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A class of slant surfaces of the nearly Kähler $\mathbf{S}^3 \times \mathbf{S}^3$

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Abstract

We investigate slant surfaces of the nearly Kähler $\mathbf{S}^3 \times \mathbf{S}^3$ which are orbits of isometric actions, classify them and show that for a prescribed angle there exists corresponding slant surface. Also, amongst them, we find the totally geodesic ones.

Keywords: slant submanifolds, nearly Kähler $\mathbf{S}^3 \times \mathbf{S}^3$, orbit manifolds.

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1. Introduction

Let $(\widetilde{M}, g, J)$ be an almost Hermitian manifold, i.e. a manifold endowed with an almost complex structure J, $J^2 = -Id$, such that $g(JX, JY) = g(X, Y)$ for arbitrary vector fields X, Y on $\widetilde{M}$. If $\widetilde{\nabla}$ is the Levi-Civita connection of the metric g, denote by $G(X, Y) = (\widetilde{\nabla}_X J)Y$, the $(2, 1)$-tensor field on $\widetilde{M}$. If the tensor field G vanishes identically we say that the manifold $\widetilde{M}$ is Kähler. If manifold satisfies the weaker condition, that G is skew symmetric, then $\widetilde{M}$ is a nearly Kähler manifold.

It is known that there exist only four six-dimensional homogeneous nearly Kähler manifolds, that are not Kähler: the sphere $\mathbf{S}^6$, the complex projective space $\mathbb{C}P^3$, the flag manifold $\mathbb{F}^3$ and $\mathbf{S}^3 \times \mathbf{S}^3$, see [4].

It is natural to investigate the submanifolds of a manifold $\widetilde{M}$ with an almost complex structure with respect to that structure. We say that M is an almost complex submanifold if $JT_pM = T_pM$ for any $p \in M$, and it is totally real if $JT_pM \subset T_pM^{\perp}$, for each $p \in M$, where by $T_pM^{\perp}$ we denote the normal space of the submanifold at a point p. Specially, if $JT_pM = T_pM^{\perp}$, M is said to be a Lagrangian submanifold. Amongst the four six-dimensional nearly Kähler manifolds mentioned before, the almost complex and

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the totally real submanifolds and their generalizations have been mostly investigated in the case of the sphere $\mathbf{S}^6$, we recall [2, 3, 7, 8, 12, 11]. Recently, the investigation of the geometry of almost complex and Lagrangian submanifolds of $\mathbf{S}^3 \times \mathbf{S}^3$ has been also initiated, we refer the reader to [1, 10, 9].

The most natural generalizations of the notions of the almost complex and the totally real subamnifolds are CR and slant submanifolds. We say that a submanifold M is slant if the angle between the vector JX and tangent space T_pM, for $p \in M$ and $X \in T_pM$ is constant, i.e. independent on the choice of the point p and the vector X. This angle is called the Wirtinger angle of X, see [5, 6]. Obviously, the almost complex and the totally real submanifolds can be considered as a special type of slant submanifolds with Wirtinger angles, respectively, 0 and $\pi/2$. If a slant submanifold does not belong to one of these two types we say that it is a proper slant submanifold. If the ambient manifold of a proper slant submanifold M is six-dimensional, then M has to be two-dimensional (see [6]). Note also that there do not exist four-dimensional almost complex submanifolds in the six-dimensional nearly Kähler manifold, see [15], so even in the case when Wirtinger angle is 0, the submanifold is a surface. There are not many known examples of the proper slant submanifolds of the nearly Kähler manifolds, we refer the reader to [13, 14] in the case of $\mathbf{S}^6$. In [11], Hashimoto and Mashimo found a family of examples of three dimensional CR submanifolds of $\mathbf{S}^6$, which were orbit submanifolds. The same approach was then used in [14], for obtaining orbit slant surfaces in $\mathbf{S}^6$. Here we investigate the slant surfaces of $\mathbf{S}^3 \times \mathbf{S}^3$ which are orbits of a two-dimensional connected Lie subgroup of the nearly Kähler isometries and prove the following theorems.

1.1. Theorem. *Let M be a slant, two-dimensional submanifold of the nearly Kähler $\mathbf{S}^3 \times \mathbf{S}^3$ which is an orbit of the point (p, q) of an isometric action of a connected Lie subgroup of the nearly Kähler isometries group $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$. Then M is congruent to an orbit immersion of $(1, 1)$ given by*

$$\begin{aligned} f(t,s) =&\big((\cos(a_1t + a_2s) + \sin(a_1t + a_2s)\alpha)(\cos(c_2s) - \sin(c_2s)\gamma),\\ &(\cos(b_1t + b_2s) + \sin(b_1t + b_2s)\beta)(\cos(c_2s) - \sin(c_2s)\gamma)\big), \end{aligned} \tag{1.1}$$

where α, β, γ are arbitrary unit imaginary quaternions. Here $(a_1, b_1, 0), (a_2, b_2, c_2)$ are unit and orthogonal vectors in the space $\mathbb{R}^3$ such that, in the case that α, β and γ are collinear (taking $\alpha = \beta = \gamma$), satisfy additional condition:

$$(a_2, b_2, c_2) \neq \pm\frac{1}{\sqrt{(a_1 - b_1)^2 + 1)}}(-(a_1 - b_1)b_1, (a_1 - b_1)a_1, 1).$$

Moreover, it holds that:
a) Any such immersion is flat.
b) The immersion f is totally geodesic if and only if α, β and γ are collinear.

