Transverse vibration of nonuniform Euler-Bernoulli beams on bounded time scales

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Article Info

Abstract

In this article, we consider Euler-Bernoulli equation of transverse vibrations of nonuniform beams on bounded time scales $\mathbb{T}$. We will give a description of all maximal dissipative, maximal accretive, self adjoint and other extensions of such operators.

1. Introduction

The theory of symmetric extensions of a symmetric operator in a Hilbert space developed by J. von Neumann [27]. Especially, it plays a central role in spectral problems associated with formally self-adjoint linear differential operators. The problem on the description of all self adjoint extensions of a symmetric operator in terms of abstract boundary conditions was given by Calkin [25]. Later, Rofe- Beketov [28] described self adjoint extensions of a symmetric operator in terms of abstract boundary conditions with aid of linear relations. Bruk [24] and Kochubei [12] are introduced the notion of a space of boundary values. They described all maximal dissipative, accretive, self adjoint extensions of symmetric operators. This problem has been investigated by many mathematicians (see [13]-[20]). For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [26].

The theory of time scales unifies continuous and discrete analysis. It was introduced by Hilger (see [1]). Recently, it has received a lot of attention. The study of dynamic equations on time scales has several important applications, e.g., in the study of heat transfer, insect population models, epidemic models stock market, and neural networks (see [1]-[5]).

On the other hand, transverse vibration of nonuniform beams is one of the important problems in mechanical and civil engineering. It has led to several applications in modern engineering, e.g., turbine blade, helicopter blades, satellites structure, even robotic arms etc. It has been studied by many investigators (see [29]-[43]).

In this article, we consider Euler-Bernoulli dynamic equation of transverse vibrations of nonuniform beans on bounded time scales. A space of boundary value is constructed for this operator. It is given a description all maximal dissipative, accretive, self adjoint and other extensions of such operators in terms of boundary conditions.

2. Preliminaries

Now, we recall some necessary fundamental concepts of time scales, and we refer to [8], [9] for more detail.

Definition 2.1. *Let $\mathbb{T}$ be a time scale, i.e., a non-empty closed subset of real numbers $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ where } t \in \mathbb{T}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ where } t \in \mathbb{T}.$$

It is convenient to have grainiess operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$. We introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2.2. *A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s| \quad \text{where } s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show (see [11])

$$f^\Delta(t) = f^\nabla(\sigma(t)), \qquad f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

Example 2.3. *If $\mathbb{T} = \mathbb{R}$, then we have*

$$\sigma(t) = t, \; f^\Delta(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then we have

$$\sigma(t) = t + 1, \; f^\Delta(t) = \Delta f(t) = f(t+1) - f(t).$$

If $\mathbb{T} = q^{\mathbb{N}_0} = \left\{ q^k : q > 1, \; k \in \mathbb{N}_0 \right\}$, then we have

$$\sigma(t) = qt, \; f^\Delta(t) = \frac{f(qt) - f(t)}{qt - t}.$$

Definition 2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula*

$$\int_a^b f(t) \Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.

Let $L_\Delta^2(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2 \Delta t \right)^{1/2} < \infty.$$

The space $L_\Delta^2(\mathbb{T}^*)$ is a Hilbert space with the inner product (see [23])

$$(f, g) := \int_a^b f(t) \overline{g(t)} \Delta t, \; f, g \in L_\Delta^2(\mathbb{T}^*).$$

3. Main results

Let us consider Euler-Bernoulli dynamic expression of transverse vibrations of nonuniform beams

$$l(y) := \left(EI^*(t) y^{\Delta\nabla} \right)^{\nabla\Delta}(t) - \rho_0 w^2 A^*(t) y(t), \; t \in \mathbb{T}_1 = \mathbb{T}^* \cap (a, b), a < b, \tag{3.1}$$

where y is the transverse displacement, E, ρ_0 and w are Young modulus, mass density, and natural frequency, respectively, $A^*(t)$ and $I^*(t)$ are the area and moment of inertia of current cross-section, respectively; t is the current longitudinal coordinate of the beam, and a and b are the coordinates of the fixed end and the free end of the beam, respectively.
For simplicity of notation, we have

$$
\begin{aligned}
y^{[0]} &= y, \\
y^{[1]} &= y^\Delta, \\
y^{[2]} &= EI^*(t) y^{\Delta\nabla}, \\
y^{[3]} &= -\left(y^{[2]} \right)^\nabla, \\
y^{[4]} &= -\rho_0 w^2 A^*(t) y - \left(y^{[3]} \right)^\Delta.
\end{aligned}
$$

