Fundamental Journal of Mathematics and Applications, 1 (1) (2018) 82-87

FUJMA

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma

Differential bubordinations and argument inequalities for certain multivalent functions defined by convolution structure

Mohamed Kamal Aouf[a], Rabha El-Ashwah[b] and Ekram Elsayed Ali[c*]

[a]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[b]*Department of Mathematics, Faculty of Science, Damietta University, New Damiette, 34517, Egypt*
[c]*Department of Mathematics and Computer Science, Faculty of Science, Port-Said University, Port-Said, 42521, Egypt*
[*]*Corresponding author E-mail: ekram_008eg@yahoo.com*

Article Info

Keywords: *Multiplier transformation , Hadamard product, Subordination*
2010 AMS: *30C45*
Received: *12 March 2018*
Accepted: *5 June 2018*
Available online: *30 June 2018*

Abstract

The main object of the present paper is to investigate certain interesting argument inequalities and differential subordinations properties of multivalent functions associated with a linear operator $D^n_{\lambda,p}(f*g)(z)$ defined by Hadamard product

1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p} \quad (p\in\mathbb{N}=\{1,2,....\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U=\{z:z\in\mathbb{C} \text{ and } |z|<1\}$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as follows:

$$f\prec g \text{ or } f(z)\prec g(z)\ ,$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$ such that $f(z)=g(w(z))$ $(z\in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [4], [13]; see also [14, p. 4]:

$$f(z)\prec g(z) \ \Leftrightarrow f(0)=g(0) \text{ and } f(U)\subset g(U).$$

For functions $f(z)\in A(p)$ given by (1.1), and $g(z)\in A(p)$ defined by

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{k+p}z^{k+p} \quad (p\in\mathbb{N}), \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}b_{k+p}z^{k+p}=(g*f)(z) \quad (p\in\mathbb{N};z\in U). \tag{1.3}$$

For functions $f,g\in A(p)$, we define the following differential operator:

Email addresses: **mkaouf127@yahoo.com (M. K. Aouf) r_elashwah@yahoo.com (R. M. El-Ashwah) ekram_008eg@yahoo.com (E. E. Ali)**

$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z), \tag{1.4}$$

$$D^1_{\lambda,p}(f*g)(z) = D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda z}{p}(f*g)'(z)\ (\lambda \geq 0), \tag{1.5}$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z)) \\ &= z^p + \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}z^{k+p} \\ (\lambda &\geq 0; p\in\mathbb{N}; n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}). \end{aligned} \tag{1.6}$$

From (1.6) it is easy to verify that

$$\frac{\lambda}{p}z(D^n_{\lambda,p}(f*g)(z))' = D^{n+1}_{\lambda,p}(f*g)(z) - (1-\lambda)D^n_{\lambda,p}(f*g)(z)\ (\lambda>0; n\in\mathbb{N}_0). \tag{1.7}$$

The operator $D^n_{\lambda,p}(f*g)(z)$, when $p=1$, was introduced and studied by Aouf and Mostafa [3].

We observe that the linear operator $D^n_{\lambda,p}(f*g)(z)$ reduces to several interesting operators for different choices of n,λ,p and the function $g(z)$:
(i) For $\lambda=1$ and $g(z)=\frac{z^p}{1-z}$ (or $b_{k+p}=1$), $D^n_{1,p}(f*g)(z)=D^n_p f(z)$, where D^n_p is the p-valent Salagean operator introduced and studied by Kamali and Orhan [9], Orhan and Kiziltunc [17] (see also [2]);
(ii) For $g(z)=\frac{z^p}{1-z}$ (or $b_{k+p}=1$), we have

$$D^n_{\lambda,p}(f*g)(z) = D^n_{\lambda,p}f(z) = z^p + \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}z^{k+p}\ \ (\lambda\geq 0);$$

