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Best proximity points for weak $\mathscr{MT}$-cyclic Kannan contractions

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Abstract

In this paper, we introduce a notion of weak $\mathscr{MT}$-cyclic Kannan contractions with respect to a $\mathscr{MT}$-function φ and then we shall prove some new convergent and existence theorems of best proximity point theorems for these contractions in uniformly Banach spaces.

1. Introduction

Let A and B be nonempty subsets of a Banach space E. A map T on $A\cup B$ into $A\cup B$ is called a *cyclic* mapping if $T(A)\subset B$ and $T(B)\subset A$. Let $T: A\cup B\to A\cup B$ be a cyclic map. For any nonempty subsets A and B of E, let $dist(A,B)=\inf\{||x-y||: x\in A, y\in B\}$. A point $x\in A\cup B$ is called to be a best proximity point for T if $||x-Tx||=dist(A,B)$.

In [2] A. Anthony Eldred and P. Veeramani introduced cyclic contraction mappings and then in a uniformly convex Banach space a theorem was established which ensures the existence of a best proximity point of cyclic contractions. Afterward, in these spaces, C. Di Bari et al. in [13] introduced the notion of cyclic Meir-Keeler contractions and proved the existence of a best proximity point for cyclic Meir-Keeler contractions in the case of two sets. After this, this result was generalized for p sets by S. Karpagam, Sushama Agrawal [11]. In [4] a new class of maps was introduced, called cyclic φ-contraction which contains the cyclic contractions maps as a subclass and for this type of contractive conditions, in uniformly convex Banach spaces, results of best proximity points were obtained. Many authors have been investigated the existence, uniqueness and convergence of iterates to the best proximity point under weaker assumptions over T; see [1]-[5], [8], [10]-[14], [16]-[18], and [22]-[24] and their references. See also [25, 26].

The notion of weak cyclic Kannan contractions (see below definition) was introduced by M. A. Petric [14]; see also [21]-[23].

Definition 1.1. *[14] Let A and B be nonempty subsets of a metric space (X,d). If a map $T: A\cup B\to A\cup B$ satisfies*

(i) $T(A)\subset B$ and $T(B)\subset A$;
(ii) there exists a $\alpha\in[0,\frac{1}{2})$ such that

$$d(Tx,Ty)\le \alpha[d(x,Tx)+d(y,Ty)]+(1-2\alpha)dist(A,B) \text{ for any } x\in A \text{ and } y\in B,$$

then T is called a weak cyclic Kannan contraction on $A\cup B$.

The existence and convergence theorems of best proximity points in uniformly convex Banach spaces is proved as follows:

Theorem 1.2. *[14] Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T: A\cup B\to A\cup B$ be a weak cyclic Kannan contraction map. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x\in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

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2. Preliminaries

Definition 2.1. *[6, 7, 20] A function $\varphi : [0,\infty) \to [0,1)$ is said to be a $\mathscr{MT}$-function if it satisfies Mizoguchi-Takahashi's condition (i.e. $\limsup_{s\to t+0}\varphi(s) < 1$ for all $t \in [0,\infty)$).*

Obviously, if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing or nonincreasing function, then φ is a $\mathscr{MT}$-function. So, in particular, if $\varphi : [0,\infty) \to [0,1)$ is defined by $\varphi(t) = c$, where $c \in [0,1)$, then φ is a $\mathscr{MT}$-function. It is known that $\varphi : [0,\infty) \to [0,1)$ is a $\mathscr{MT}$-function *if and only if* for each $t \in [0,\infty)$, there exist $r_t \in [0,1)$ and $\varepsilon_t > 0$ such that $\varphi(s) \le r_t$ for all $s \in [t, t+\varepsilon_t)$. For more details, one can see Remark 2.5 in [7].
Note that if φ is a $\mathscr{MT}$-function then clearly $\psi := \frac{\varphi}{2-\varphi}$ is a $\mathscr{MT}$-function.
The notion of $\mathscr{MT}$-cyclic contraction with respect to a $\mathscr{MT}$-function φ (see below definition) is introduced by W.-S. Du et al [8] that contain cyclic contractions as a subclass. Some new existence and convergence theorems of iterates of best proximity points for $\mathscr{MT}$-cyclic contractions has been proved.