Let $y_i, 1 \leq i \leq 4$, be solutions of Eq. (3.1). The Wronskian of y_1, y_2, y_3 and y_4 is defined to be (see [6])

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1^{[1]} & y_2^{[1]} & y_3^{[1]} & y_4^{[1]} \\ y_1^{[2]} & y_2^{[2]} & y_3^{[2]} & y_4^{[2]} \\ y_1^{[3]} & y_2^{[3]} & y_3^{[3]} & y_4^{[3]} \end{vmatrix}.$$

We will denote by $Dom_{\max}$ the set of all functions $y(t)$ in $L^2_\Delta(\mathbb{T}_1)$ such that first three Δ derivatives are locally Δ absolutely continuous in $\mathbb{T}_1$, and $l(y) \in L^2_\Delta(\mathbb{T}_1)$. We define the maximal operator $L_{\max}$ on $Dom_{\max}$ by the equality $L_{\max}y = ly$.

For every $y, z \in Dom_{\max}$, we have Green's formula

$$\int_a^b (ly)(t)\overline{z(t)}\Delta t - \int_a^b y(t)\overline{(lz)(t)}\Delta t = [y,z]_b - [y,z]_a, \tag{3.2}$$

where $[y,z]_t := y^{[0]}(t)\bar{z}^{[3]}(t) - y^{[3]}(t)\bar{z}^{[0]}(t) + y^{[1]}(t)\bar{z}^{[2]}(t) - y^{[2]}(t)\bar{z}^{[1]}(t)$ (see [6]).

Let $Dom_{\min}$ denote the linear set of all vectors $y \in Dom_{\max}$ satisfying the conditions

$$
\begin{aligned}
y^{[0]}(a) &= y^{[1]}(a) = y^{[2]}(a) = y^{[3]}(a) = 0, \\
y^{[0]}(b) &= y^{[1]}(b) = y^{[2]}(b) = y^{[3]}(b) = 0.
\end{aligned}
$$

If we restrict the operator $L_{\max}$ to the set $Dom_{\min}$, then we obtain the minimal operator $L_{\min}$. It is clear that $L^*_{\min} = L_{\max}$, and $L_{\min}$ is a closed symmetric operator (see [6]). Now we recall the following definitions.

Definition 3.1. *A linear operator M (with dense domain $D(M)$) acting on some Hilbert space H is called dissipative (accumulative) if $\Im(Mf,f) \geq 0$ ($\Im(Mf,f) \leq 0$) for all $f \in D(M)$ and maximal dissipative (maximal accumulative) if it does not have a proper dissipative (accumulative) extension (see [14], [16]-[19]).*

Definition 3.2. *A triplet $(\mathbb{H}, \Phi_1, \Phi_2)$ is called a space of boundary values of a closed symmetric operator M on a Hilbert space H if Φ_1 and Φ_2 are linear maps from $D(M^*)$ to H, with equal deficiency numbers and such that:*

i) For every $f, g \in D(M^*)$ we have

$$(M^*f,g)_H - (f, M^*g)_H = (\Phi_1 f, \Phi_2 g)_{\mathbb{H}} - (\Phi_2 f, \Phi_1 g)_{\mathbb{H}};$$

ii) For any $F_1, F_2 \in H$ there is a vector $f \in D(M^*)$ such that $\Phi_1 f = F_1$ and $\Phi_2 f = F_2$ (see [10]).

Let's define by Φ_1, Φ_2 the linear maps from D to $\mathbb{C}^4$ by the formula

$$\Phi_1 y = \begin{pmatrix} -y^{[0]}(a) \\ -y^{[1]}(a) \\ y^{[0]}(b) \\ y^{[1]}(b) \end{pmatrix}, \quad \Phi_2 y = \begin{pmatrix} y^{[3]}(a) \\ y^{[2]}(a) \\ y^{[3]}(b) \\ y^{[2]}(b) \end{pmatrix}. \tag{3.3}$$

Now we will state and prove a theorem.