1.2. Theorem. *For any prescribed angle θ there exists a slant immersion of the form (1.1) with the Wirtinger angle θ.*

2. Preliminaries

Let $\mathbf{S}^3 \subset \mathbb{R}^4 = \mathbb{H}$ be a unit three-dimensional sphere, which we can regard as the set of all unit quaternions. It is well known that $\mathbf{S}^3$ is a Lie group isometric to $SU(2)$ and that it admits a global moving frame. If we denote by $p = x_1 + ix_2 + x_3j + x_4k \in \mathbf{S}^3$ an arbitrary point of the sphere, one of such moving frames is given by $X_1(p) = pi, X_2(p) = pj, X_3(p) = -pk$. Straightforwardly, we get that these vector fields form an orthonormal moving frame and that $[X_i, X_j] = -2\varepsilon_{ijk}X_k$, where ε_{ijk} are the Levi-Civita symbols.

This relation directly implies that there are no two-dimensional Lie subalgebras of the Lie algebra $T_1\mathbf{S}^3 = \mathfrak{su}(2)$.

Obviously, an arbitrary tangent vector field can be represented in the form $X = p\alpha$, where p denotes the position vector field, and $\alpha(p)$ is purely imaginary quaternion.

The isometry of the spaces $T_{(p,q)}(\mathbf{S}^3 \times \mathbf{S}^3) \cong T_p\mathbf{S}^3 \oplus T_q\mathbf{S}^3$ allows us to represent an arbitrary tangent vector at a point $(p,q) \in \mathbf{S}^3 \times \mathbf{S}^3$ by $Z = (U,V) = (p\alpha, q\beta)$, where α and β are imaginary quaternions. Recall that the Cayley product of the imaginary quaternions satisfies

$$\alpha \cdot \beta = -\langle \alpha, \beta \rangle + \alpha \times \beta,$$

where we denote by $\langle , \rangle$ and $\times$ the standard metric of $\mathbb{R}^4$ and the cross product in $\mathbb{R}^3$, respectively.

One of the useful moving frames of $\mathbf{S}^3 \times \mathbf{S}^3$, obtained in a natural way from the quaternionic structure is the following, see [1]:

$$\begin{aligned} &E_1(p,q) = (pi, 0), && E_2(p,q) = (pj, 0), && E_3(p,q) = -(pk, 0),\\ &F_1(p,q) = (0, qi), && F_2(p,q) = (0, qj), && F_3(p,q) = -(0, qk). \end{aligned}$$

The almost complex structure on $\mathbf{S}^3 \times \mathbf{S}^3$ is given by, see [1, 4]:

$$JZ_{(p,q)} = \frac{1}{\sqrt{3}}(2pq^{-1}V - U, -2qp^{-1}U + V).$$

Since the almost complex structure is not an isometry with regard to the standard product metric inherited from the space $\mathbb{R}^8$ which we also denote by $\langle , \rangle$, we define another metric g by

$$g(Z, Z') = \frac{1}{2}(\langle Z, Z' \rangle + \langle JZ, JZ' \rangle).$$

Then $(\mathbf{S}^3 \times \mathbf{S}^3, g, J)$ is an almost complex manifold. We denote the Levi-Civita connection of g by $\widetilde{\nabla}$ and by $G(X,Y) = (\widetilde{\nabla}_X J)Y$. Then, see [1], we have that

$$\begin{aligned} &\widetilde{\nabla}_{E_i} E_j = -\varepsilon_{ijk} E_k, && \widetilde{\nabla}_{E_i} F_j = \frac{\varepsilon_{ijk}}{3}(E_k - F_k),\\ &\widetilde{\nabla}_{F_i} E_j = \frac{\varepsilon_{ijk}}{3}(F_k - E_k), && \widetilde{\nabla}_{F_i} F_j = -\varepsilon_{ijk} F_k. \end{aligned}$$

Moreover, straightforwardly, it holds that

$$\begin{aligned} &G(E_i, E_j) = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k + 2F_k), && G(E_i, F_j) = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k - F_k),\\ &G(F_i, E_j) = -\frac{2}{3\sqrt{3}}\varepsilon_{ijk}(E_k - F_k), && G(F_i, F_j) = \frac{2}{3\sqrt{3}}\varepsilon_{ijk}(2E_k + F_k), \end{aligned} \tag{2.1}$$

which further implies that G is a skew-symmetric tensor field, and $\mathbf{S}^3 \times \mathbf{S}^3$ is a nearly Kähler manifold.

