Fundamental Journal of Mathematics and Applications, 1 (1) (2018) 18-24

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma

A horizontal endomorphism of the canonical superspray

Esmaeil Azizpour[a*] **and Mohammad Hassan Zarifi**[a]

[a]*Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, PO Box 1914, Rasht, Iran*
[*]*Corresponding author E-mail: eazizpour@gmail.com*

Article Info

Keywords: *Horizontal endomorphism, Riemann-Finsler supermetric, Superspray*
2010 AMS: *53C60, 58A50*
Received: *12 March 2018*
Accepted: *15 April 2018*
Available online: *30 June 2018*

Abstract

Giving up the homogeneity condition of a Lagrange superfunction, we prove that there is a unique horizontal endomorphism h (nonlinear connection) on a supermanifold $\mathscr{M}$, such that h is conservative and its torsion vanishes. There are several examples for nonhomogeneous Lagrangians such that this result is not true.

1. Introduction

The fundamental relation between the horizontal endomorphisms and semisprays was discovered, independently, by M. Crampin [3] and J. Grifone [6, 7]. The conditions for a system of second order differential equations to be derivable from a Lagrangian are related to the differential geometry of the tangent bundle of configuration space. These conditions are simply expressed in terms of the horizontal distribution which is associated with any vector field representing a system of second-order differential equations.
In supergeometry, relationship between nonlinear connections and supersprays structures to be discussed. Also it was shown that there exists a homogeneous superspray, so called the Euler-Lagrange supervector field, which is induced by a Finsler metric [8, 13]. This superspray can help us to introduce a horizontal endomorphism which will be used to obtain the main result. So we will show that on a Finsler supermanifold $(\mathscr{M},F)$, there is a unique horizontal endomorphism h which is conservative (see theorem 3.6) i.e. $d_hL = 0$. The property $d_hL = 0$ tells us that the Lagrangian L is constant along the horizontal curves of the nonlinear connection and hence it is constant along the geodesics of the superspace. This result is not true for an arbitrary Lagrangian L. We will find non homogeneous Lagrangian superfunctions for which $d_hL \neq 0$.
The paper is organized as follows: Section 2 deals with the vertical and complete lift of supervector fields to the tangent superbundle. It contains a brief review of the notion of superspray and the relationship between supersprays and nonlinear connections. We also introduce the notion of Euler-Lagrange supervector field which is an important tool to construct the horizontal endomorphism. In section three, we introduce a horizontal endomorphism h on a supermanifold $\mathscr{M}$, such that h is conservative and its torsion vanishes. We consider an example for a nonhomogeneous Lagrangian such that this result is not true.

2. Preliminary

The basic structure for building up supermanifolds is the Grassmann algebra. With $B_L = (B_L)_0 + (B_L)_1$ we shall denote a real Grassmann algebra with L generators. If $L = \infty, B_L$ is given a suitable Banach norm, making B_∞ a Banach-Grassmann algebra as defined in [9]. Here B_L is a graded commutative algebra, namely ,

$$ab \in (B_L)_{|a|+|b|}, \quad ab = (-1)^{|a||b|}ba,$$

where the element $a, b \in B_L$ are the homogeneous. A $(m,n)-$dimensional supermanifold $\mathscr{M}$ is defined on $B_L^{m,n}$(see details in [4]). Throughout this paper, $\mathscr{M}$ will denote an (m,n)-dimensional supermanifold.
The concept of nonlinear connection (N-connection) was introduced in component form in a number of works by Cartan [2], Kawaguchi [10, 11] and Ehresmann [5]. But the first global definition is due to Barthel [1]. The geometry of N-connection in superspaces are considered in detail in [16], [14].

Email addresses: **eazizpour@gmail.com (E. Azizpour) amir.zarifi.m@gmail.com (M. H. Zarifi)**

Let us consider a vector superbundle $\mathscr{E} = (E, \pi_E, \mathscr{M})$ whose type fiber is $\mathscr{F}$ and $\pi^T : T\mathscr{E} \to T\mathscr{M}$ is the superdifferential of the map π_E. The kernel of this vector superbundle morphism being a subbundle of (TE, τ_E, E) is called the vertical subbundle over $\mathscr{E}$ and is denoted by $V\mathscr{E} = (VE, \tau_V, E)$. Its total space is $V\mathscr{E} = \bigcup_{u\in\mathscr{E}} \quad V_u, \quad$ where $V_u = ker\pi^T, \quad u\in\mathscr{E}$.
A nonlinear connection, N-connection [15, 16], in vector superbundle $\mathscr{E}$ is a splitting on the left of the exact sequence

$$0 \longmapsto V\mathscr{E} \overset{i}{\longmapsto} T\mathscr{E} \longmapsto T\mathscr{E}/V\mathscr{E} \longmapsto 0, \tag{2.1}$$

i.e. a morphism of vector superbundles $N : T\mathscr{E} \to V\mathscr{E}$ such that $N\circ i$ is the identity on $V\mathscr{E}$.
The kernel of the morphism N is called the horizontal subbundle and is denoted by (HE, τ_H, E). From the exact sequence (2.1) it follows that N-connection structure can be equivalently defined as a distribution $T_uE = H_uE \oplus V_uE, \quad u \in E$ on E defining a global decomposition, as a Whitney sum,

$$T\mathscr{E} = H\mathscr{E} \oplus V\mathscr{E}. \tag{2.2}$$

Locally a nonlinear connection in $\mathscr{E}$ is given by its coefficients

$$N_i^j(x,y,\eta,\theta), N_i^\beta(x,y,\eta,\theta), N_\alpha^j(x,y,\eta,\theta), N_\alpha^\beta(x,y,\eta,\theta).$$