Theorem 3.3. *The triple $(\mathbb{C}^4, \Phi_1, \Phi_2)$ defined by (3.3) is a boundary spaces of the operator $L_{\min}$.*

Proof. For every $y, z \in Dom_{\max}$, we have

$$
\begin{aligned}
(\Phi_1 y, \Phi_2 z)_{\mathbb{C}^4} - (\Phi_1 z, \Phi_2 y)_{\mathbb{C}^4} &= -y^{[0]}(a)\bar{z}^{[3]}(a) - y^{[1]}(a)\bar{z}^{[2]}(a) \\[6pt]
&\quad + y^{[0]}(b)\bar{z}^{[3]}(b) + y^{[1]}(b)\bar{z}^{[2]}(b) \\[6pt]
&\quad - (-\bar{z}^{[0]}(a)y^{[3]}(a) - \bar{z}^{[1]}(a)y^{[2]}(a)) \\[6pt]
&\quad - (\bar{z}^{[0]}(b)y^{[3]}(b) + \bar{z}^{[1]}(b)y^{[2]}(b)) \\[6pt]
&= y^{[0]}(b)\bar{z}^{[3]}(b) - \bar{z}^{[0]}(b)y^{[3]}(b) \\[6pt]
&\quad + y^{[1]}(b)\bar{z}^{[2]}(b) - \bar{z}^{[1]}(b)y^{[2]}(b) \\[6pt]
&\quad + \bar{z}^{[0]}(a)y^{[3]}(a) - y^{[0]}(a)\bar{z}^{[3]}(a) \\[6pt]
&\quad - y^{[1]}(a)\bar{z}^{[2]}(a) + \bar{z}^{[1]}(a)y^{[2]}(a) \\[6pt]
&= [y,z](b) - [y,z](a).
\end{aligned}
$$

$\square$

From Green's formula (3.2), we obtain the following equation

$$(\Phi_1 y, \Phi_2 z)_{\mathbb{C}^4} - (\Phi_1 z, \Phi_2 y)_{\mathbb{C}^4} = [y,z](b) - [y,z](a) = (L_{\max}y,z) - (y, L_{\max}z).$$

So, we proved the first requirement of the definition of a space of boundary values.

Now, we will prove the second requirement of the definition of a space of boundary values. Let $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \mathbb{C}^4$. Then the

vector-valued function

$$y(t) = \alpha_1(t)u_1 + \alpha_2(t)v_1 + \alpha_3(t)u_2 + \alpha_4(t)v_2 + \alpha_5(t)u_3 + \alpha_6(t)v_3 + \alpha_7(t)u_4 + \alpha_8(t)v_4,$$

where $\alpha_i(t) \in H$ $(i = 1, ..., 8)$ satisfy the conditions

$$
\begin{array}{llll}
\alpha_1^{[0]}(a) = 1 & \alpha_1^{[1]}(a) = 0 & \alpha_1^{[2]}(a) = 0 & \alpha_1^{[3]}(a) = 0 \\
\alpha_2^{[0]}(a) = 0 & \alpha_2^{[1]}(a) = 0 & \alpha_2^{[2]}(a) = 0 & \alpha_2^{[3]}(a) = 1 \\
\alpha_3^{[0]}(a) = 0 & \alpha_3^{[1]}(a) = -1 & \alpha_3^{[2]}(a) = 0 & \alpha_3^{[3]}(a) = 0 \\
\alpha_4^{[0]}(a) = 0 & \alpha_4^{[1]}(a) = 0 & \alpha_4^{[2]}(a) = 1 & \alpha_4^{[3]}(a) = 0 \\
\alpha_5^{[0]}(a) = 0 & \alpha_5^{[1]}(a) = 0 & \alpha_5^{[2]}(a) = 0 & \alpha_5^{[3]}(a) = 0 \\
\alpha_6^{[0]}(a) = 0 & \alpha_6^{[1]}(a) = 0 & \alpha_6^{[2]}(a) = 0 & \alpha_6^{[3]}(a) = 0 \\
\alpha_7^{[0]}(a) = 0 & \alpha_7^{[1]}(a) = 0 & \alpha_7^{[2]}(a) = 0 & \alpha_7^{[3]}(a) = 0 \\
\alpha_8^{[0]}(a) = 0 & \alpha_8^{[1]}(a) = 0 & \alpha_8^{[2]}(a) = 0 & \alpha_8^{[3]}(a) = 0
\end{array}
$$

