Investigation of Natural Frequencies of Piles Partially Embedded in Elastic Soil Using Transfer Matrix Method

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ABSTRACT

This study aims to calculate exact natural frequencies of piles partially embedded in elastic soil by using transfer matrix method (TMM). The elastic soil is considered as Winkler foundation model using linear springs. The piles are modelled as Timoshenko beam-columns. The transfer matrices of piles are constructed after obtaining end forces and end displacements of piles using governing equation of motion of Timoshenko beam-columns. Different soil conditions are used to show their effects on natural frequencies of piles partially embedded in Winkler foundation. Moreover, the results are tabulated comparatively with finite element solutions that obtained from structural analysis software SAP2000.

Keywords: Natural frequency, partially embedded pile, transfer matrix method, winkler foundation.

INTRODUCTION

Free vibration analysis of piles embedded in elastic soil plays an important role in civil engineering applications. Therefore, the calculation of exact natural frequencies of partially embedded piles becomes an interesting research area. Free vibration analysis of partially embedded piles is performed in limited number of literature. Catal (2006) investigated free vibrations of semi-rigid connected and partially embedded piles considering the effects of bending moment, axial force and shear force. Yesilce and Catal (2008a) calculated natural frequencies of partially embedded piles using shear theory via TMM. Yesilce and Catal (2008b) performed free vibration analysis of semi-rigid connected piles embedded in soil by using Reddy-Bickford beam theory and Winkler foundation model. Lu and Yuan (2014) investigated free vibrations of a periodic viaduct supported by pile foundations using boundary element method (BEM). Ai et al. (2016) researched vibrations of a partially embedded pile group in transversly isotropic soils. Hu et al. (2016) obtained lateral dynamic response of a partially embedded pile subjected to combined loads in saturated soil.

The TMM provides exact results for vibration analysis of beams and beam assembly structures. The TMM uses end forces and end displacement of members that obtained analytically. In open literature, several studies about application of TMM are found for continous beams (Lin and Chang, 2005; Ceaşu et al., 2010; Attar, 2012; Wu and Chang, 2015; Lee and Lee 2016; Lee and Lee, 2017; Lee and Lee, 2018).

In this study, first five exact natural frequencies of partially embedded piles are obtained by using TMM. The Winkler foundation model is used via linear springs along the embedded region of pile. The pile segments are considered as Timoshenko beam-columns. Different Winkler spring stiffness values are used to observe the effects on the free vibrations of partially embedded piles. The exact TMM results are presented with finite element results of SAP2000.

MODEL AND FORMULATION

The mathematical models of partially embedded piles considered in the study are presented in Figure 1 where k_s is Winkler spring stiffness, d is diameter of pile cross-section, 1,2 and 3 are node numbers, L_1 and L_2 are length of the free region and embedded region, respectively. The boundary conditions of Model-a and Model-b are fixed-free and fixed-simply supported, respectively.