for $p=1$, the operator D^n_λ is the generalized Sălăgean operator introduced and studied by Al-Oboudi [1]) which in turn contains as special case the Sălăgean operator see [20];
(iii) For $n=0$ and

$$g(z) = z^p + \sum_{k=1}^{\infty}\left[\frac{p+\ell+\lambda k}{p+\ell}\right]^m z^{k+p}\ \ (\lambda\geq 0; p\in\mathbb{N}; \ell,m\in\mathbb{N}_0),$$

we see that $D^0_{\lambda,p}(f*g)(z)=(f*g)(z)=I^m_p(\lambda,\ell)f(z)$, where $I^m_p(\lambda,\ell)$ is the generalized multiplier transformation which was introduced and studied by Cătaş [5], the operator $I^m_p(\lambda,\ell)$, contains as special cases, the multiplier transformation $I^m_p(\ell)$ (see Kumar et al. [11] and Srivastava et al. [23]);
(iv) For $n=0$,

$$g(z) = z^p + \sum_{k=1}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}.\frac{z^{k+p}}{k!} \tag{1.8}$$

$$\left(\alpha_i\in\mathbb{C}; i=1,\ldots,q; \beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,\ldots\}; j=1,\ldots,s; \atop q\leq s+1; q,s\in\mathbb{N}_0, p\in\mathbb{N}; z\in U\right)$$

and

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu=0;\theta\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}), \\ \theta(\theta-1)\ldots(\theta+\nu-1) & (\nu\in\mathbb{N};\theta\in\mathbb{C}), \end{cases}$$

we have $D^0_{\lambda,p}(f*g)(z)=(f*g)(z)=H_{p,q,s}(\alpha_1)f(z)$, where $H_{p,q,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied by Dziok and Srivastava [8]. The operator $H_{p,q,s}(\alpha_1)$ contains in turn many interesting operators such as, Carlson and Shaffer linear operator (see [19]), the Ruscheweyh derivative operator (see [10]), the Choi-Saigo-Srivastava operator (see [7]), the Cho-Kwon-Srivastava operator (see [6]), the differeintegral operator (see Srivastava and Aouf [22] and Patel and Mishra [18]) and the Noor integral operator (see Liu and Noor [12]);
(v) For $p=1$ and $g(z)$ of the form (1.8), the operator $D^n_\lambda(f*g)(z)$ inroduced and studied by Selvaraj and Karthikeyan [21].
For $f,g\in A(p),\lambda>0,\delta\geq 0,p\in\mathbb{N}$ and $n\in\mathbb{N}_0$, we define a function $H(z)$ by

$$H(z) = H^n_{\lambda,p,\delta}(f*g)(z) = \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]D^n_{\lambda,p}(f*g)(z) + \delta\frac{p}{\lambda}D^{n+1}_{\lambda,p}(f*g)(z). \tag{1.9}$$

We note that:
(i) For $\lambda=1$ and $g(z)=\frac{z^p}{1-z}$ in (1.9), we obtain

$$H^n_{1,p,\delta}(f*\frac{z^p}{1-z})(z)=G^n_{p,\delta}f(z)=G(z)=(1-\delta)D^n_pf(z)+\delta pD^{n+1}_pf(z); \tag{1.10}$$

(ii) For $g(z)=\frac{z^p}{1-z}$ in (1.9), we obtain

$$\begin{aligned} H^n_{\lambda,p,\delta}(f*\frac{z^p}{1-z})(z) &= K^n_{\lambda,p,\delta}f(z)=K(z) \\ &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]D^n_{\lambda,p}f(z)+\delta\frac{p}{\lambda}D^{n+1}_{\lambda,p}f(z). \end{aligned} \tag{1.11}$$

In this paper, we investigate some interesting argument inequalities and differential subordinations properties of the function $H(z)$ given by (1.9). The following lemma will be required in our investigation.