and

$$
\begin{array}{llll}
\alpha_1^{[0]}(b) = 0 & \alpha_1^{[1]}(b) = 0 & \alpha_1^{[2]}(b) = 0 & \alpha_1^{[3]}(b) = 0 \\
\alpha_2^{[0]}(b) = 0 & \alpha_2^{[1]}(b) = 0 & \alpha_2^{[2]}(b) = 0 & \alpha_2^{[3]}(b) = 0 \\
\alpha_3^{[0]}(b) = 0 & \alpha_3^{[1]}(b) = 0 & \alpha_3^{[2]}(b) = 0 & \alpha_3^{[3]}(b) = 0 \\
\alpha_4^{[0]}(b) = 0 & \alpha_4^{[1]}(b) = 0 & \alpha_4^{[2]}(b) = 0 & \alpha_4^{[3]}(b) = 0 \\
\alpha_5^{[0]}(b) = -1 & \alpha_5^{[1]}(b) = 0 & \alpha_5^{[2]}(b) = 0 & \alpha_5^{[3]}(b) = 0 \\
\alpha_6^{[0]}(b) = 0 & \alpha_6^{[1]}(b) = 0 & \alpha_6^{[2]}(b) = 0 & \alpha_6^{[3]}(b) = 1 \\
\alpha_7^{[0]}(b) = 0 & \alpha_7^{[1]}(b) = 1 & \alpha_7^{[2]}(b) = 0 & \alpha_7^{[3]}(b) = 0 \\
\alpha_8^{[0]}(b) = 0 & \alpha_8^{[1]}(b) = 0 & \alpha_8^{[2]}(b) = 1 & \alpha_8^{[3]}(b) = 0
\end{array}
$$

belongs to the set $Dom_{\max}$ and $\Phi_1 y = u$, $\Phi_2 y = v$

Corollary 3.4. *For any contraction T in $\mathbb{C}^4$ the restriction of the operator $L_{\max}$ to the set of functions $y \in Dom_{\max}$ satisfying either*

$$(T - I)\Phi_1 y + i(T + I)\Phi_2 y = 0 \tag{3.4}$$

or

$$(T - I)\Phi_1 y - i(T + I)\Phi_2 y = 0 \tag{3.5}$$

is respectively the maximal dissipative and accretive extension of the operator $L_{\min}$. Conversely, every maximal dissipative (accretive) extension of the operator $L_{\min}$ is the restriction of $L_{\max}$ to the set of functions $y \in Dom_{\max}$ satisfying (3.4) ((3.5)), and the extension uniquely determines the contraction T. If T is an isometry in $\mathbb{C}^4$, then the conditions (3.4) ((3.5)) describe the maximal symmetric extensions of $L_{\min}$ in $L_\Delta^2(\mathbb{T}_1)$.

The general form of dissipative and accretive extensions of an operator L is given by the conditions

$$
\begin{aligned}
T(\Phi_1 y + i\Phi_2 y) &= \Phi_1 y - i\Phi_2 y, \ \Phi_1 y + i\Phi_2 y \in Dom(T), \\
T(\Phi_1 y - i\Phi_2 y) &= \Phi_1 y + i\Phi_2 y, \ \Phi_1 y - i\Phi_2 y \in Dom(T),
\end{aligned}
\tag{3.6}
$$

respectively, where T is a linear operator with

$$\|Tf\| \le \|f\|, \ f \in Dom(T).$$

4. Conclusion

In this paper, we have considered Euler-Bernoulli equation of transverse vibrations of nonuniform beams on bounded time scales $\mathbb{T}$.. In this context, we have constructed a space of boundary values of the minimal operator and described all maximal dissipative, maximal accretive, self-adjoint extensions of of such operators.

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