Lemma 2.2. *[2] Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||z_n - y_n|| \to dist(A,B)$.
(ii) For every $\varepsilon > 0$ there exists N_0 such that for all $m > n \ge N_0$, $||x_m - y_n|| \le dist(A,B) + \varepsilon$.
Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \ge N_1$, $||x_m - z_n|| \le \varepsilon$.

Lemma 2.3. *[2] Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*
(i) $||x_n - y_n|| \to dist(A,B)$.
(ii) $||z_n - y_n|| \to dist(A,B)$.
Then $||x_n - z_n|| \to 0$.

In this paper, we first define weak $\mathscr{MT}$-cyclic Kannan contractions with respect to a $\mathscr{MT}$-function φ and then we generalized Theorem P for these contractions in uniformly convex Banach spaces.

3. Main results

Definition 3.1. *Let A and B be nonempty subsets of a metric space (X,d). If a map $T : A\cup B \to A\cup B$ satisfies*

(MTK1) $T(A) \subset B$ and $T(B) \subset A$;
(MTK2) there exists a $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that

$$d(Tx,Ty) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx)+d(y,Ty)] + (1-\varphi(d(x,y)))dist(A,B) \text{ for any } x \in A \text{ and } y \in B,$$

then T is called a weak $\mathscr{MT}$-cyclic Kannan contraction with respect to φ on $A\cup B$.

Remark 3.2. *It is obvious that (MTK2) implies that for any $x \in A$ and $y \in B$, T satisfies $d(Tx,Ty) - dist(A,B) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx) + d(y,Ty) - 2dist(A,B)] \le 0$ and so $d(Tx,Ty) \le d(x,y)$, for any $x \in A$ and $y \in B$.*

In the case that $dist(A,B) = 0$, we can obtain the following theorem that generalize Kannan theorem [19] and Theorem 2 in [14].

Theorem 3.3. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that $A\cap B \ne \emptyset$ and $T : A\cup B \to A\cup B$ be a weak $\mathscr{MT}$-cyclic Kannan contraction with respect to φ such that*

$$d(Tx,Ty) \le \frac{1}{2}\varphi(d(x,y))[d(x,Tx)+d(y,Ty)] \text{ for any } x \in A \text{ and } y \in B. \tag{3.1}$$

Then T has a unique fixed point z in $A\cap B$.

Proof. Suppose that x is an arbitrary point in A. Then by (3.1), we have

$$d(T^n x, T^{n+1}x) \le \frac{1}{2}\varphi(d(T^{n-1}x, T^n x))[d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x)],$$

so,

$$d(T^n x, T^{n+1}x) \le \psi(d(T^{n-1}x, T^n x))d(T^{n-1}x, T^n x), \tag{3.2}$$

where $\psi := \frac{\varphi}{2-\varphi}$; by Definition 2.1 ψ is a $\mathscr{MT}$-function, so $\psi(t) < 1$ for any $t > 0$; therefore we have,

$$d(T^n x, T^{n+1}x) < d(T^{n-1}x, T^n x),$$

for any $n \in N$. Thus the sequence $\{d(T^n x, T^{n+1}x)\}$ is decreasing in $[0,\infty)$. Then

$$t_0 := \lim_{n\to\infty} d(T^n x, T^{n+1}x) = \inf_{n\to\infty} d(T^n x, T^{n+1}x) \ge 0. \tag{3.3}$$

Since ψ is a $\mathscr{MT}$-function, there exist $r_{t_0} \in [0,1)$ and $\varepsilon_{t_0} > 0$ such that $\psi(s) \le r_{t_0}$ for all $s \in [t_0, t_0+\varepsilon_{t_0})$. By (3.3), there exists $\ell \in N$, such that

$$t_0 \leq d(T^n x, T^{n+1}x) < t_0 + \varepsilon_{t_0}$$

for all $n \in N$ with $n \geq \ell$. Hence $\psi(d(T^n x, T^{n+1}x)) \leq r_{t_0}$ for all $n \geq \ell$. Let

$$\lambda := \max\{\psi(d(T^1 x, T^2 x)), \psi(d(T^2 x, T^3 x)), \cdots, \psi(d(T^{\ell-1}x, T^{\ell}x)), r_{t_0}\}.$$