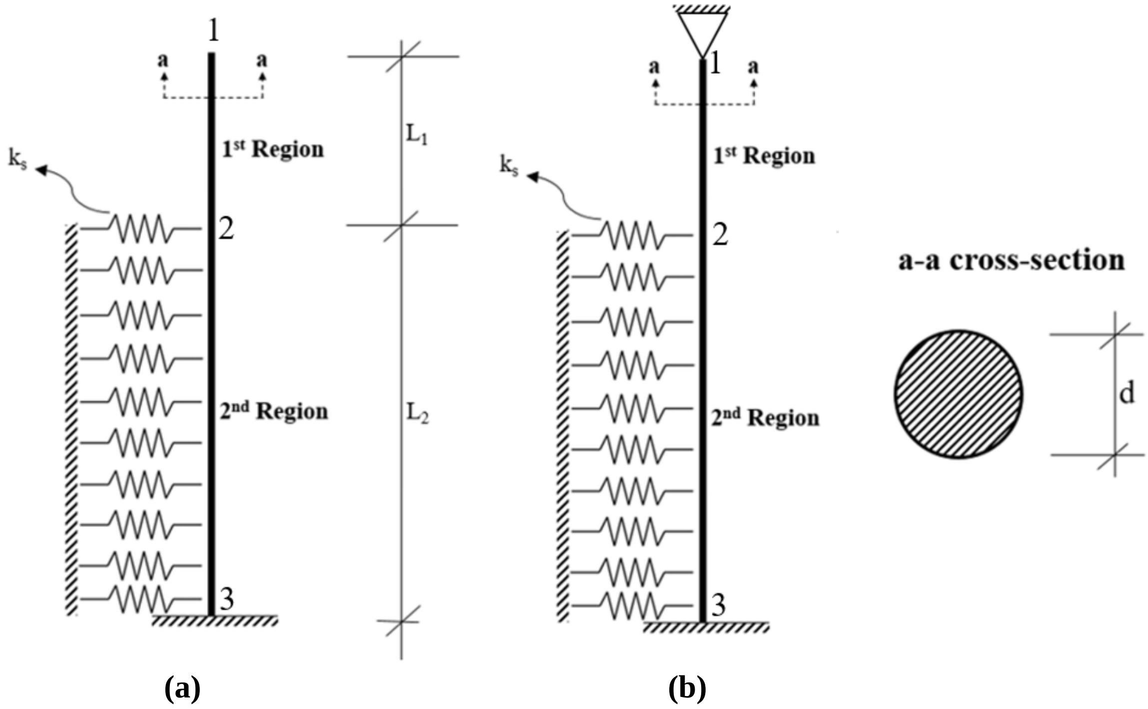

Figure 1. Partially embedded pile with free cap (a) and partially embedded pile with simply supported cap (b)

Some assumptions are considered to clarify and simplificate the analysis procedure as follows:

1. The material of the pile is isotropic and homogenous.
2. The cross-section of the pile is uniform.
3. The pile behaves linear and elastic.
4. The damping is neglected.
5. The linear springs are distributed along the embedded pile length.

The governing equations of motion of vibrating Timoshenko beam-column embedded in elastic soil can be written as:

$$\begin{aligned}&\frac{AG}{\bar{k}}\left(\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{\partial\theta(x,t)}{\partial x}\right)-\bar{m}\frac{\partial^2 y(x,t)}{\partial t^2}-k_s y(x,t)=0\\&EI\frac{\partial^2\theta(x,t)}{\partial x^2}-\bar{m}\frac{I}{A}\frac{\partial^2\theta(x,t)}{\partial t^2}+\frac{AG}{\bar{k}}\left(\frac{\partial y(x,t)}{\partial x}-\theta(x,t)\right)=0\end{aligned}\tag{1}$$

where x is axis of beam-column , t is time, A is cross-sectional area, I is area moment of inertia, G is shear modulus, E is Young's modulus, $\bar{k}$ is shear coefficient, $\bar{m}$ is mass per unit length, L is beam-column lenght, y(x,t) and θ(x,t) are transverse deflection function and rotation function, respectively.

Eq.(2) is obtained by assuming the motion is harmonic and applying separation of variables method:

$$\begin{aligned}&\frac{AG}{\bar{k}L^2}\frac{d^2y(z)}{dz^2}-\frac{AG}{\bar{k}L}\frac{d\theta(z)}{dz}+\bar{m}\omega^2 y(z)-k_s y(z)=0\\&\frac{EI}{L^2}\frac{d^2\theta(z)}{dz^2}+\frac{AG}{\bar{k}L}\frac{dy(z)}{dz}+\left(\frac{\bar{m}I\omega^2}{A}-\frac{AG}{\bar{k}}\right)\theta(z)=0\end{aligned}\tag{2}$$

where z=x/L and ω is angular natural frequency.

The solution is assumed as:

$$\begin{aligned}&y(z)=\left\{\bar{C}\right\}e^{isz}\\&\theta(z)=\left\{\bar{D}\right\}e^{isz}\end{aligned}\tag{3}$$

By substituting Eq.(3) into Eq.(2), y(z) and θ(z) are written as Eq.(4) and Eq.(5), respectively.

$$y(z)=(\bar{C}_1e^{is_1z}+\bar{C}_2e^{is_2z}+\bar{C}_3e^{is_3z}+\bar{C}_4e^{is_4z})\tag{4}$$

$$\theta(z)=(K_1\bar{C}_1e^{is_1z}+K_2\bar{C}_2e^{is_2z}+K_3\bar{C}_3e^{is_3z}+K_4\bar{C}_4e^{is_4z})\tag{5}$$

where $K_p=\left(-\left(\frac{AG}{\bar{k}L^2}\right)+\left(\bar{m}\omega^2\right)-k_s\right)/\left(\left(\frac{AG}{\bar{k}L^2}\right)is_j\right)$; p=1,2,3,4; j=1,2,3,4.

The bending moment function and shear force function are given in Eq.(6) and Eq.(7), respectively.

$$M(z)=\frac{EI}{L}\frac{d\theta(z)}{dz}\tag{6}$$

$$Q(z)=\frac{AG}{\bar{k}L}\frac{dy(z)}{dz}-\frac{AG}{\bar{k}}\theta(z)\tag{7}$$

The displacement functions written in Eqs. (4)-(5) and force functions written in Eqs.(6)-(7) are used to obtain transfer matrix formulations of embedded Timoshenko element pile. For the free part of the pile that is denominated as 1st Region, k_s is equated to zero.

TRANSFER MATRIX METHOD (TMM)

For the 1st Region:

The state vector of node 1 can be can be written as Eq.(8):

$$\{Z_1\} = [H_1]\{\overline{B}\} \tag{8}$$

where

$\{Z_1\} = \{y_{(z=1)} \quad \theta_{(z=1)} \quad Q_{(z=1)} \quad M_{(z=1)}\}^T$ and $[H_1]$ is 4x4 coefficient matrix that constructed by using end displacements and end forces of Timoshenko beam-column.