Lemma 1.1. *[15], [16] Let a function $\phi(z)=1+b_1z+\dots$ be analytic in U and $\phi(z)\neq 0$ $(z\in U)$. If there exists a point $z_0\in U$ such that*

$$|\arg\phi(z)|<\frac{\pi}{2}\beta\quad(|z|<|z_0|)\quad \text{and}\quad |\arg\phi(z_0)|=\frac{\pi}{2}\beta\quad(0<\beta\leq 1),$$

then we have $z_0\phi'(z_0)/\phi(z_0)=ik\beta$, where

$$\begin{aligned} k &\geq \frac{1}{2}(a+\frac{1}{a})\quad (\text{where } \arg\phi(z_0)=\frac{\pi\beta}{2}), \\ k &\leq -\frac{1}{2}(a+\frac{1}{a})\quad (\text{where } \arg\phi(z_0)=-\frac{\pi\beta}{2}), \end{aligned}$$

and $(\phi(z_0))^{\frac{1}{\beta}}=\pm ia$ $(a>0)$.

2. Main results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda>0,\delta\geq 0,p\in\mathbb{N},n\in\mathbb{N}_0$ and g(z) is given by (1.2).

Theorem 2.1. *Let $f,g\in A(p)$ and let H be defined by (1.9). If*

$$\left|\arg\left(\frac{H^{(q)}(z)}{z^{p-q}}\right)\right|<\frac{\pi}{2}\beta\quad(z\in U), \tag{2.1}$$

then

$$\left|\arg\left(\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}}\right)\right|<\frac{\pi}{2}\beta\quad(z\in U),$$

where $0<\beta\leq 1$ and $0\leq q\leq p$.

Proof. Let

$$\phi(z)=\frac{(p-q)!}{p!}\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}}\quad(z\in U). \tag{2.2}$$

Then $\phi(z)$ is analytic in U, $\phi\neq 0$ for all $z\in U$ and $\phi(z)$ can be written as $\phi(z)=1+b_1z+\dots$. Since

$$\left(z\left(D^n_{\lambda,p}(f*g)(z)\right)'\right)^{(q)}=q\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+z\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q+1)}, \tag{2.3}$$

we have from (1.7), (1.9) and (2.3) that

$$\begin{aligned} H^{(q)}(z) &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+\delta\frac{p}{\lambda}\left(D^{n+1}_{\lambda,p}(f*g)(z)\right)^{(q)} \\ &= \left[1-\delta\left(1+\frac{p}{\lambda}-p\right)\right]\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}+\delta\left(z\left(D^n_{\lambda,p}(f*g)(z)\right)'\right)^{(q)} \\ &\quad +\delta\frac{p}{\lambda}(1-\lambda)\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)} \\ &= (1-\delta+\delta q)(D^n_{\lambda,p}(f*g)(z))^{(q)}+\delta z(D^n_{\lambda,p}(f*g)(z))^{(q+1)}. \end{aligned}$$

(2.4)

It is easy to see from (2.4) and (2.2) that

$$\begin{aligned} \frac{H^{(q)}(z)}{z^{p-q}} &= (1-\delta+\delta q)\frac{(D^n_{\lambda,p}(f*g)(z))^{(q)}}{z^{p-q}} + \delta\frac{z(D^n_{\lambda,p}(f*g)(z))^{(q+1)}}{z^{p-q}} \\ &= \frac{p!(1-\delta+\delta q)}{(p-q)!}\phi(z) + \frac{\delta p!}{(p-q)!}\left((p-q)\phi(z)+z\phi^{'}(z)\right) \\ &= \frac{p!(1-\delta+\delta p)}{(p-q)!}\left(\phi(z)+\frac{\delta}{1-\delta+\delta p}z\phi^{'}(z)\right). \end{aligned} \tag{2.5}$$