Then

$$0 \leq \psi(d(T^n x, T^{n+1}x)) \leq \lambda < 1 \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Now, by (3.2) and (3.4), we have $d(T^n x, T^{n+1}x) \leq \lambda d(T^{n-1}x, T^n x)$ and by induction, we conclude that $d(T^n x, T^{n+1}x) \leq \lambda^n d(x, Tx)$, for any $n \in N$.
Now, if $m > n$,

$$\begin{aligned} d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + \cdots + d(T^{m-1}x, T^m x) \\ &\leq \lambda^n d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &\leq \frac{\lambda^n}{1-\lambda} d(x, Tx), \end{aligned}$$

Since $\lambda \in (0,1)$, $\lim_{n\to\infty} \lambda^n = 0$. Thus, $\{T^n x\}$ is a Cauchy sequence. Since A is closed, there exists $z \in A$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \tag{3.5}$$

Now, we show that $Tz = z$.
By (3.2), we have $d(Tz, T^{n+1}x) \leq \psi(d(z, T^n x)) d(z, T^n x)$, and so

$$\lim_{n\to\infty} d(Tz, T^{n+1}x) = 0. \tag{3.6}$$

Hence, by (3.5), (3.6) and Lemma 2.3, $d(Tz, z) = 0$, or $Tz = z$. We prove z is unique. Let v be another point such that $Tv = v$. Then by (3.1),

$$d(v,z) = d(Tv, Tz) \leq \frac{1}{2}\varphi(d(v,z))[d(Tv,v) + d(z,Tz)] = 0.$$

So, $v = z$. □

For the main results of this paper we need the following lemma.

Lemma 3.4. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of uniformly convex Banach space X and $T : A\cup B \to A\cup B$ cyclic map with respect to $\mathscr{MT}$-function φ satisfying*

$$||Tx - T^2x|| \leq \varphi(||x - Tx||)||x - Tx|| + (1 - \varphi(||x - Tx||))dist(A,B) \tag{3.7}$$

for all $x \in A\cup B$. Then
(i) $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = dist(A,B)$ for all $x \in A\cup B$.
(ii) $\lim_{n\to\infty} ||T^{2n}x - T^{2n\pm 2}x|| = 0$ for all $x \in A\cup B$.
(iii) z is a best proximity point if and only if z is a fixed point of T^2.

Proof. First we prove (i). This proof follows similar patterns as Theorem 2.1 in [8]. We include the proof for completeness reasons. Let $x \in A\cup B$ be given. Clearly, $dist(A,B) \leq ||T^n x - T^{n+1}x||$ for all $n \in \mathbb{N}$. If there exists $j \in \mathbb{N}$ such that $T^j x = T^{j+1}x \in A\cap B$, then $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$ and $dist(A,B) = 0$; therefore (i). So it suffices to consider the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By Remark 3.2, it is easy to see that the sequence $\{||T^n x - T^{n+1}x||\}$ is nonincreasing in $(0,\infty)$ and so it is convergent. Set

$$\widehat{t} := \lim_{n\to\infty} ||T^n x - T^{n+1}x||. \tag{3.8}$$

Since φ is a $\mathscr{MT}$-function, there exist $r_{\widehat{t}} \in [0,1)$ and $\varepsilon_{\widehat{t}} > 0$ such that $\varphi(s) \leq r_{\widehat{t}}$ for all $s \in [\widehat{t}, \widehat{t} + \varepsilon_{\widehat{t}})$. By (3.8), there exists $\ell \in N$, such that

$$\widehat{t} \leq ||T^n x - T^{n+1}x|| < \widehat{t} + \varepsilon_{\widehat{t}}$$

for all $n \in N$ with $n \geq \ell$. Hence $\varphi(||T^n x - T^{n+1}x||) \leq r_{\widehat{t}}$ for all $n \geq \ell$. Let

$$\lambda := \max\{\varphi(||T^1 x - T^2 x||), \varphi(||T^2 x - T^3 x||), \cdots, \varphi(||T^{\ell-1}x - T^{\ell}x||), r_{\widehat{t}}\}.$$