In the tangent superbundle a local basis adapted to the given nonlinear connection N is introduced by

$$\left(\frac{\delta}{\delta x_i}, \frac{\delta}{\delta \eta_\alpha}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial \theta_\alpha}\right),$$

where

$$\frac{\delta}{\delta x_i} := \frac{\partial}{\partial x_i} - N_i^j \frac{\partial}{\partial y_j} - N_i^\alpha \frac{\partial}{\partial \theta_\alpha} \tag{2.3}$$

and

$$\frac{\delta}{\delta \eta_\alpha} := \frac{\partial}{\partial \eta_\alpha} - N_\alpha^i \frac{\partial}{\partial y_i} - N_\alpha^\beta \frac{\partial}{\partial \theta_\beta}. \tag{2.4}$$

Let $X = X^i \frac{\partial}{\partial x_i} + X^\alpha \frac{\partial}{\partial \eta_\alpha}$ be a supervector field in a coordinate neighborhood $\mathscr{U}$ of $\mathscr{M}$, then the vertical lift X^v and the complete lift X^c of X have the form

$$X^v = X^i \frac{\partial}{\partial y_i} + X^\alpha \frac{\partial}{\partial \theta_\alpha},$$

and

$$\begin{aligned} X^c \quad &= \quad \sum_{i=1}^{m} \left(X^i \frac{\partial}{\partial x_i} + \left(\sum_{j=1}^{m} y_j \frac{\partial X^i}{\partial x_j} + \sum_{\gamma=1}^{n} \theta_\gamma \frac{\partial X^i}{\partial \eta_\gamma} \right) \frac{\partial}{\partial y_i} \right) \\ &+ \quad \sum_{\alpha=1}^{n} \left(X^\alpha \frac{\partial}{\partial \eta_\alpha} + \left(\sum_{j=1}^{m} y_j \frac{\partial X^\alpha}{\partial x_j} + \sum_{\gamma=1}^{n} \theta_\gamma \frac{\partial X^\alpha}{\partial \eta_\gamma} \right) \frac{\partial}{\partial \theta_\gamma} \right). \end{aligned}$$

Definition 2.1. *A vertical endomorphism on the tangent superbundle $T\mathscr{M}$ is a (super) tensor field*

$$J : \mathscr{X}(T\mathscr{M}) \mapsto \mathscr{X}(T\mathscr{M})$$

satisfies in $ImJ = KerJ, \quad J^2 = 0$.

If J is a vertical endomorphism, the vertical differentiation d_J is the mapping $d_J = [i_J, d] = i_J od - doi_J$. In particular, for any superfunction f on $\mathscr{M}$, we have $d_J f = i_j df$.
Let $(x_i; \eta_\alpha)$ be local coordinates on $\mathscr{M}$ and $(x_i, y_i; \eta_\alpha, \theta_\alpha)$ the corresponding local coordinates on $T\mathscr{M}$. The Liouville supervector field C on $\mathscr{X}(T\mathscr{M})$ defined by

$$C = y_i \frac{\partial}{\partial y_i} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}. \tag{2.5}$$

Definition 2.2. *A morphism $h : \mathscr{X}(T\mathscr{M}) \mapsto \mathscr{X}(T\mathscr{M})$ is said to be a horizontal endomorphism on $\mathscr{M}$ if it satisfies the following conditions:*
(i) $h^2 = h$
(ii)$Kerh = \mathscr{X}^v(T\mathscr{M})$.

Assume h is a horizontal endomorphism. The supervector 1-form, or simply the vector 1-form, $[h, C]$ is said to be the tension of h. The vector 2-form $[J, h]$ is said to be the torsion of h.
Let h be a horizontal endomorphism. If $\mathscr{X}^h(T\mathscr{M}) := Imh$, then $\mathscr{X}(T\mathscr{M}) = \mathscr{X}^h(T\mathscr{M}) \oplus \mathscr{X}^v(T\mathscr{M})$ and $\mathscr{X}^h(T\mathscr{M})$ is called the supermodule of horizontal supervector fields. $v := (id - h) : \mathscr{X}(T\mathscr{M}) \mapsto \mathscr{X}(T\mathscr{M})$, is the vertical projection on $\mathscr{X}^v(T\mathscr{M})$ along $\mathscr{X}^h(T\mathscr{M})$. Also, we have $hoJ = 0$ and $Joh = J$.

