (0_A)_{\sim_f}$. Then $(x \cdot y)_{\sim_f} = (y \cdot x)_{\sim_f} = (0_A)_{\sim_f}$, it follows that $f(x) \bullet f(y) = f(x \cdot y) = f(0_A) = f(y \cdot x) = f(y) \bullet f(x)$. By Theorem 1.16 (1), we have $f(x) \bullet f(y) = f(y) \bullet f(x) = 0_B$. By (UP-4), we have $f(x) = f(y)$. Thus $x \sim_f y$, so $(x)_{\sim_f} = (y)_{\sim_f}$.
Hence, $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra. □

Theorem 2.4. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f \colon A \to B$ a UP-homomorphism. Then the mapping $\pi_f \colon A \to A/\sim_f$ defined by $\pi_f(x) = (x)_{\sim_f}$ for all $x \in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_f$.*

Proof. Let $x, y \in A$ be such that $x = y$. Then $(x)_{\sim_f} = (y)_{\sim_f}$, so $\pi_f(x) = \pi_f(y)$. Thus π_f is well defined. Note that by the definition of π_f, we have π_f is surjective. Let $x, y \in A$. Then

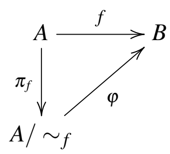
$$\pi_f(x \cdot y) = (x \cdot y)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = \pi_f(x) * \pi_f(y).$$

Thus π_f is a UP-homomorphism. So we conclude that π_f is a UP-epimorphism. □

Theorem 2.5. *(Fundamental Theorem of UP-homomorphisms) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f : A \to B$ a UP-homomorphism. Then there exists uniquely a UP-homomorphism φ from $A/\sim_f$ to B such that $f = \varphi \circ \pi_f$. Moreover,*

(1) π_f is a UP-epimorphism and φ a UP-monomorphism, and
(2) f is a UP-epimorphism if and only if φ is a UP-isomorphism.

As f makes the following diagram commute,



Proof. By Theorem 2.3, we have $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra. Define a mapping $\varphi : A/\sim_f \to B$ by

$$\varphi((x)_{\sim_f}) = f(x) \text{ for all } (x)_{\sim_f} \in A/\sim_f . \tag{2.2}$$

Indeed, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $(x)_{\sim_f} = (y)_{\sim_f}$. Then $x \sim_f y$, so

$$\varphi((x)_{\sim_f}) = f(x) = f(y) = \varphi((y)_{\sim_f}).$$

For any $x, y \in A$, we see that

$$\begin{aligned} \varphi((x)_{\sim_f} * (y)_{\sim_f}) &= \varphi((x \cdot y)_{\sim_f}) \\ &= f(x \cdot y) \\ &= f(x) \bullet f(y) \\ &= \varphi((x)_{\sim_f}) \bullet \varphi((y)_{\sim_f}). \end{aligned}$$

Thus φ is a UP-homomorphism. Also, since

$$(\varphi \circ \pi_f)(x) = \varphi(\pi_f(x)) = \varphi((x)_{\sim_f}) = f(x) \text{ for all } x \in A,$$

we obtain $f = \varphi \circ \pi_f$. We have shown the existence. Let φ' be a mapping from $A/\sim_f$ to B such that $f = \varphi' \circ \pi_f$. Then for any $(x)_{\sim_f} \in A/\sim_f$, we have

$$\begin{aligned} \varphi'((x)_{\sim_f}) &= \varphi'(\pi_f(x)) \\ &= (\varphi' \circ \pi_f)(x) \\ &= f(x) \\ &= (\varphi \circ \pi_f)(x) \\ &= \varphi(\pi_f(x)) \\ &= \varphi((x)_{\sim_f}). \end{aligned}$$