Then $0 \leq \varphi(||T^n x - T^{n+1}x||) \leq \lambda < 1$ for all $n \in N$. If $x \in A$, then, by $(MTK1)$, we have $T^{2n-1}x \in A$ and $T^{2n}x \in B$ for all $n \in N$. Notice first that $(MTK2)$ implies that

$$||Tx - T^2x|| \leq \varphi(||x - Tx||)||x - Tx|| + (1 - \varphi(||x - Tx||))dist(A,B) \leq \lambda||x - Tx|| + dist(A,B)$$

and

$$\begin{aligned}||T^3x-T^4x|| &\leq \varphi(||T^2x-T^3x||)||T^2x-T^3x||+(1-\varphi(||T^2x-T^3x||))dist(A,B)\\ &\leq \varphi(||T^2x-T^3x||)\left[\lambda||x-Tx||+dist(A,B)\right]+(1-\varphi(||T^2x-T^3x||))dist(A,B)\\ &= \varphi(||T^2x-T^3x||)\lambda||x-Tx||+dist(A,B)\\ &\leq \lambda^2||x-Tx||+dist(A,B).\end{aligned}$$

Hence, by induction, one can obtain

$$dist(A,B)\leq ||T^{n+1}x-T^{n+2}x||\leq \lambda^n||x-Tx||+dist(A,B) \text{ for all } n\in\mathbb{N} \tag{3.9}$$

Since $\lambda\in(0,1)$, $\lim_{n\to\infty}\lambda^n=0$. Using (3.8) and (3.9), we obtain $\lim_{n\to\infty}||T^nx-T^{n+1}x||=dist(A,B)$. So (i) is proved.
To see (ii), let $x\in A\cup B$. By using (i), we have $\lim_{n\to\infty}||T^{2n}x-T^{2n-1}x||=dist(A,B)$ and $\lim_{n\to\infty}||T^{2n-2}x-T^{2n-1}x||=dist(A,B)$. Lemma 2.3 concludes that

$$\lim_{n\to\infty}||T^{2n}x-T^{2n-2}x||=0,$$

for any $x\in A\cup B$. In the same way, from $\lim_{n\to\infty}||T^{2n}x-T^{2n+1}x||=dist(A,B)$, $\lim_{n\to\infty}||T^{2n+2}x-T^{2n+1}x||=dist(A,B)$ and Lemma 2.3 we can obtain

$$\lim_{n\to\infty}||T^{2n}x-T^{2n+2}x||=0,$$

for any $x\in A\cup B$.
Now we prove (iii). Let z be a fixed point of T^2 but it is not a best proximity point of T, i.e. $dist(A,B)\leq||z-Tz||$. Then by (3.1) we have

$$\begin{aligned}||z-Tz|| &= ||T^2z-Tz||\leq\varphi(||z-Tz||)||z-Tz||+(1-\varphi(||z-Tz||))dist(A,B)\\ &< \varphi(||z-Tz||)||z-Tz||+(1-\varphi(||z-Tz||))||z-Tz||=||z-Tz||,\end{aligned}$$

a contradiction.
Now, if z is a best proximity point of T i.e. $||z-Tz||=dist(A,B)$ then from (3.1) we have $||T^2z-Tz||=dist(A,B)$. So by Lemma 2.3, $T^2z=z$ which shows that (iii) is true. □

The following lemma can be obtained immediately from Lemma 3.4.

Lemma 3.5. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of uniformly convex Banach space X and $T: A\cup B\to A\cup B$ cyclic map. Suppose that there exists a nondecreasing (or nonincreasing) function $\tau:[0,\infty)\to[0,1)$ such that*

$$||Tx-T^2x||\leq\tau(||x-Tx||)||x-Tx||+(1-\tau(||x-Tx||))dist(A,B) \text{ for any } x\in A \text{ and } y\in B.$$

Then (i) $\lim_{n\to\infty}||T^nx-T^{n+1}x||=dist(A,B)$ for all $x\in A\cup B$.
(ii) $\lim_{n\to\infty}||T^{2n}x-T^{2n\pm2}x||=0$ for all $x\in A\cup B$.
(iii) z is a best proximity point if and only if z is a fixed point of T^2.

The following result is indeed proved in [8], but we give the proof for the sake of completeness.

Theorem 3.6. *[8] Let A and B be nonempty subsets of a metric space X and $T: A\cup B\to A\cup B$ be a cyclic map. Let $x_1\in A$ be given. Define an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1}=Tx_n$ for $n\in\mathbb{N}$. Suppose that*

(i) $d(Tx,Ty)\leq d(x,y)$ for any $x\in A$ and $y\in B$;
(ii) $\{x_{2n-1}\}$ has a convergent subsequence in A;
(iii) $\lim_{n\to\infty}d(x_n,x_{n+1})=dist(A,B)$.