Definition 2.3. *A morphism $\mathscr{F} : \mathscr{X}(T\mathscr{M}) \mapsto \mathscr{X}(T\mathscr{M})$ is said to be an almost complex structure on $\mathscr{M}$ if $\mathscr{F}^2 = -1$.*

Definition 2.4. *A supervector field S on $T\mathscr{M}$ is a superspray if*

$$J(S) = y_i \frac{\partial}{\partial y_i} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}. \tag{2.6}$$

When the coefficients of a superspray S are homogeneous of degree 2, we say that S is a homogeneous superspray.
If S is a homogeneous superspray and C the Liouville supervector field, then $[C,S]=S$. It is not difficult to show that if h is a horizontal endomorphism on $\mathscr{M}$ and S' an arbitrary superspray then $S:=hS'$ is also a superspray on $\mathscr{M}$. It satisfies the relation $h[C,S]=S$. So S is called the superspray associated to h.
A generalized Lagrange superspace is a pair $GL^{m,n}=(\mathscr{M},g(x,y,;\eta,\theta))$, where $g(x,y;\eta,\theta)$ is a distinguished tensor field on $T\mathscr{M}^o=T\mathscr{M}-\{0\}$, supersymmetric of superrank (m,n) . A Lagrange superspace is defined as a particular case of generalize Lagrange superspace when the distinguished tensor field on $\mathscr{M}$ can be expressed as

$$g_{ij}=\frac{1}{2}\frac{\partial^2 L}{\partial y_i\partial y_j},\; g_{i\beta}=\frac{1}{2}\frac{\partial^2 L}{\partial y_i\partial \theta_\beta},\; g_{\alpha j}=\frac{1}{2}\frac{\partial^2 L}{\partial \theta_\alpha\partial y_j},\; g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2 L}{\partial \theta_\alpha\partial \theta_\beta} \tag{2.7}$$

where $L:T\mathscr{M}\mapsto B_L$, is a superfunction called a Lagrangian on $\mathscr{M}$ (see [15]).
Locally, L is regular if and only if the matrix

$$g=\begin{bmatrix} g_{ij} & g_{i\beta}\\ g_{\alpha j} & g_{\alpha\beta}\end{bmatrix}$$

is invertible. For example, if $L=F^2$, where F will be defined in the following definition, then L is a regular Lagrangian. In this case L is a homogeneous superfunction of degree 2.
To define a (super) metric on a supermanifold, We consider the base manifold M of a vector superbundle $\mathscr{E}=(E,\pi_E,\mathscr{M})$ to be a connected and paracompact manifold.

Definition 2.5. *A metric structure on the total space E of a vector superbundle $\mathscr{E}$ is a supersymmetric, second order, covariant supertensor field g which in every point $u\in\mathscr{E}$ is given by nondegenerate supermatrix $g_{ab}=g(\partial_a,\partial_b)$ (with nonvanishing superdeterminant, $\det g\neq 0$).*

Definition 2.6. *A function $F:T\mathscr{M}\to B_L$ is called a Finsler metric (see [15]) if the following conditions are satisfied:*
(1) The restriction of F to $T\mathscr{M}^o=T\mathscr{M}-\{0\}$ is of the class G^∞ and F is only supersmooth on the image of the null cross–section in the tangent supermanifold to M.
(2) $F(x,\lambda y;\eta,\lambda\theta)=\lambda F(x,y;\eta,\theta)$, where λ is a real positive number.
(3) The restriction of F to the even subspace of $T\mathscr{M}^o$ is a positive function.
(4) If we put

$$g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y_i\partial y_j},\; g_{i\beta}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y_i\partial \theta_\beta},\; g_{\alpha j}=\frac{1}{2}\frac{\partial^2 F^2}{\partial \theta_\alpha\partial y_j},\; g_{\alpha\beta}=\frac{1}{2}\frac{\partial^2 F^2}{\partial \theta_\alpha\partial \theta_\beta} \tag{2.8}$$

then

$$g=\begin{bmatrix} g_{ij} & g_{i\beta}\\ g_{\alpha j} & g_{\alpha\beta}\end{bmatrix}$$

is invertible .

A pair $(\mathscr{M},F)$ is called a Finsler Supermanifold.
It is obvious that Finsler superspaces form a particular class of Lagrange superspaces with Lagrangian $\mathscr{L}=F^2$.

Definition 2.7. *The dynamics of a system $(T\mathscr{M},\omega,L)$, associated to a Lagrangian $L\in T\mathscr{M}$, is given by a supervector field $X\in\mathscr{X}(T\mathscr{M})$ satisfying the equation*

$$i_X\omega=-dL \tag{2.9}$$

where $\omega=dd_JL$.

It is shown that the Euler-Lagrange supervector field is a superspray [13].