Hence, $\varphi = \varphi'$, showing the uniqueness.
(1) By Theorem 2.4, we have π_f is a UP-epimorphism. Also, let $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$ be such that $\varphi((x)_{\sim_f}) = \varphi((y)_{\sim_f})$. Then $f(x) = f(y)$, so $x \sim_f y$. Thus $(x)_{\sim_f} = (y)_{\sim_f}$. Therefore, φ a UP-monomorphism.
(2) Assume that f is a UP-epimorphism. By (1), it suffices to prove φ is surjective. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus $y = f(x) = \varphi((x)_{\sim_f})$, so φ is surjective. Hence, φ is a UP-isomorphism.
Conversely, assume that φ is a UP-isomorphism. Then φ is surjective. Let $y \in B$. Then there exists $(x)_{\sim_f} \in A/\sim_f$ such that $\varphi((x)_{\sim_f}) = y$. Thus $f(x) = \varphi((x)_{\sim_f}) = y$, so f is surjective. Hence, f is a UP-epimorphism. □

Theorem 2.6. *(First UP-isomorphism Theorem) Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f : A \to B$ a UP-homomorphism. Then*

$$A/\sim_f \cong \mathrm{Im}(f).$$

Proof. By Theorem 1.16 (3), we have $\mathrm{Im}(f)$ is a UP-subalgebra of B. Thus $f : A \to \mathrm{Im}(f)$ is a UP-epimorphism. Applying Theorem 2.5 (2), we obtain $A/\sim_f \cong \mathrm{Im}(f)$. □

Lemma 2.7. *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, $f : A \to B$ a UP-homomorphism, and H a UP-subalgebra of A. Denote $H_{\sim_f} = \bigcup_{h \in H} (h)_{\sim_f}$. Then $H_{\sim_f}$ is a UP-subalgebra of A.*

Proof. Clearly, $\emptyset \neq H_{\sim_f} \subseteq A$. Let $a,b \in H_{\sim_f}$. Then $a \in (x)_{\sim_f}$ and $b \in (y)_{\sim_f}$ for some $x,y \in H$, so $(a)_{\sim_f} = (x)_{\sim_f}$ and $(b)_{\sim_f} = (y)_{\sim_f}$. Theorem 2.3 gives $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra, so

$$(a \cdot b)_{\sim_f} = (a)_{\sim_f} * (b)_{\sim_f} = (x)_{\sim_f} * (y)_{\sim_f} = (x \cdot y)_{\sim_f}.$$

Thus $a \cdot b \in (x \cdot y)_{\sim_f}$. Since $x,y \in H$, it follows from Proposition 1.7 that $x \cdot y \in H$. Thus $a \cdot b \in (x \cdot y)_{\sim_f} \subseteq H_{\sim_f}$. Hence, $H_{\sim_f}$ is a UP-subalgebra of A. □

Theorem 2.8. *(Second UP-isomorphism Theorem) Let $(A,\cdot,0_A)$ and $(B,\bullet,0_B)$ be UP-algebras, $f\colon A \to B$ a UP-homomorphism, and H a UP-subalgebra of A. Denote $H_{\sim_f}/\sim_f = \{(x)_{\sim_f} \mid x \in H_{\sim_f}\}$. Then*

$$H/\sim_{\pi_f|_H} \cong H_{\sim_f}/\sim_f .$$

Proof. By Lemma 2.7, we have $H_{\sim_f}$ is a UP-subalgebra of A. Then it is easy to check that $H_{\sim_f}/\sim_f$ is a UP-subalgebra of $A/\sim_f$, thus $(H_{\sim_f}/\sim_f, *, (0_A)_{\sim_f})$ itself is a UP-algebra. Also, it is obvious that $H \subseteq H_{\sim_f}$, then

$$(\pi_f|_H =) g\colon H \to H_{\sim_f}/\sim_f, x \mapsto (x)_{\sim_f}, \tag{2.3}$$

is a mapping. Indeed, g is the restriction of π_f to H. Thus g is a UP-epimorphism. Indeed, $H_{\sim_f}/\sim_f = H/\sim_f$. Theorem 2.6 gives $H/\sim_{\pi_f|_H} \cong H_{\sim_f}/\sim_f$. □