Then there exists $v\in A$ such that $d(v,Tv)=dist(A,B)$.

Proof. Since T is a cyclic map and $x_1\in A$, $x_{2n-1}\in A$ and $x_{2n}\in B$ for all $n\in\mathbb{N}$. By (ii), $\{x_{2n-1}\}$ has a convergent subsequence $\{x_{2n_k-1}\}$ and $x_{2n_k-1}\to v$ as $k\to\infty$ for some $v\in A$. Since

$$dist(A,B)\leq d(v,x_{2n_k})\leq d(v,x_{2n_k-1})+d(x_{2n_k-1},x_{2n_k}) \text{ for all } k\in\mathbb{N},$$

it follows from $\lim_{n\to\infty}d(v,x_{2n_k-1})=0$ and the condition (iii) that $\lim_{n\to\infty}d(v,x_{2n_k})=dist(A,B)$. By (i), we have

$$dist(A,B)\leq d(Tv,x_{2n_k+1})\leq d(v,x_{2n_k}) \text{ for all } k\in\mathbb{N},$$

which implies $d(v,Tv)=dist(A,B)$. □

In the following theorem we prove a new existence theorem for weak $\mathcal{MT}$-cyclic Kannan contractions.

Theorem 3.7. *Let (X,d) be a metric space, let A and B be nonempty subsets of X. Let $T: A\cup B\to A\cup B$ be a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to a $\mathcal{MT}$-function φ. Let $x\in A$ such that the sequence $\{T^{2n}x\}$ has a convergent subsequence in A. Then there exists a unique point $z\in A$ such that $d(z,Tz)=dist(A,B)$.*

Proof. The existence of best proximity point z by using of Lemma 3.4 and Theorem 3.3 is concluded. We prove z is unique. Let v be another point such that $d(v,Tv) = dist(A,B)$. Then by Lemma 3.4 we have $v = T^2v$. If $d(v,Tz) > dist(A,B)$, then by $(MTK2)$ we have

$$\begin{aligned} d(v,Tz) &= d(T^2v,Tz) \\ &\leq \frac{1}{2}\varphi(d(Tv,Tz))[d(Tv,v)+d(z,Tz)]+(1-\varphi(d(Tv,Tz)))dist(A,B) \\ &< \varphi(d(Tv,Tz))dist(A,B)+(1-\varphi(d(Tv,Tz)))dist(A,B) = dist(A,B). \end{aligned}$$

So, $d(v,Tz) = dist(A,B)$. On the other hand $d(z,Tz) = dist(A,B)$. Hence by Lemma 2.3 we have $d(z,v) = 0$ or $z = v$. □

For weak $\mathcal{MT}$-cyclic Kannan contractions, we establish the following convergence theorem, which is our main result in this paper.

Theorem 3.8. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T : A\cup B \to A\cup B$ be a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to a $\mathcal{MT}$-function φ. Then*

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x \in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

Proof. We divide the proof of theorem into two cases:
case 1: $dist(A,B) = 0$.
For proof of this case see Theorem 3.6.
case 2: $dist(A,B) \neq 0$. Let x be an arbitrary point in A. Since T is a weak $\mathcal{MT}$-cyclic Kannan contraction, by part (i) of Lemma 3.4, $\lim_{n\to\infty} ||T^{2n}x - T^{2n+1}x|| = dist(A,B)$.
Now, we claim that for every $\varepsilon > 0$ there exists $N_0 \in N$ such that for $m > n > N_0$,

$$||T^{2m}x - T^{2n+1}x|| < dist(A,B)+\varepsilon.$$

Hence by Lemma 2.3 and for given $\varepsilon > 0$ there exists $N_1 \in N$ such that for $n > N_1$,

$$||T^{2m}x - T^{2n}x|| \leq \varepsilon;$$

it follows that $\{T^{2n}x\}$ is a Cauchy sequence and so there exists $z \in A$ such that $T^{2n}x \to z$ as $n \to \infty$. Using Theorem 3.6, z is a unique best proximity point of T in A.
Lemma 3.4-(iii) concludes that z is a unique fixed point of T^2, since z is unique.
Now, we prove the claim. Suppose not. Then there exists $\varepsilon > 0$ such that for any $k \in N$ there exists $m_k > n_k > k$ such that