Theorem 2.8. *([13]) On any Finsler supermanifold $(\mathscr{M},F)$, there is a homogeneous superspray*

$$S=y_j\frac{\partial}{\partial x_j}+\theta_\beta\frac{\partial}{\partial\eta_\beta}-2G^j(x,y;\eta,\theta)\frac{\partial}{\partial y_j}-2G^\beta(x,y;\eta,\theta)\frac{\partial}{\partial\theta_\beta}$$

where

$$G^j=\frac{1}{4}g^{jm}(y^k\frac{\partial^2F^2}{\partial x_k\partial y_m}-\frac{\partial^2F^2}{\partial\eta_\alpha\partial y_m}\theta_\alpha-\frac{\partial F^2}{\partial x_m})$$
$$-\frac{1}{4}g^{m\beta}(y^j\frac{\partial^2F^2}{\partial x_j\partial\theta_\gamma}+\frac{\partial^2F^2}{\partial\eta_\mu\partial\theta_\gamma}\theta_\mu-\frac{\partial F^2}{\partial\eta_\gamma}) \tag{2.10}$$

and

$$G^\beta=\frac{1}{4}g^{\beta m}(y^k\frac{\partial^2F^2}{\partial x_k\partial y_m}-\frac{\partial^2F^2}{\partial\eta_\alpha\partial y_m}\theta_\alpha-\frac{\partial F^2}{\partial x_m})$$
$$+\frac{1}{4}g^{\beta\gamma}(y^j\frac{\partial^2F^2}{\partial x_j\partial\theta_\gamma}+\frac{\partial^2F^2}{\partial\eta_\mu\partial\theta_\gamma}\theta_\mu-\frac{\partial F^2}{\partial\eta_\gamma}). \tag{2.11}$$

We call this superspray the **canonical superspray of a Finsler metric**.
Let $(\mathscr{M}, F)$ be a Finsler supermanifold and consider $T\mathscr{M}^o = T\mathscr{M} - \{0\}$ and denote by $VT\mathscr{M}^o$ the vertical superbundle over $T\mathscr{M}^o$. It is easy to show that a Finsler metric F allows to define a (super) metric g on the vertical superbundle $VT\mathscr{M}^o$, by setting $L = F^2$ and

$$g(JX, JY) = \omega(JX, Y) \tag{2.12}$$

for $X, Y \in T(TM)$. So the coefficients of this metric are superfunctions defined in (2.8).
If h is a horizontal endomorphism on $\mathscr{M}$ and $v = id - h$, g can be extended to $T\mathscr{M}$ by putting

$$G(X,Y) = g(JX, JY) + g(vX, vY),$$

where J is the vertical endomorphism.

3. A Horizontal endomorphism

We are now in position to define a horizontal endomorphism which is conservative and torsion-free. To do it we need to define a supervector 1-form $[J, X]$, where J is a vector 1-form and X a supervector field. Since J is a vector form of degree 0, for each supervector field Y on $T\mathscr{M}$ we have

$$\begin{aligned}[J,X]Y &= (-1)^{|X||Y|}\left(Y^i[\frac{\partial}{\partial y_i}, X] + Y^\alpha[\frac{\partial}{\partial \theta_\alpha}, X]\right) \\ &- (-1)^{|X||Y|}\left(Y(X^i)\frac{\partial}{\partial y_i} + Y(X^\alpha)\frac{\partial}{\partial \theta_\alpha}\right).\end{aligned}$$

An easy computation shows that

$$[J,X]Y = (-1)^{|X||Y|}[JY, X] - (-1)^{|X||Y|}J[Y, X]. \tag{3.1}$$

Theorem 3.1. *(1) Any superspray S generates a torsion-free horizontal endomorphism*

$$h = \frac{1}{2}(id + [J, S]), \tag{3.2}$$

where id is the identity map on $T(TM)$. The horizontal lift of a supervector field X on $\mathscr{M}$ is

$$X^h := hX^c = \frac{1}{2}(X^c + [X^v, S]). \tag{3.3}$$

(2) A superspray associated to h is

$$S_h = \frac{1}{2}(S + [C, S]). \tag{3.4}$$

If S is a homogeneous superspray, then $S_h = S$.
(3) The torsion of h vanishes.

Proof. (1) First, we show that h is a horizontal endomorphism. So let X be a homogeneous supervector field on $\mathscr{M}$. Since S is an even supervector field, thus

$$\begin{aligned}h(X^v) &= \frac{1}{2}\Big(X^v - J\Big\{X^i(\frac{\partial}{\partial x_i} - 2\frac{\partial G^j}{\partial y_i}\frac{\partial}{\partial y_j} - 2\frac{\partial G^\beta}{\partial y_i}\frac{\partial}{\partial \theta_\beta}) \\ &+ X^\alpha(\frac{\partial}{\partial \eta_\alpha} - 2\frac{\partial G^i}{\partial \theta_\alpha}\frac{\partial}{\partial y_i} - 2\frac{\partial G^\beta}{\partial \theta_\alpha}\frac{\partial}{\partial \theta_\beta})\Big\} - y_j(\frac{\partial X^i}{\partial x_j}\frac{\partial}{\partial y_i} + \frac{\partial X^\alpha}{\partial x_j}\frac{\partial}{\partial \theta_\alpha}) \\ &- \theta^\beta(\frac{\partial X^i}{\partial \eta_\beta}\frac{\partial}{\partial y_i} + \frac{\partial X^\alpha}{\partial \eta_\beta}\frac{\partial}{\partial \theta_\beta})\Big) = \frac{1}{2}(X^v - X^i\frac{\partial}{\partial y_i} - X^\alpha\frac{\partial}{\partial \theta_\alpha}) = 0.\end{aligned}$$