Suppose there exists a point $z_0 \in U$ such that

$$|\arg\phi(z)| < \frac{\pi}{2}\beta \quad (|z|<|z_0|)$$

and

$$|\arg\phi(z_0)| = \frac{\pi}{2}\beta \,.$$

Then, by using Lemma 1.1, we can write that $z_0\phi^{'}(z_0)/\phi(z_0) = ik\beta$ and $(\phi(z_0))^{\frac{1}{\beta}} = \pm ia$ $(a>0)$. Therefore, if $\arg\phi(z_0) = \frac{\pi}{2}\beta$, then by using (2.5), we have

$$\begin{aligned} \frac{H^{(q)}(z_0)}{z_0^{p-q}} &= \frac{p!(1-\delta+\delta p)}{(p-q)!}\phi(z_0)\left(1+\frac{\delta}{1-\delta+\delta p}\frac{z_0\phi^{'}(z_0)}{\phi(z_0)}\right) \\ &= \frac{p!(1-\delta+\delta p)}{(p-q)!}a^{\beta}e^{i\pi\beta/2}\left(1+\frac{\delta}{1-\delta+\delta p}ik\beta\right). \end{aligned}$$

This shows that

$$\begin{aligned} \arg\left(\frac{H^{(q)}(z_0)}{z_0^{p-q}}\right) &= \frac{\pi}{2}\beta + \arg\left(1+\frac{\delta k\beta i}{1-\delta+\delta p}\right) \\ &= \frac{\pi}{2}\beta + \tan^{-1}\left(\frac{\delta k\beta}{1-\delta+\delta p}\right) \\ &\geq \frac{\pi}{2}\beta, \ \ (\textit{where } k \geq \frac{1}{2}(a+\frac{1}{a}) \geq 1), \end{aligned}$$

which contradicts the condition (2.1). Similarly, if $\arg\phi(z_0) = \frac{-\pi\beta}{2}$, then we obtain

$$\arg\left(\frac{H^{(q)}(z_0)}{z_0^{p-q}}\right) \leq -\frac{\pi}{2}\beta \,,$$

which also contradicts the condition (2.1). Thus, the function $\phi(z)$ satisfies $|\arg\phi(z)| < \frac{\pi\beta}{2}$ $(z\in U)$. This shows that

$$\left|\arg\left(\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\,.$$

This completes the proof of Theorem 2.1. □

Putting $n=0$ and $\lambda=1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $f,g \in A(p)$ and let Q be defined by*

$$Q(z) = (1-\delta)(f*g)(z) + \delta\frac{z}{p}\left((f*g)(z)\right)^{'}. \tag{2.6}$$

If

$$\left|\arg\left(\frac{Q^{(q)}(z)}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\,,$$

then

$$\left|\arg\left(\frac{((f*g)(z))^{(q)}}{z^{p-q}}\right)\right| < \frac{\pi}{2}\beta \quad (z\in U)\,,$$

where $0<\beta\leq 1$ and $0\leq q\leq p$.

Theorem 2.3. *Let $f,g \in A(p)$ and let H be defined by (1.9). If*

$$\frac{\left(D^{n}_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}} \prec \frac{p!}{(p-q)!}\frac{1+(1-2\alpha)z}{1-z} \quad (z\in U)\,. \tag{2.7}$$

Then

$$\frac{H^{(q)}(z)}{z^{p-q}} \prec \frac{p!(1-\delta+\delta p)}{(p-q)!}\frac{1+(1-2\alpha)z}{1-z} \quad (|z|<\rho)\,, \tag{2.8}$$

where $0\le q\le p, 0\le \alpha<1$, and

$$\rho=\left[1+\left(\frac{\delta}{1-\delta+\delta p}\right)^{2}\right]^{\frac{1}{2}}-\frac{\delta}{1-\delta+\delta p}\,. \tag{2.9}$$

The bound $\rho\in(0,1)$ is the best possible.