In [1] the following almost product structure P was introduced

$$P(U,V) = (pq^{-1}V, qp^{-1}U),$$

and it was also shown that it holds

$$\begin{aligned} &P^2 = Id, && PJ = -JP,\\ &g(PZ, PZ') = g(Z, Z'), && g(PZ, Z') = g(Z, PZ'), \end{aligned}$$

$$PG(X,Y) + G(PX, PY) = 0. \tag{2.2}$$

Denote for an imaginary quaternion $\alpha = \alpha_1 i + \alpha_2 j + \alpha_3 k$

$$\begin{aligned} V_\alpha &= (p\alpha, 0) = \alpha_1 E_1 + \alpha_2 E_2 - \alpha_3 E_3, \\ W_\alpha &= (0, q\alpha) = \alpha_1 F_1 + \alpha_2 F_2 - \alpha_3 F_3. \end{aligned}$$

Then, we have $PV_\alpha = W_\alpha$, and

$$JV_\alpha = -\frac{1}{\sqrt{3}}(V_\alpha + 2W_\alpha), \quad JW_\alpha = \frac{1}{\sqrt{3}}(2V_\alpha + W_\alpha).$$

By using (2.1) and the last relation of (2.2) straightforwardly we obtain that

$$\begin{aligned} G(V_\alpha, V_\beta) &= \frac{2}{3\sqrt{3}}(V_{\alpha\times\beta} + 2W_{\alpha\times\beta}), \\ G(V_\alpha, W_\beta) &= \frac{2}{3\sqrt{3}}(V_{\alpha\times\beta} - W_{\alpha\times\beta}), \\ G(W_\alpha, V_\beta) &= \frac{2}{3\sqrt{3}}(V_{\alpha\times\beta} - W_{\alpha\times\beta}), \\ G(W_\alpha, W_\beta) &= -\frac{2}{3\sqrt{3}}(2V_{\alpha\times\beta} + W_{\alpha\times\beta}). \end{aligned} \tag{2.3}$$

Also, directly we have

$$\langle V_\alpha, V_\beta\rangle = \langle W_\alpha, W_\beta\rangle = \langle \alpha, \beta\rangle, \quad \langle V_\alpha, W_\beta\rangle = 0,$$

which further implies

$$\langle JV_\alpha, JV_\beta\rangle = \langle JW_\alpha, JW_\beta\rangle = \frac{5}{3}\langle \alpha, \beta\rangle, \quad \langle JV_\alpha, JW_\beta\rangle = -\frac{4}{3}\langle \alpha, \beta\rangle.$$

Then we obtain

$$g(V_\alpha, V_\beta) = g(W_\alpha, W_\beta) = \frac{4}{3}\langle \alpha, \beta\rangle, \quad g(V_\alpha, W_\beta) = -\frac{2}{3}\langle \alpha, \beta\rangle. \tag{2.4}$$

In [9] it was shown that the relation between the Euclidean connection ∇ and $\widetilde{\nabla}$ is given by

$$\nabla_X Y = \widetilde{\nabla}_X Y + K(X, Y),$$

where by $K(X, Y) = \frac{1}{2}(JG(X, PY) + JG(Y, PX))$ we denote the difference tensor of the two connections.

The isometries preserving the nearly Kähler structure are given by

$$F_{a,b,c} : (p, q) \mapsto (apc^{-1}, bqc^{-1}), \tag{2.5}$$

for unit quaternions a, b, c, see [15]. Straightforwardly, the group of isometries of $(\mathbf{S}^3 \times \mathbf{S}^3, g, J)$ is isomorphic to the Lie group $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$.

We consider the orbits of two-dimensional connected Lie subgroup of $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$. Let (p, q) be a point of a surface M immersed into $\mathbf{S}^3 \times \mathbf{S}^3$ and X, Y orthonormal basis of $T_{(p,q)}M$. Since $g(X, JX) = 0$, the projection of JX to the tangent plane is collinear with Y. Therefore, we have

$$\begin{aligned} |\cos\angle(JX, T_{(p,q)}M)| &= |\cos\angle(JX, Y)| = |\cos\angle(X, JY)| \\ &= |\cos\angle(JY, T_{(p,q)}M)|, \end{aligned}$$

so the Wirtinger angle does not depend on a choice of tangent vector at a point. Assume, now that M is also an orbit of isometric action, preserving nearly Kähler structure. Then it is obvious that Wirtinger angle is also independent of the choice of the point and that M is a slant surface.

3. Proof of the Theorems 1.1 and 1.2

Let us find the two-dimensional connected Lie subgroups of $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$. An element of the corresponding Lie algebra $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is of the form $E + F + G$, $E = (\alpha, 0, 0)$, $F = (0, \beta, 0)$, $G = (0, 0, \gamma)$, where α, β, γ are imaginary quaternions. The Lie bracket of the direct sum of algebras is taken componentwise so, for two elements of the Lie algebra $\mathfrak{h}$ we have $[E_1 + F_1 + G_1, E_2 + F_2 + G_2] = [E_1, E_2] + [F_1, F_2] + [G_1, G_2]$. Therefore a two-dimensional subspace of $\mathfrak{h}$ is a subalgebra if, respectively, E_1 and E_2, F_1 and F_2, G_1 and G_2 span algebras of $\mathfrak{su}(2)$. If E_1 and E_2 are not collinear, the Lie bracket $[E_1, E_2]$ is in the direction of the vector orthogonal to both E_1, E_2. Therefore, E_1 and E_2, and similarly F_1 and F_2, G_1 and G_2 are, respectively, collinear, their Lie brackets vanish, and the two-dimensional subalgebra $\mathfrak{g}$ of $\mathfrak{h}$ is Abelian.