This shows that $X^v(T\mathscr{M}) \subset kerh$.
Now, let $Y \in kerh$, then

$$0 = 2h(Y) = Y + [JY, S] - J[Y, S],$$

so $Y = -[JY, S] + J[Y, S]$. If we compute JY, it follows that

$$JY = -J[JY, S] = 0.$$

Thus $kerh \subset X^v(T\mathscr{M})$ and therefore $X^v(T\mathscr{M}) = kerh$.
It is clear that for any supervector field $X^v \in \mathscr{X}(T\mathscr{M})$, we have $h^2(X^v) = 0$. On the other hand

$$\begin{aligned}h^2(X^c) &= \frac{1}{2}\Big(hX^c + h[JX^c, S] - hoJ[X^c, S]\Big) \\ &= \frac{1}{2}\Big(hX^c + h[X^v, S]\Big) = hX^c.\end{aligned}$$

This shows that on $\mathscr{X}(T\mathscr{M})$ we have $h^2 = h$.

(2) If $\tilde{S}$ is an arbitrary superspray on $\mathscr{M}$ and h is the horizontal endomorphism defined by (3.2), then $Joh(\tilde{S}) = C$. So $S_h = h(\tilde{S})$ is a superspray.
Now let $\tilde{S}$ has the local form

$$\tilde{S} = y_i \frac{\partial}{\partial x_i} + \theta_\alpha \frac{\partial}{\partial \eta_\alpha} - 2\tilde{G}^i \frac{\partial}{\partial y_i} - 2\tilde{G}^\alpha \frac{\partial}{\partial \theta_\alpha}.$$

It is not difficult to show that $J[\tilde{S}, S] = -S + \tilde{S}$. If S is a homogeneous superspray, i.e. G^i and G^α are superfunctions of degree two, then $[C, S] = S$ and

$$h(\tilde{S}) = \frac{1}{2}(\tilde{S} + [J\tilde{S}, S] - J[\tilde{S}, S]) = S.$$

(3) We begin this part of proof with the definition of horizontal endomorphism h, thus we have

$$[J, h] = \frac{1}{2}[J, id] + \frac{1}{2}[J, [J, S]].$$

It is clear that $[J, id] = 0$, so we show that $[J, [J, S]] = 0$. Note that in this case J is an even 1-vector valued form and S an even supervector field. From the Bianchi identities for the lie superalgebra of vector-valued forms, we have

$$(-1)^{1.0}[J, [J, S]] + (-1)^{1.1}[J, [S, J]] + (-1)^{0.1}[S, [J, J]] = 0.$$

Apply (3.1) to $[S, J]$, we see that $[S, J] = -[J, S]$. Since $[J, J] = 0$, therefore $[J, [J, S]] = 0$ and the torsion of h is zero. □

Lemma 3.2. *If h is the horizontal endomorphism defined by (3.2), then there is a unique almost complex structure $\mathscr{F}$ on $T\mathscr{M}$ such that*

$$\mathscr{F} oJ = h, \quad \mathscr{F} oh = -J.$$

Proof. If we use the above conditions, it is easy to see that $\mathscr{F}$ permutes the vertical and horizontal superspaces if and only if

$$\begin{aligned} \mathscr{F}(\frac{\partial}{\partial x_i}) &= -\frac{\partial}{\partial y_i} + N_i^j \frac{\delta}{\delta x_j} + N_i^\alpha \frac{\delta}{\delta \eta_\alpha} \quad , \quad \mathscr{F}(\frac{\partial}{\partial y_i}) = \frac{\delta}{\delta x_i}, \\ \mathscr{F}(\frac{\partial}{\partial \eta_\alpha}) &= -\frac{\partial}{\partial \theta_\alpha} + N_\alpha^i \frac{\delta}{\delta x_i} + N_\alpha^\beta \frac{\delta}{\delta \eta_\beta} \quad , \quad \mathscr{F}(\frac{\partial}{\partial \theta_\alpha}) = \frac{\delta}{\delta \eta_\alpha}. \end{aligned}$$

For example $\mathscr{F} oJ = h$ implies that $\mathscr{F} oJ(\frac{\partial}{\partial x_i}) = \frac{\delta}{\delta x_i}$, so $\mathscr{F}(\frac{\partial}{\partial y_i}) = \frac{\delta}{\delta x_i}$. Similarly, $\mathscr{F}(\frac{\partial}{\partial \theta_\alpha}) = \frac{\delta}{\delta \eta_\alpha}$. Also $\mathscr{F} oh = -J$ implies that $\mathscr{F}(\frac{\delta}{\delta x_i}) = -\frac{\partial}{\partial y_i}$, so $\mathscr{F}(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i} + N_i^j \frac{\delta}{\delta x_j} + N_i^\alpha \frac{\delta}{\delta \eta_\alpha}$. □