Theorem 2.9. *Let $(A,\cdot,0_A)$ and $(B,\bullet,0_B)$ be UP-algebras, $f\colon A \to B$ and $g\colon A \to B$ UP-homomorphisms with $\sim_f \subseteq \sim_g$. Define the binary relation $\sim_g/\sim_f$ on $A/\sim_f$ as follows: for all $x,y \in A$,*

$$(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f} \text{ if and only if } x \sim_g y. \tag{2.4}$$

Then $\sim_g/\sim_f$ is a congruence on $A/\sim_f$.

Proof. By Theorem 2.3, we have $(A/\sim_f, *, (0_A)_{\sim_f})$ is a UP-algebra.
Reflexive: For all $x \in A$, we have $x \sim_g x$. Thus $(x)_{\sim_f} \sim_g/\sim_f (x)_{\sim_f}$.
Symmetric: Let $x,y \in A$ be such that $(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f}$. Then $x \sim_g y$, so $y \sim_g x$. Thus $(y)_{\sim_f} \sim_g/\sim_f (x)_{\sim_f}$.
Transitive: Let x,y,z be such that $(x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f}$ and $(y)_{\sim_f} \sim_g/\sim_f (z)_{\sim_f}$. Then $x \sim_g y$ and $y \sim_g z$, so $x \sim_g z$. Thus $(x)_{\sim_f} \sim_g/\sim_f (z)_{\sim_f}$.
Therefore, $\sim_g/\sim_f$ is an equivalence relation on $A/\sim_f$. Finally, let $x,y,u,v \in A$ be such that $(x)_{\sim_f} \sim_g/\sim_f (u)_{\sim_f}$ and $(y)_{\sim_f} \sim_g/\sim_f (v)_{\sim_f}$. Then $x \sim_g u$ and $y \sim_g v$. The binary relation $\sim_g$ is a congruence on A by Theorem 2.2, that is $x \cdot y \sim_g u \cdot v$. Thus $(x \cdot y)_{\sim_f} \sim_g/\sim_f (u \cdot v)_{\sim_f}$, so $(x)_{\sim_f} * (y)_{\sim_f} \sim_g/\sim_f (u)_{\sim_f} * (v)_{\sim_f}$. Hence, $\sim_g/\sim_f$ is a congruence on $A/\sim_f$. □

Theorem 2.10. *(Third UP-isomorphism Theorem) Let $(A,\cdot,0_A)$ and $(B,\bullet,0_B)$ be UP-algebras, $f\colon A \to B$ and $g\colon A \to B$ UP-homomorphisms with $\sim_f \subseteq \sim_g$. Then*

$$(A/\sim_f)/(\sim_g/\sim_f) \cong A/\sim_g .$$

Proof. By Theorem 2.3, we obtain $(A/\sim_f, *, (0_A)_{\sim_f})$ and $(A/\sim_g, *', (0_A)_{\sim_g})$ are UP-algebras. By Theorem 2.4, we obtain

$$\pi_f\colon A \to A/\sim_f, x \mapsto (x)_{\sim_f}$$

and

$$\pi_g\colon A \to A/\sim_g, x \mapsto (x)_{\sim_g}$$

are UP-epimorphisms. Applying Theorem 2.5 (2), there exists a UP-isomorphism

$$g/f\colon A/\sim_f \to A/\sim_g, (x)_{\sim_f} \mapsto (x)_{\sim_g}. \tag{2.5}$$

Indeed, $A/\sim_f \cong A/\sim_g$. By Theorem 2.9 and 2.3, we have $(A/\sim_f)/\sim_{g/f}$ is a UP-algebra. By Theorem 2.4, we obtain

$$\pi_{g/f}\colon A/\sim_f \to (A/\sim_f)/\sim_{g/f}, (x)_{\sim_f} \mapsto ((x)_{\sim_f})_{\sim_{g/f}}$$

is a UP-epimorphism. Applying Theorem 2.5 (2), there exists a UP-isomorphism

$$\varphi\colon (A/\sim_f)/\sim_{g/f} \to A/\sim_g, ((x)_{\sim_f})_{\sim_{g/f}} \mapsto (x)_{\sim_g}. \tag{2.6}$$