$$||T^{2m_k}x - T^{2n_k+1}x|| \geq dist(A,B)+\varepsilon.$$

We can assume that m_k is minimal index such that $||T^{2m}x - T^{2n+1}x|| \geq dist(A,B)+\varepsilon$ but $||T^hx - T^{2n_k+1}x|| < dist(A,B)+\varepsilon, h \in \{2n_{k+1},\cdots,2m_k - 1\}$. We have

$$dist(A,B)+\varepsilon \leq ||T^{2m_k}x - T^{2n_k+1}x|| \leq ||T^{2m_k}x - T^{2m_k-2}x|| + ||T^{2m_k-2}x - T^{2n_k+1}x||.$$

Using part of (ii) in Lemma 3.4 concludes that $||T^{2m_k}x - T^{2m_k-2}x|| \to 0$ as $k \to \infty$ this implies $\lim_k ||T^{2m_k}x - T^{2n_k+1}x|| = dist(A,B)+\varepsilon$.
Now,

$$||T^{2m_k}x - T^{2n_k+1}x|| \leq ||T^{2m_k}x - T^{2m_k+2}x|| + ||T^{2m_k+2}x - T^{2n_k+3}x|| + ||T^{2n_k+3}x - T^{2n_k+1}x|| = ||T^{2m_k+2}x - T^{2n_k+3}x||.$$

So, $dist(A,B)+\varepsilon \leq \limsup_{n\to\infty} ||T^{2m_k+2}x - T^{2n_k+3}x||$. But,

$$\begin{aligned} ||T^{2m_k+2}x - T^{2n_k+3}x|| &\leq \frac{1}{2}\varphi(||T^{2m_k+1}x - T^{2n_k+2}x||)[||T^{2m_k+1}x - T^{2m_k+2}x|| \\ &+ ||T^{2n_k+2}x - T^{2n_k+3}x||](1-\varphi(||T^{2m_k+1}x - T^{2n_k+2}x||))dist(A,B). \end{aligned}$$

Hence, by "lim sup" from the above inequality, as $(n \to \infty)$, we conclude that $dist(A,B)+\varepsilon \leq dist(A,B)$ and so $\varepsilon \leq 0$, a contradiction.
Now we prove (iv). z is best proximity point of T and so $||z - Tz|| = dist(A,B)$. Since T is a weak $\mathcal{MT}$-cyclic Kannan contraction, we have

$$||Tz - T^2z|| \leq \frac{1}{2}\varphi(||z-Tz||)[||z-Tz|| + ||Tz - T^2z||] + (1-\varphi(||z-Tz||))dist(A,B),$$

and so

$$dist(A,B) \leq ||Tz - T^2z|| \leq \frac{\varphi(||z-Tz||)}{2-\varphi(||z-Tz||)}||z-Tz|| + \frac{2-2\varphi(||z-Tz||)}{2-\varphi(||z-Tz||)}dist(A,B) = dist(A,B).$$

Therefore $||Tz - T^2z|| = dist(A,B)$, i.e. Tz is best proximity point of T in B. This complete the proof. □

The following theorem can be obtain immediately from Lemma 3.5 and Theorem 3.7.

Theorem 3.9. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Let $T : A\cup B \to A\cup B$ be a a cyclic map. Suppose that there exists a nondecreasing (or nonincreasing) function $\tau : [0,\infty) \to [0,1)$ such that*

$$||Tx-Ty|| \leq \frac{1}{2}\tau(||x-y||)||x-y|| + (1-\tau(||x-y||))dist(A,B) \text{ for any } x\in A \text{ and } y\in B.$$

Then

(i) T has a unique best proximity point z in A.
(ii) The sequence $\{T^{2n}x\}$ converges to z for any starting point $x\in A$.
(iii) z is the unique fixed point of T^2.
(iv) Tz is a best proximity point of T in B.

Remark 3.10. *In Theorems 3.6 and 3.8 if we define $\varphi(t) = c$, where $c\in[0,1)$ and for all $t\in[0,\infty)$, then φ is a $\mathcal{MT}$-function and so we can obtain Theorems P [12] as the special cases.*

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