Definition 3.3. *With respect to the (super) metric G on $T\mathscr{M}$, we define the Kahler form*

$$K(X, Y) = G(X, JY) - G(JX, Y). \tag{3.5}$$

Theorem 3.4. *Let h be a horizontal endomorphism defined by (3.2). So*

$$i_v \omega = K.$$

Proof. The canonical expression of the vertical projection $v = 1 - h$ is

$$v = (N_i^j \frac{\partial}{\partial y_j} + N_i^\beta \frac{\partial}{\partial \theta_\beta}) \otimes dx_i - (N_\alpha^i \frac{\partial}{\partial y_i} + N_\alpha^\beta \frac{\partial}{\partial \theta_\beta}) \otimes d\eta_\alpha + \frac{\partial}{\partial y_i} \otimes dy_i - \frac{\partial}{\partial \theta_\alpha} \otimes d\theta_\alpha.$$

A long but standard computation shows that

$$\begin{aligned} i_v \omega = & \frac{\partial^2 L}{\partial y_j \partial y_i} N_k^j dx_k \wedge dx_i - \frac{\partial^2 L}{\partial y_j \partial y_i} N_\alpha^j d\eta_\alpha \wedge dx_i + \frac{\partial^2 \mathscr{L}}{\partial y_j \partial y_i} dy_j \wedge dx_i \\ - & (-1)^L \left\{ \frac{\partial^2 L}{\partial \theta_\alpha \partial y_i} N_k^\alpha dx_k \wedge dx_i + \frac{\partial^2 L}{\partial \theta_\alpha \partial y_i} N_\alpha^\alpha d\eta_\beta \wedge dx_i + \frac{\partial^2 \mathscr{L}}{\partial \theta_\alpha \partial y_i} d\theta_\alpha \wedge dx_i \right\} \\ - & (-1)^L \left\{ \frac{\partial^2 L}{\partial y_i \partial \theta_\alpha} N_j^i dx_j \wedge d\eta_\alpha - \frac{\partial^2 L}{\partial y_i \partial \theta_\alpha} N_\beta^i d\eta_\beta \wedge d\eta_\alpha + \frac{\partial^2 \mathscr{L}}{\partial y_i \partial \theta_\alpha} dy_i \wedge d\eta_\alpha \right\} \\ - & \left\{ \frac{\partial^2 L}{\partial \theta_\beta \partial \theta_\alpha} N_i^\beta dx_i \wedge d\eta_\alpha + \frac{\partial^2 L}{\partial \theta_\beta \partial \theta_\alpha} N_\gamma^\beta d\eta_\gamma \wedge d\eta_\alpha + \frac{\partial^2 \mathscr{L}}{\partial \theta_\beta \partial \theta_\alpha} d\theta_\beta \wedge d\eta_\alpha \right\}. \end{aligned}$$

Now, it is easy to check that for two supervector fields $X, Y \in \mathscr{X}(T\mathscr{M})$, we have

$$(i_v \omega)(X, Y) = \omega(vX, Y) + \omega(X, vY). \tag{3.6}$$

Since $\omega(X, vY) = -(-1)^{XY} \omega(vX, Y)$ so

$$\begin{aligned} (i_v \omega)(X, Y) &= g(vX, JY) - (-1)^{XY} g(vY, JX) = \{g(JvX, JJY)\} \\ &+ g(vvX, vJY) - (-1)^{XY} \{g(vvY, vJX) + g(JvY, JJX)\} \\ &= G(vX, JY) - (-1)^{XY} G(vY, JX) \end{aligned}$$

But $G(vX, JY) = G(X, JY)$, thus $(i_v \omega)(X, Y) = K(X, Y)$. □

Definition 3.5. *A nonlinear connection is called Lagrangian if the horizontal superspace is Lagrangian with respect to the 2-form $\omega = dd_J L$, i.e. if $\omega(hX, hY) = 0$ for any $X, Y \in \mathscr{X}(T\mathscr{M})$.*

An easy computation will show that if a nonlinear connection is Lagrangian then $i_h\omega = \omega$. So from the above proposition we have

$$2\omega = i_{id}\omega = i_h\omega + i_v\omega$$

therefore $K = \omega$.

Theorem 3.6. *Consider a regular homogeneous Lagrangian L and N a Lagrangian connection. There exist a unique horizontal endomorphism h on $\mathscr{M}$ such that*
(i) h is conservative, i.e. $d_h L = 0$,
(ii) h is torsion-free,
(iii) The tension of h is zero, i.e. $[h, C] = 0$.
Explicitly, h is given by

$$h = \frac{1}{2}(id + [J, S]) \tag{3.7}$$

where S is the canonical superspray of a Finsler metric.