Therefore, if W_1 and W_2 span the two-dimensional subalgebra $\mathfrak{g}$, there exist unit vectors, with respect to standard metric, E, F and G in each copy of $\mathfrak{su}(2)$ such that $W_i = a_i E + b_i F + c_i G, i = 1, 2$. Denote by α, β, γ the unit imaginary quaternions such that $E = (\alpha, 0, 0)$, $F = (0, \beta, 0)$, $G = (0, 0, \gamma)$. Vectors W_1 and W_2 commute, and moreover, the algebra $\mathfrak{g}$ contains a vector orthogonal to G, so from now on, we can take that it is W_1, i.e. $c_1 = 0$. Also we may assume that W_1, W_2 are unit and orthogonal. Here we also consider the standard metrics. Note that two connected subalgebras having the same Lie algebra are equal, see [16].

We have that the flow corresponding to the vector $aE + bF + cG$ is given by $Fl(t) = (\cos(at) + \sin(at)\alpha, \cos(bt) + \sin(bt)\beta, \cos(ct) + \sin(ct)\gamma)$. Hence, a two-dimensional Lie subgroup with the algebra $\mathfrak{g}$ has elements of the form

$$\begin{aligned} I_{t,s} =& (\cos(a_1 t + a_2 s) + \sin(a_1 t + a_2 s)\alpha, \\ & \cos(b_1 t + b_2 s) + \sin(b_1 t + b_2 s)\beta, \cos(c_2 s) + \sin(c_2 s)\gamma). \end{aligned}$$

If we denote by $\mathrm{a} = a_1 t + a_2 s, \mathrm{b} = b_1 t + b_2 s, \mathrm{c} = c_2 s$, taking (2.5), we obtain that the orbit of the point $(p, q) \in \mathbf{S}^3 \times \mathbf{S}^3$, is given by

$$\begin{aligned} & f(t,s) = (f_1, f_2)(t,s) \\ & = \big((\cos \mathrm{a} + \sin \mathrm{a}\,\alpha)\, p\, (\cos \mathrm{c} - \sin \mathrm{c}\,\gamma), (\cos \mathrm{b} + \sin \mathrm{b}\,\beta)\, q\, (\cos \mathrm{c} - \sin \mathrm{c}\,\gamma)\big). \end{aligned} \tag{3.1}$$

Notice that this orbit is congruent by the isometry $F_{p^{-1},q^{-1},1}$, see (2.5), to an orbit of the point $(1,1)$ of the same form, determined by the imaginary quaternions $p^{-1}\alpha p, q^{-1}\beta q$ and γ. Hence we can consider only the orbits of the point $(p, q) = (1, 1)$.

Then, directly, we obtain that

$$\begin{aligned} & \partial_t f = f_t \\ & = (a_1(-\sin \mathrm{a} + \cos \mathrm{a}\alpha)(\cos \mathrm{c} - \sin \mathrm{c}\gamma), b_1(-\sin \mathrm{b} + \cos \mathrm{b}\beta)(\cos \mathrm{c} - \sin \mathrm{c}\gamma)) \\ & = (f_1 a_1(\cos \mathrm{c} + \sin \mathrm{c}\gamma)\alpha(\cos \mathrm{c} - \sin \mathrm{c}\gamma), f_2 b_1(\cos \mathrm{c} + \sin \mathrm{c}\gamma)\beta(\cos \mathrm{c} - \sin \mathrm{c}\gamma)), \end{aligned}$$

$$\begin{aligned} & \partial_s f = f_s \\ & = (a_2(-\sin \mathrm{a} + \cos \mathrm{a}\alpha)(\cos \mathrm{c} - \sin \mathrm{c}\gamma), b_2(-\sin \mathrm{b} + \cos \mathrm{b}\beta)(\cos \mathrm{c} - \sin \mathrm{c}\gamma) \\ & - c_2((\cos \mathrm{a} + \sin \mathrm{a}\alpha)(\sin \mathrm{c} + \cos \mathrm{c}\gamma), (\cos \mathrm{b} + \sin \mathrm{b}\beta)(\sin \mathrm{c} + \cos \mathrm{c}\gamma)) \\ & = (f_1 a_2(\cos \mathrm{c} + \sin \mathrm{c}\gamma)\alpha(\cos \mathrm{c} - \sin \mathrm{c}\gamma), f_2 b_2(\cos \mathrm{c} + \sin \mathrm{c}\gamma)\beta(\cos \mathrm{c} - \sin \mathrm{c}\gamma)) \\ & - c_2(f_1\gamma, f_2\gamma). \end{aligned}$$

For an arbitrary imaginary quaternion σ we denote by

$$\sigma^*(s) = (\cos \mathrm{c} + \sin \mathrm{c}\gamma)\sigma(\cos \mathrm{c} - \sin \mathrm{c}\gamma)$$

a curve in the space of imaginary quaternions. Note that $\gamma^* = \gamma$ is a constant curve.