Proof. Set

$$\psi(z)=(1-\gamma)\frac{z}{1-z}+\gamma\frac{z}{(1-z)^{2}} \quad (z\in U)\,,$$

where $\gamma=\frac{\delta}{1-\delta+\delta p}>0$. We need to show that

$$\mathrm{Re}\left\{\frac{\psi(\rho z)}{\rho z}\right\}>\frac{1}{2} \quad (z\in U)\,, \tag{2.10}$$

where $\rho=(1+\gamma^{2})^{\frac{1}{2}}-\gamma$ and $0<\rho<1$. Let $\frac{1}{1-z}=R\,e^{i\theta}$ and $|z|=r<1$. In view of

$$\cos\theta=\frac{1+R^{2}(1-r^{2})}{2R}\,,\ R\ge\frac{1}{1+r}\,,$$

we have

$$\begin{aligned}2\,\mathrm{Re}\left\{\frac{\psi(z)}{z}-\frac{1}{2}\right\} &= 2(1-\gamma)R\cos\theta+2\gamma R^{2}\cos 2\theta-1\\ &= R^{4}\gamma(1-r^{2})^{2}+R^{2}\left((1-\gamma)(1-r^{2})\right)-2\gamma r^{2})\\ &\ge R^{2}(\gamma(1-r)^{2}+(1-\gamma)(1-r^{2})-2\gamma r^{2})\\ &= R^{2}(1-2\gamma r-r^{2})>0\end{aligned}$$

for $|z|=r<\rho$, which gives (2.10). Thus the function ψ has the integral representation

$$\frac{\psi(\rho z)}{\rho z}=\int\limits_{1x1=1}\frac{d\mu(x)}{1-xz} \quad (z\in U)\,, \tag{2.11}$$

where $\mu(x)$ is a prabability measure on $|x|=1$.
Now letting $\phi(z)$ be in the form (2.2), we see that $\phi(z)=1+b_{1}z+\ldots$ is analytic in U and it follows from (2.7) that

$$\mathrm{Re}\,\phi(z)>\alpha \quad (0\le\alpha<1;z\in U)\,. \tag{2.12}$$

Since we can write

$$\phi(z)+\gamma z\phi^{'}(z)=\left(\frac{\psi(z)}{z}\right)*\phi(z)\,,$$

it follows from (2.11) that

$$\mathrm{Re}\left\{\phi(\rho z)+\gamma\rho z\phi^{'}(\rho z)\right\}=\mathrm{Re}\left\{\left(\frac{\psi(\rho z)}{\rho z}\right)*\phi(z)\right\}$$

$$=\mathrm{Re}\left\{\int\limits_{1x1=1}\phi(xz)d\mu(x)\right\}>\alpha \quad (z\in U)\,. \tag{2.13}$$

Thus, from (2.3) and (2.13), we conclude that (2.8)) holds. To show that the bound ρ is sharp we take $f,g \in A(p)$ defined by

$$\frac{(p-q)!}{(p)_q}\frac{\left(D^n_{\lambda,p}(f*g)(z)\right)^{(q)}}{z^{p-q}} = \alpha + (1-\alpha)\frac{1+z}{1-z}\,.$$

Since

$$\begin{aligned}\frac{(p-q)!}{(p)_q(1-\delta+\delta p)}\frac{H^{(q)}(z)}{z^{p-q}} &= \alpha + (1-\alpha)\frac{1+z}{1-z} + \gamma(1-\alpha)z\left(\frac{1+z}{1-z}\right)' \\ &= \alpha + (1-\alpha)\frac{1+2\gamma z - z^2}{(1-z)^2} = \alpha\end{aligned}$$

for $z = -\rho$, it follows that ρ is sharp. □

Remark 2.4. *(i) Putting $\lambda = 1$ and $g(z) = \frac{z^p}{1-z}$ in the above results we obtain the results for function $G(z)$ defined by* (1.10).
(ii) Putting $g(z) = \frac{z^p}{1-z}$ in the above results we obtain the results for function $K(z)$ defined by (1.11).