Proof. Let $(x_i; \eta_\alpha)$ be local coordinates on $\mathscr{M}$ and $(x_i, y_i; \eta_\alpha, \theta_\alpha)$ the corresponding local coordinates on $T\mathscr{M}$. It should be mentioned that we assume L is a homogeneous Lagrangian superfunction of degree $K > 1$ with respect to (y, θ). We proved before that $h = \frac{1}{2}(id + [J, S])$ is a torsion-free horizontal endomorphism. Given the local forms of $h = dx_i \otimes \frac{\delta}{\delta x_i} + d\eta_\alpha \otimes \frac{\delta}{\delta \eta_\alpha}$ and $C = y_i \frac{\partial}{\partial y_i} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha}$ and using the method used in Lemma 3.5, it is easy to see that $[h, C] = 0$. To complete the proof, we only need to prove that $d_h L = 0$. Let S be the canonical superspray introduced in theorem 2.8. As we mentioned earlier, S is even supervector field, so for any supervector field X on $T\mathscr{M}$, we have $(i_S\omega)(X) = \omega(S, X)$. Since $K = \omega$ thus

$$\begin{aligned}(i_S\omega)(X) &= G(S, JX) - G(JS, X) = -g(vC, vX) \\ &= -g(vC, J\mathscr{F}X) = -\omega(C, \mathscr{F}X).\end{aligned}$$

Now, we show that for any homogeneous supervector field $X \in \mathscr{X}(T\mathscr{M})$, $\omega(X, \mathscr{F}X) = i_v dL(X)$. So if X has a local form $X = X^i \frac{\partial}{\partial x_i} + \bar{X}^i \frac{\partial}{\partial y_i} + X^\alpha \frac{\partial}{\partial \eta_\alpha} + \bar{X}^\alpha \frac{\partial}{\partial \theta_\alpha}$, then we have

$$\begin{aligned}(i_v dL)(X) &= \frac{\partial L}{\partial y_i}\left(N^i_k dx_k - N^i_\alpha d\eta_\alpha + dy_i\right)(X) \\ &\quad -(-1)^{|L|}\frac{\partial L}{\partial \theta_\alpha}\left(N^\alpha_i dx_i + N^\alpha_\beta d\eta_\beta + d\theta_\alpha\right)(X) \\ &= \frac{\partial L}{\partial y_i}\left(N^i_k X^k - (-1)^{|X|} N^i_\alpha X^\alpha + \bar{X}^i\right) \\ &\quad -(-1)^{|L|}\frac{\partial L}{\partial \theta_\alpha}\left(N^\alpha_i X^i + (-1)^{|X|} N^\alpha_\beta X^\beta + (-1)^{|X|}\bar{X}^\alpha\right).\end{aligned}$$

One can easily check that $\omega(X, \mathscr{F}X) = i_v dL(X)$. Now, $i_S\omega = -dL$, because S is the canonical superspray and $dL = d_v L + d_h L$ then $d_h L = 0$. □

Let h be the horizontal endomorphism (3.2), the horizontal differential operator is defined by

$$d_h L(X) := dL(hX),$$

where X is a homogeneous supervector field on $\mathscr{M}$.
The horizontal covariant derivatives of a Lagrange superfunction L with respect to even or odd coordinates are denoted respectively by $L_{|i} = \frac{\delta L}{\delta x_i}$ and $L_{|\alpha} = \frac{\delta L}{\delta \eta_\alpha}$. In the following theorem, we use the canonical superspray to have a local expression for the horizontal covariant derivative of a Lagrange superfunction.

Theorem 3.7. *Let h be the horizontal endomorphism (3.2). The horizontal covariant derivatives of a Lagrange superfunction L are*

$$L_{|i} = \frac{1}{2}\frac{\partial(S(L))}{\partial y_i}, \tag{3.8}$$

$$L_{|\alpha} = \frac{1}{2}\frac{\partial(S(L))}{\partial \theta_\alpha} \tag{3.9}$$

Proof. First we compute the right hand of the above formulas. Then we have

$$\begin{aligned}\frac{\partial(S(L))}{\partial y_i} &= \frac{\partial L}{\partial x_i} + y_j \frac{\partial^2 L}{\partial y_i \partial x_j} + \theta_\alpha \frac{\partial^2 L}{\partial y_i \partial \eta_\alpha} - 2N^j_i \frac{\partial L}{\partial y_j} \\ &- 4G^j \frac{\partial^2 L}{\partial y_i \partial y_j} - 2N^\alpha_i \frac{\partial L}{\partial \theta_\alpha} - 4G^\alpha \frac{\partial^2 L}{\partial y_i \partial \theta_\alpha},\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{\partial(S(L))}{\partial\theta_\alpha} &= \frac{\partial L}{\partial\eta_\alpha}+y_j\frac{\partial^2 L}{\partial\theta_\alpha\partial x_j}-\theta_\beta\frac{\partial^2 L}{\partial\theta_\alpha\partial\eta_\beta}-2N^j_\alpha\frac{\partial L}{\partial y_j}\\ &- 4G^j\frac{\partial^2 L}{\partial\theta_\alpha\partial y_j}-2N^\beta_\alpha\frac{\partial L}{\partial\theta_\beta}+4G^\beta\frac{\partial^2 L}{\partial\theta_\alpha\partial\theta_\beta}.\end{aligned} \tag{3.11}$$