3.1. Lemma. *The mapping* (3.1) *is of the rank less then two at arbitrary point if and only if* α, β, γ *are collinear* ($\alpha = \beta = \gamma$) *and*

$$(a_2, b_2, c_2) = \pm \frac{1}{\sqrt{(a_1 - b_1)^2 + 1)}} (-(a_1 - b_1)b_1, (a_1 - b_1)a_1, 1).$$

Proof. We can write now $f_t = (f_1 a_1 \alpha^*, f_2 b_1 \beta^*)$ and $f_s = (f_1 a_2 \alpha^*, f_2 b_2 \beta^*) - (f_1 c_2 \gamma, f_2 c_2 \gamma)$. Because of the orthogonality of $(a_1, b_1, 0)$ and (a_2, b_2, c_2) for some $\lambda \in \mathbb{R}$ we have $(a_2, b_2) = \lambda(-b_1, a_1)$, where $\lambda^2 + c_2^2 = 1$. Also, since $a_1^2 + b_1^2 = 1$ we have that $f_t \neq 0$.

Assume that f_t and f_s are collinear. Then there exists a coefficient k such that $f_s = k f_t$, i.e.

$$(3.2) \qquad k a_1 \alpha^* = a_2 \alpha^* - c_2 \gamma, \quad k b_1 \beta^* = b_2 \beta^* - c_2 \gamma.$$

Assume first that $c_2 = 0$ which implies $\lambda = \pm 1$. Then (3.2) reduce to

$$(3.3) \qquad k a_1 = a_2 = -\lambda b_1, \quad k b_1 = b_2 = \lambda a_1.$$

If $b_1 \neq 0$ then the first relation of (3.3) implies $a_1, k \neq 0$ and similarly from $a_1 \neq 0$ we get $b_1 \neq 0$. Since $a_1^2 + b_1^2 = 1$ we have that $a_1, b_1, k \neq 0$. However, then relations (3.3) also imply that the sign of $a_1 \cdot b_1$ satisfies $sgn(a_1 \cdot b_1) = sgn(-k\lambda) = sgn(k\lambda)$ which is impossible.

Therefore, we have $c_2 \neq 0$, and further that α^*, β^* and γ are collinear, which then implies that α, β and γ are collinear. By change of signs of $a_i, b_i, i = 1, 2$, if necessary, we can assume that $\alpha = \beta = \gamma$. Then (3.2) reduces to $k(a_1, b_1) = (a_2 - c_2, b_2 - c_2)$ and we have that $k \in \mathbb{R}$. Further, it holds

$$(3.4) \qquad c_2 = \lambda a_1 - k b_1 = -k a_1 - \lambda b_1.$$

The determinant of the system (3.4) over λ and k is $a_1^2 + b_1^2 = 1$, different from zero, and therefore we have $\lambda = c_2(a_1 - b_1), k = -c_2(a_1 + b_1)$, and $c_2 = \pm 1/\sqrt{(a_1 - b_1)^2 + 1}$.

Straightforwardly it follows that

$$\begin{aligned} (a_2, b_2, c_2) &= (-\lambda b_1, \lambda a_1, c_2) \\ &= \pm \frac{1}{\sqrt{(a_1 - b_1)^2 + 1)}} (-(a_1 - b_1)b_1, (a_1 - b_1)a_1, 1). \end{aligned} \tag{3.5}$$

Straightforward computation shows that $\alpha = \beta = \gamma$ and (3.5) implies that mapping (3.1) is of rank less then two. □

3.2. Remark. We have that ${f_1}_t = f_1 a_1 \alpha^*, {f_1}_s = f_1(a_2 \alpha^* - c_2 \gamma)$ and that imaginary curves ${f_1}^{-1} {f_1}_t$ and ${f_1}^{-1} {f_1}_s$ defining these vector fields depend only on parameter s. Moreover, if we assume that $f_1 = const.$ it follows that $a_1 = 0$ and $a_2 \alpha^* = c_2 \gamma$. We then obtain that $\alpha^* = \gamma$, $a_2 = c_2$ and further $(a_1, b_1, 0) = (0, 1, 0), (a_2, b_2, c_2) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ which implies that (f_1, f_2) is not an immersion. Same conclusion is obtained for assumption $f_2 = const.$ Therefore, neither f_1 nor f_2 can be constant points.

From now on, we assume that either α, β, γ are not collinear or that

$$(a_2, b_2, c_2) \neq \pm \frac{1}{\sqrt{(a_1 - b_1)^2 + 1)}} (-(a_1 - b_1)b_1, (a_1 - b_1)a_1, 1),$$

so that, by Lemma 3.1, we have that the mapping (3.1) is an immersion.