That is,

$$(A/\sim_f)/\sim_{g/f} \cong A/\sim_g .$$

We shall show that $\sim_{g/f} = \sim_g/\sim_f$. For any $(x)_{\sim_f}, (y)_{\sim_f} \in A/\sim_f$,

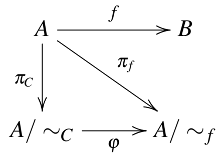
$$\begin{aligned} (x)_{\sim_f} \sim_{g/f} (y)_{\sim_f} &\Leftrightarrow (g/f)((x)_{\sim_f}) = (g/f)((y)_{\sim_f}) \\ &\Leftrightarrow (x)_{\sim_g} = (y)_{\sim_g} \\ &\Leftrightarrow x \sim_g y \\ &\Leftrightarrow (x)_{\sim_f} \sim_g/\sim_f (y)_{\sim_f} \end{aligned}$$

by (2.1) and (2.4). Thus $\sim_{g/f} = \sim_g/\sim_f$. Hence, $(A/\sim_f)/(\sim_g/\sim_f) \cong A/\sim_g$. □

Corollary 2.11. *Let* $(A,\cdot,0_A)$ *and* $(B,\bullet,0_B)$ *be UP-algebras,* $f: A\to B$ *a UP-homomorphism, and* C *a UP-ideal of* A*. Then*

$$A/\sim_C\cong A/\sim_f.$$

As π_f *makes the following diagram commute,*



Proof. It is straightforward by Theorem 1.12, 1.14, 2.4, and 2.5 (2). □

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References

[1] S. Asawasamrit, *KK-isomorphism and its properties*, Int. J. Pure Appl. Math. **78** (2012), no. 1, 65–73.
[2] K. H. Dar and M. Akram, *On K-homomorphisms of K-algebras*, Int. Math. Forum **2** (2007), no. 46, 2283–2293.
[3] J. Hao and C. X. Li, *On ideals of an ideal in a BCI-algebra*, Sci. Math. Jpn. (in Editione Electronica) **10** (2004), no. 16, 493–500.
[4] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Semin. Notes, Kobe Univ. **11** (1983), 313–320.
[5] A. Iampan, *The UP-isomorphism theorems for UP-algebras*, Manuscript submitted for publication, April 2015.
[6] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Top. **5** (2017), no. 1, 35–54.
[7] Y. Imai and K. Iséki, *On axiom system of propositional calculi, XIV*, Proc. Japan Acad. **42** (1966), no. 1, 19–22.
[8] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), no. 1, 26–29.
[9] Y. B. Jun, S. M. Hong, X. L. Xin, and E. H. Roh, *Chinese remainder theorems in BCI-algebras*, Soochow J. Math. **24** (1998), no. 3, 219–230.
[10] S. Keawrahun and U. Leerawat, *On isomorphisms of SU-algebras*, Sci. Magna **7** (2011), no. 2, 39–44.
[11] C. B. Kim and H. S. Kim, *On BG-algebras*, Demonstr. Math. **41** (2008), no. 3, 497–505.
[12] K. H. Kim, *On structure of KS-semigroup*, Int. Math. Forum **1** (2006), no. 2, 67–76.
[13] J. S. Paradero-Vilela and M. Cawi, *On KS-semigroup homomorphism*, Int. Math. Forum **4** (2009), no. 23, 1129–1138.
[14] J. K. Park, W. H. Shim, and E. H. Roh, *On isomorphism theorems in IS-algebras*, Soochow J. Math. **27** (2001), no. 2, 153–160.
[15] C. Prabpayak and U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna **5** (2009), no. 1, 54–57.