If we now replace the superfunctions G^i and G^α with (2.10) and (2.11) respectively, then some terms of (3.10) and (3.11) cancel with some terms of the replaced sentences and the only terms that survive are $L_{|i}=2\frac{\delta L}{\delta x_i}$ and $L_{|\alpha}=2\frac{\delta L}{\delta\eta_\alpha}$, and the theorem is proved. □

From the above theorem we found a condition under which the horizontal differential of a Lagrangian L is vanishes. In other words we found that $S(L)=0$ implies $d_hL=0$.
In the previous theorem, we showed that if L is a homogeneous superfunction then there exist a unique horizontal endomorphism h on $\mathscr{M}$ such that $d_hL=0$. In the following, we will show that this result is not true for an arbitrary Lagrangian L. We will find non homogeneous Lagrangian superfunctions for which $d_hL\neq 0$.
Let $\mathscr{M}$ be a Riemannian supermanifold with a supermetric $\tilde{g}$. In the standard local coordinate system (x,η) in $\mathscr{M}$, $\tilde{g}$ is expressed in the form

$$\tilde{g}=\tilde{g}_{ij}dx_i\otimes dx_j+\tilde{g}_{i\alpha}dx_i\otimes d\eta_\alpha+\tilde{g}_{\alpha i}d\eta_\alpha\otimes dx_i+\tilde{g}_{\alpha\beta}d\eta_\alpha\otimes d\eta_\beta$$

where $\tilde{g}_{ij},\tilde{g}_{i\alpha}$ and $\tilde{g}_{\alpha\beta}$ are superfunctions on $\mathscr{M}$ and $\tilde{g}_{ij}=\tilde{g}_{ji},\tilde{g}_{\alpha\beta}=-\tilde{g}_{\beta\alpha},\tilde{g}_{i\alpha}=\tilde{g}_{\alpha i}$. The superfunction

$$L(x,y,\eta,\theta)=\tilde{g}_{ij}(x,\eta)y_iy_j+\tilde{g}_{i\alpha}y_i\theta_\alpha+\tilde{g}_{\alpha\beta}\theta_\alpha\theta_\beta \tag{3.12}$$

is a regular Lagrangian on $T\mathscr{M}$.
Now we are ready to introduce a Lagrangian superfunction which is not homogeneous and its horizontal differential is not zero. To construct this superfunction, let L be the superfunction (3.12) and ϕ an even homogeneous superfunction on the supermanifold $\mathscr{M}$, then

$$L'=L(x,y,\eta,\theta)+\frac{\partial\phi}{\partial x_i}(x,\eta)y_i+\frac{\partial\phi}{\partial\eta_\alpha}(x,\eta)\theta_\alpha \tag{3.13}$$

is a regular Lagrangian on $T\mathscr{M}$. Using (2.9), it is easy to check that the Cartan 2-forms associated to the superfunctions L and L' are equal (see [8]), then the canonical superspray associated to these superfunctions are equal (see (2.10) and (2.11)). On the other hand, in the definition of the endomorphism (3.7) we see that it depends on the canonical superspray, so we conclude that L and L' have the same horizontal endomorphism.
In local coordinates, let $X=X^i\frac{\partial}{\partial x_i}+\bar{X}^i\frac{\partial}{\partial y_i}+X^\alpha\frac{\partial}{\partial\eta_\alpha}+\bar{X}^\alpha\frac{\partial}{\partial\theta_\alpha}$ be a homogeneous supervector field on $T\mathscr{M}$. We have showed that $d_hL=0$, so

$$\begin{aligned} d_hL'(X) &= d_h\left(\frac{\partial\phi}{\partial x_i}(x,\eta)y_i+\frac{\partial\phi}{\partial\eta_\alpha}(x,\eta)\theta_\alpha\right)(X)\\ &= d\left(\frac{\partial\phi}{\partial x_i}y_i+\frac{\partial\phi}{\partial\eta_\alpha}\theta_\alpha)\right)(h(X))\\ &= \left(\frac{\partial^2\phi}{\partial x_j\partial x_i}y_i-\frac{\partial\phi}{\partial x_k}N^k_j+\frac{\partial^2\phi}{\partial x_j\partial\eta_\alpha}\theta_\alpha-\frac{\partial\phi}{\partial\eta_\beta}N^\beta_j\right)X^j\\ &\quad -(-1)^{|X|}\left(\frac{\partial^2\phi}{\partial\eta_\beta\partial x_i}y_i-\frac{\partial\phi}{\partial x_j}N^j_\beta+\frac{\partial^2\phi}{\partial\eta_\beta\partial\eta_\alpha}\theta_\alpha+\frac{\partial\phi}{\partial\eta_\gamma}N^\gamma_\beta\right)X^\beta\end{aligned}$$

Now we need to get the coefficients of X^j and X^β in the last equation to be nonzero. We can do this using a linear type of the superfunction ϕ in x and η. Then $d_hL'\neq 0$.

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