Further we have that

$$
\begin{aligned}
(3.6)\ f_{tt} &= -(a_1^2(\cos \mathrm{a} + \sin \mathrm{a}\alpha)(\cos \mathrm{c} - \sin \mathrm{c}\gamma), b_1^2(\cos \mathrm{b} + \sin \mathrm{b}\beta)(\cos \mathrm{c} - \sin \mathrm{c}\gamma)),\\
f_{ts}&\\
&= -(a_1a_2(\cos \mathrm{a} + \sin \mathrm{a}\alpha)(\cos \mathrm{c} - \sin \mathrm{c}\gamma), b_1b_2(\cos \mathrm{b} + \sin \mathrm{b}\beta)(\cos \mathrm{c} - \sin \mathrm{c}\gamma))\\
&- c_2(a_1(-\sin \mathrm{a} + \cos \mathrm{a}\alpha)(\sin \mathrm{c} + \cos \mathrm{c}\gamma), b_1(-\sin \mathrm{b} + \cos \mathrm{b}\beta)(\sin \mathrm{c} + \cos \mathrm{c}\gamma)),\\
f_{ss} &= 2c_2(a_2(\sin \mathrm{a} - \cos \mathrm{a}\alpha)(\sin \mathrm{c} + \cos \mathrm{c}\gamma), b_2(\sin \mathrm{b} - \cos \mathrm{b}\beta)(\sin \mathrm{c} + \cos \mathrm{c}\gamma))\\
&- c_2^2 f - (a_2^2(\cos \mathrm{a} + \sin \mathrm{a}\alpha)(\cos \mathrm{c} - \sin \mathrm{c}\gamma), b_2^2(\cos \mathrm{b} + \sin \mathrm{b}\beta)(\cos \mathrm{c} - \sin \mathrm{c}\gamma)).
\end{aligned}
$$

Denote by $e_1 = \frac{1}{\sqrt{g(f_t,f_t)}} f_t$ and by $V = f_s - g(f_s, e_1)e_1 = f_s - \frac{g(f_s,f_t)}{g(f_t,f_t)} f_t$ the vector field orthogonal to e_1. Then, straightforwardly, we get $g(V,V) = g(f_s, f_s) - \frac{g(f_s,f_t)^2}{g(f_t,f_t)}$. The vector fields e_1 and $e_2 = \frac{1}{\sqrt{g(V,V)}} V$ form an orthonormal frame. Therefore, the Wirtinger angle of M is given by $\cos\theta = g(Je_1, e_2)$, i.e.

$$
(3.7) \qquad \cos^2\theta = \frac{g(Jf_t, f_s)^2}{g(f_t, f_t)g(f_s, f_s) - g(f_t, f_s)^2}.
$$

We can write

$$
\begin{aligned}
f_t &= a_1 V_{\alpha^*} + b_1 W_{\beta^*}, \quad f_s = a_2 V_{\alpha^*} + b_2 W_{\beta^*} - c_2(V_\gamma + W_\gamma),\\
Jf_t &= \frac{1}{\sqrt{3}}\big(-a_1(V_{\alpha^*} + 2W_{\alpha^*}) + b_1(2V_{\beta^*} + W_{\beta^*})\big),\\
(3.8) \qquad Jf_s &= \frac{1}{\sqrt{3}}\big(-a_2(V_{\alpha^*} + 2W_{\alpha^*}) + b_2(2V_{\beta^*} + W_{\beta^*}) + c_2(-V_\gamma + W_\gamma)\big).
\end{aligned}
$$

Since we have that $\langle \alpha^*, \beta^* \rangle = \langle \alpha, \beta \rangle$, $\gamma^* = \gamma$ and $a_1^2 + b_1^2 = a_2^2 + b_2^2 + c_2^2 = 1$ it follows

$$
\begin{aligned}
g(f_t, f_t) &= \frac{4}{3}(1 - a_1b_1\langle \alpha, \beta \rangle),\\
g(f_s, f_s) &= \frac{4}{3}(1 - a_2c_2\langle \alpha, \gamma \rangle - b_2c_2\langle \beta, \gamma \rangle - a_2b_2\langle \alpha, \beta \rangle),\\
g(f_t, f_s) &= -\frac{2}{3}((a_1b_2 + b_1a_2)\langle \alpha, \beta \rangle + a_1c_2\langle \alpha, \gamma \rangle + b_1c_2\langle \beta, \gamma \rangle),\\
g(Jf_t, f_s) &= \frac{2}{\sqrt{3}}((b_1a_2 - a_1b_2)\langle \alpha, \beta \rangle + a_1c_2\langle \alpha, \gamma \rangle - b_1c_2\langle \beta, \gamma \rangle).
\end{aligned}
$$

Notice that, from $\alpha^* \cdot \beta^* = (\alpha\beta)^*$, and $\langle \alpha, \beta \rangle = \langle \alpha^*, \beta^* \rangle$ we have that $(\alpha \times \beta)^* = \alpha^* \times \beta^*$. Moreover, we have that

$$
\begin{aligned}
&f_1^{-1}(-\sin \mathrm{a} + \cos \mathrm{a}\alpha)(\sin \mathrm{c} + \cos \mathrm{c}\gamma) + \langle \alpha, \gamma \rangle\\
&= (\cos \mathrm{c} + \sin \mathrm{c}\gamma)\alpha(\sin \mathrm{c} + \cos \mathrm{c}\gamma)(\cos \mathrm{c} + \sin \mathrm{c}\gamma)(\cos \mathrm{c} - \sin \mathrm{c}\gamma) + \langle \alpha, \gamma \rangle\\
(3.9) \qquad &= (\alpha\gamma)^* + \langle \alpha, \gamma \rangle = (\alpha\gamma + \langle \alpha, \gamma \rangle)^* = (\alpha \times \gamma)^*.
\end{aligned}
$$