3. Conclusion

In this paper, three subclasses $H^n_{\lambda,p,\delta}(f*g)(z)$, $G^n_{p,\delta}f(z)$ and $K^n_{\lambda,p,\delta}f(z)$ are introduced and certain interesting argument inequalities and differential subordinations properties are investigated.

Acknowledgements

The authors thank the referees for their valuable suggestions to improve the paper.

References

[1] F. M. Al-Oboudi, On univalent function defined by a generalized Salagean operator, Internat. J. Math. Math. Sci., 27(2004),1429-1436.
[2] M. K. Aouf and A.O. Mostaafa, On a subclass of p-valent functions prestarlike functions, Comput. Math. Appl., (2008), no. 55, 851-861.
[3] M.K.Aouf and A.O.Mostafa, Sandwich theorems for analytic functions defined by convolution, Acta Univ. Apulensis, 21(2010), 7-20.
[4] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.
[5] A. Catas, On certain classes of p-valent functions defined by multiplier transformations, in Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (İstanbul, Turkey; 20-24 August 2007) (S. Owa and Y. Polatoglu, Editors), pp. 241–250, TC İstanbul Kűltűr University Publications, Vol. 91, TC İstanbul Kűltűr University, İstanbul, Turkey, 2008.
[6] N. E.Cho, O. S. Kwon and H. M Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292(2004), 470-483.
[7] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276(2002), 432-445.
[8] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999), no. 1, 1–13.
[9] M. Kamali and H. Orhan, On a subclass of certian starlike functions with negative coefficients, Bull. Korean Math. Soc. 41 (2004), no. 1, 53-71.
[10] Vinod Kumar and S. L. Shukla, Multivalent functions defined by Ruscheweyh derivatives, I and II, Indian J. Pure Appl. Math. 15(1984), no. 11, 1216-1227, 15(1984), no. 1, 1228-1238.
[11] S. S. Kumar, H. C. Taneja and V. Ravichandran, Classes multivalent functions defined by Dziok-Srivastava linear operator and multiplier transformations, Kyungpook Math. J. (2006), no. 46, 97-109.
[12] J. L. Liu and K. I. Noor, Some properties of Noor integral operator, J. Natur. Geom. 21(2002), 81-90.
[13] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(1981), 157-171.
[14] S. S. Miller and P. T. Mocanu, Differential Subordinations : Theory and Applications, Series on Monographs and Texbooks in Pure and Applied Mathematics, Vol.225, Marcel Dekker, New York and Basel, 2000.
[15] N. Nunokawa, On properties of non- caratheodory functions, Proc. Jpn Acad. Ser.A Math. Sci. 68(1992), 152-153.
[16] N. Nunokawa, S. Owa, H. Saitoh, A. Ikeda and N. Koike, Some results for strongly starlike functions, J. Math. Anal. Appl. 212(1997), 98-106.
[17] H. Orhan and H. Kiziltunc, A generalization on subfamily of p-valent functions with negative coefficients, Appl. Math. Comput., 155(2004), 521-530.
[18] J. Patel and A. K. Mishra, On certain subclasses of multivalent functions associated with an extended fractional differintegral operator, J. Math. Anal. Appl. 332(2007), 109-122.
[19] H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japon. 44(1996), 31-38.
[20] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math. (Springer-Verlag) 1013(1983), 362-372.
[21] C. Selvaraj and K. R. Karthikeyan, Differential subordination and superordination for certain subclasses of analytic functions, Far East J. Math. Sci. (FJMS), 29 (2008), no. 2, 419-430.
[22] H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, J. Math. Anal. Appl. 171(1992), 1-13, 192(1995), 673-688.
[23] H. M. Srivastava, K. Suchithra, B. Adolf Stephen and S. Sivasubramanian, Inclusion and neighborhood properties of certian subclasses of multivalent functions of complex order, J. Ineq. Pure Appl. Math. 7(2006), no. 5, Art. 191, 1-8.