Note that, in case $c_2 = 0$, which means that both W_1 and W_2 are orthogonal to G, with regard to standard metric, we can choose W_1 and W_2, so that W_1 is orhtogonal to F. Than we can take $W_1 = E$, $W_2 = F$, i.e.

$$
(3.10) \qquad a_1 = 1, b_1 = 0, a_2 = 0, b_2 = 1, c_1 = c_2 = 0.
$$

Let us now show that Theorem 1.2 holds, i.e. that for arbitrary prescribed angle θ, there exists a slant surface M in $\mathbf{S}^3 \times \mathbf{S}^3$ such that θ is its Wirtinger angle. For simplicity, we consider the surfaces that satisfy (3.10). Then (3.7) reduces to $\cos^2\theta = \frac{3\langle \alpha,\beta \rangle^2}{4-\langle \alpha,\beta \rangle^2}$, so it is sufficient to take the unit imaginary quaternions α and β that satisfy $\langle \alpha, \beta \rangle^2 = \frac{4\cos^2\theta}{3+\cos^2\theta}$.

Now, let us investigate the second fundamental form of the submanifold M. Recall that the second fundamental form of the immersion $f_1 : \mathbf{S}^3 \to \mathbb{R}^4$ with respect to the standard connection is given by $h^4(X,Y) = -\langle X, Y\rangle f_1$. Therefore, the second fundamental form of the immersion $(f_1, f_2) : \mathbf{S}^3 \times \mathbf{S}^3 \to \mathbb{R}^8$ with respect to the standard Euclidean, i.e. product connection is given by $h^D(X,Y) = -\langle \pi_1(X), \pi_1(Y)\rangle (f_1, 0) - \langle \pi_2(X), \pi_2(Y)\rangle (0, f_2)$, where by $\pi_1(X)$ and $\pi_2(X)$ we denote the projections of the vector X at a point on the first or second four coordinates. If we denote by D the standard connection in space $\mathbb{R}^8$, we then have, for the vector fields X, Y tangent to $\mathbf{S}^3 \times \mathbf{S}^3$, that

$$D_X Y = h^D(X,Y) + K(X,Y) + \widetilde{\nabla}_X Y. \tag{3.11}$$

First, the direct computation by using (2.3) and $G(V_\alpha, W_\alpha) = 0$ we obtain that

$$\begin{aligned}
G(f_t, Pf_t) &= \frac{2}{\sqrt{3}} a_1 b_1 (V_{\alpha^* \times \beta^*} + W_{\alpha^* \times \beta^*}),\\
G(f_t, Pf_s) &= \frac{2}{3\sqrt{3}} \big((2b_1a_2 + a_1b_2) V_{\alpha^* \times \beta^*} + (2a_1b_2 + b_1a_2) W_{\alpha^* \times \beta^*}\\
&\quad + a_1c_2(2V_{\gamma \times \alpha^*} + W_{\gamma \times \alpha^*}) + b_1c_2(V_{\beta^* \times \gamma} + 2W_{\beta^* \times \gamma})\big),\\
G(f_s, Pf_t) &= \frac{2}{3\sqrt{3}} \big((2b_2a_1 + a_2b_1) V_{\alpha^* \times \beta^*} + (2a_2b_1 + b_2a_1) W_{\alpha^* \times \beta^*}\\
&\quad + a_1c_2(V_{\gamma \times \alpha^*} + 2W_{\gamma \times \alpha^*}) + b_1c_2(2V_{\beta^* \times \gamma} + W_{\beta^* \times \gamma})\big),\\
G(f_s, Pf_s) &= \frac{2}{\sqrt{3}} (a_2b_2(V_{\alpha^* \times \beta^*} + W_{\alpha^* \times \beta^*}) + a_2c_2(V_{\gamma \times \alpha^*} + W_{\gamma \times \alpha^*})\\
&\quad + b_2c_2(V_{\beta^* \times \gamma} + W_{\beta^* \times \gamma})).
\end{aligned}$$

Then the difference tensor is given by

$$\begin{aligned}
K(f_t, f_t) &= \frac{2}{3} a_1 b_1 (V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}),\\
K(f_t, f_s) &= \frac{1}{3} ((a_1b_2 + a_2b_1)(V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}) + b_1c_2(V_{\beta^* \times \gamma} - W_{\beta^* \times \gamma})\\
&\quad + a_1c_2(V_{\gamma \times \alpha^*} - W_{\gamma \times \alpha^*})),\\
K(f_s, f_s) &= \frac{2}{3} (a_2b_2(V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}) + a_2c_2(V_{\gamma \times \alpha^*} - W_{\gamma \times \alpha^*})\\
&\quad + b_2c_2(V_{\beta^* \times \gamma} - W_{\beta^* \times \gamma})).
\end{aligned} \tag{3.12}$$

Now, straightforwardly, by using (3.9) in (3.6), and further using (3.11) and (3.12) we obtain that

$$\begin{aligned}
\widetilde{\nabla}_{\partial_t} \partial_t &= -\frac{2}{3} a_1 b_1 (V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}),\\
\widetilde{\nabla}_{\partial_t} \partial_s &= \frac{1}{3} (-(a_1b_2 + a_2b_1)(V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}) + a_1c_2(2V_{\gamma \times \alpha^*} + W_{\gamma \times \alpha^*})\\
&\quad - b_1c_2(V_{\beta^* \times \gamma} + 2W_{\beta^* \times \gamma})),\\
\widetilde{\nabla}_{\partial_s} \partial_s &= \frac{2}{3} (-a_2b_2(V_{\alpha^* \times \beta^*} - W_{\alpha^* \times \beta^*}) + a_2c_2(2V_{\gamma \times \alpha^*} + W_{\gamma \times \alpha^*})\\
&\quad - b_2c_2(V_{\beta^* \times \gamma} + 2W_{\beta^* \times \gamma})).
\end{aligned} \tag{3.13}$$

3.3. Lemma. *The given coordinates (t, s) are flat.*

Proof. We denote $\partial_t = f_t, \partial_s = f_s$. By using (3.8), (2.4) and that the cross product of two vectors is orthogonal to the components with respect to the metric $\langle , \rangle$ we have that

$$\begin{aligned} g(\widetilde{\nabla}_{\partial t}\partial_t, \partial_s) &= -\frac{2}{3}a_1 b_1 \big(g(V_{\alpha^*\times\beta^*} - W_{\alpha^*\times\beta^*}, a_2 V_{\alpha^*} + b_2 W_{\beta^*} - c_2(V_\gamma + W_\gamma)) \big) \\ &= -\frac{2}{3}a_1 b_1 \big(a_2(\frac{4}{3}\langle \alpha^* \times \beta^*, \alpha^* \rangle - (-\frac{2}{3})\langle \alpha^* \times \beta^*, \alpha^* \rangle) \\ &+ b_2((-\frac{2}{3})\langle \alpha^* \times \beta^*, \beta^* \rangle - \frac{4}{3}\langle \alpha^* \times \beta^*, \beta^* \rangle) \\ &- c_2(\frac{4}{3} - \frac{2}{3} + \frac{2}{3} - \frac{4}{3})\langle \alpha^* \times \beta^*, \gamma \rangle \big) = 0. \end{aligned}$$

In the same manner we obtain that $g(\widetilde{\nabla}_X Y, Z) = 0$, for all $X, Y, Z \in \{\partial_t, \partial_s\}$ which concludes the proof. □

Hence, the expressions on the right hand side of (3.13) then represent the corresponding components of the second fundamental form of the immersion into $\mathbf{S}^3 \times \mathbf{S}^3$.

Let us find the necessary and sufficient conditions for the immersion (1.1) to be totally geodesic.

First, assume that $\alpha^* \times \beta^* = 0$. Obviously, then for $c_2 = 0$ we obtain the totally geodesic immersion. Then we can trivially take γ collinear to α and β. Therefore, let's take $c_2 \neq 0$. Then, we obtain that $(2a_i\alpha^* + b_i\beta^*)$ and $(a_i\alpha^* + 2b_i\beta^*)$ for $i = 1, 2$ are collinear to γ. Hence, we have $a_i\alpha^*, b_i\beta^* || \gamma$. Therefore, we have $\alpha^*, \beta^* || \gamma$, which we can also take in the case that $a_1 = a_2 = 0$ or $b_1 = b_2 = 0$.

Assume that it holds $\alpha^* \times \beta^* \neq 0$. Therefore we have $c_2 \neq 0$ and exactly one of the coefficients a_1, b_1 vanishes. For instance, take $a_1 = 1, b_1 = 0$, the other case is similar. Then, in the same manner as before, there exist coefficients k_i such that

$$\begin{aligned} &\gamma \times (2a_i\alpha^* + b_i\beta^*) = k_i(\alpha^* \times \beta^*), \\ &\gamma \times (a_i\alpha^* + 2b_i\beta^*) = -k_i(\alpha^* \times \beta^*), \end{aligned}$$

so $\gamma \times (a_i\alpha^* + b_i\beta^*) = 0$. Therefore γ is collinear to $a_1\alpha^*$ and $a_2\alpha^* + b_2\beta^*$. Further we have that $a_2 = 0$, so we can again take $\alpha^*, \beta^* || \gamma$. Now, recall that α^* is collinear to γ if and only if α is collinear to γ. This ends the proof of the Theorem 1.1.

3.4. Example. Let us now give an example of the minimal, non totally geodesic immersion. Straightforwardly, we have that the immersion is minimal if and only if

$$g(f_t, f_t)h(\partial_s, \partial_s) + g(f_s, f_s)h(\partial_t, \partial_t) - 2g(f_t, f_s)h(\partial_t, \partial_s) = 0.$$

Let $\alpha = -\beta = \frac{1}{\sqrt{2}}(-i + j), \gamma = i$ and $a_1 = b_1 = \frac{1}{\sqrt{2}}$, $a_2 = b_2 = 0$, $c_2 = 1$. Then we have $g(f_s, f_t) = 0$, $h(\partial_t, \partial_t) = h(\partial_s, \partial_s) = 0$, so the immersion is minimal. The corresponding Wirtinger angle is given by $\cos^2\theta = \frac{1}{2}$.

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