The state vector of node 2 can be can be written as Eqs.(9):

$$\{Z_{21}\} = [H_2]\{\overline{B}\} \tag{9}$$

where

$\{Z_{21}\} = \{y_{(z=0)} \quad \theta_{(z=0)} \quad Q_{(z=0)} \quad M_{(z=0)}\}^T$ and $[H_2]$ is 4x4 coefficient matrix of Timoshenko beam-column.

By using the relation between Eqs.(8)-(9), the following equation can be obtained.

$$\begin{gathered} \{Z_{21}\} = [T_1]\{Z_1\} \\ [T_1] = [H_2][H_1]^{-1} \end{gathered} \tag{10}$$

where $[T_1]$ is transfer matrix of 1st Region.

For the 2nd Region:

The state vector of node 3 is given in Eq.(11):

$$\{Z_3\} = [H_3]\{\overline{C}\} \tag{11}$$

where

$\{Z_3\} = \{y_{(z=0)} \quad \theta_{(z=0)} \quad Q_{(z=0)} \quad M_{(z=0)}\}^T$ and $[H_4]$ is 4x4 coefficient matrix that constructed by using end displacements and end forces of embedded Timoshenko beam-column.

The state vector of node 2 can be can be written as Eqs.(12):

$$\{Z_{22}\} = [H_4]\{\overline{C}\} \tag{12}$$

where

$\{Z_{22}\} = \{y_{(z=1)} \quad \theta_{(z=1)} \quad Q_{(z=1)} \quad M_{(z=1)}\}^T$ and $[H_4]$ is 4x4 coefficient matrix of embedded Timoshenko beam-column.

By using the relation between Eqs.(11)-(12), the fequation below is obtained:

$$\begin{gathered} \{Z_3\} = [T_2]\{Z_{22}\} \\ [T_2] = [H_4][H_3]^{-1} \end{gathered} \tag{13}$$

where $[T_2]$ is transfer matrix of 1st Region.

The global transfer matrix of the system is constructed as follows:

$$\{Z_1\} = [T_G]\{Z_3\}$$
$$[T_G] = [T_1][T_2] \tag{14}$$

where $[T_G]$ is global transfer matrix of the partially embedded Timoshenko element pile.

The natural frequencies can be calculated by equating the determinant of the global transfer matrix to zero. It should be noted that the global transfer matrix of the system is reduced due to boundary conditions of Model-a and Model-b.

NUMERICAL ANALYSIS AND DISCUSSION

The numerical analysis is performed according to following material and geometrical properties: L_1 = 4 m, L_2 = 10 m, d = 0.8 m, unit weight of pile = 24.5 kN/m^3, E = 2.94x10^7 kN/m^2, G = 1.23x 10^7 kN/m^2, $\bar{k}$ = 1.18.

The natural frequencies that obtained via TMM and finite element method (FEM) from SAP2000 commercial software are presented in Table 1 and Table 2 for Model-a and Model-b, respectively. It should be noted that the partially embedded pile is meshed using 0.1 m long segments in SAP2000.

Table 1. First five natural frequencies of Model-a

Method	**Natural Frequency (Hz)**	**k_s (kN/m)** 5000	10000	15000	20000	25000
TMM	1st	**4.6876**	**5.7228**	**6.3591**	**6.8216**	**7.1876**
FEM(SAP2000)		4.6913	5.7292	6.3677	6.8319	7.1994
TMM	2nd	**14.9895**	**17.5638**	**19.8533**	**21.9133**	**23.7899**
FEM(SAP2000)		15.0352	17.6128	19.9065	21.9712	23.8530
TMM	3rd	**34.4394**	**35.4633**	**36.4642**	**37.4438**	**38.4040**
FEM(SAP2000)		34.7205	35.7488	36.7538	37.7373	38.7010
TMM	4th	**64.4551**	**64.9734**	**65.4910**	**66.0079**	**66.5242**
FEM(SAP2000)		65.3907	65.9121	66.4325	66.9522	67.4711
TMM	5th	**103.1076**	**103.4536**	**103.7996**	**104.1456**	**104.4915**
FEM(SAP2000)		105.3402	105.6890	106.0377	106.3863	106.7349

Table 2. First five natural frequencies of Model-b

Method	**Natural Frequency (Hz)**	**k_s (kN/m)** 5000	10000	15000	20000	25000
TMM	1st	**12.3231**	**15.1213**	**17.3972**	**19.3316**	**21.0158**

FEM(SAP2000)		12.3379	15.1392	17.4186	19.3568	21.0454
TMM	2nd	**28.3796**	**29.5583**	**30.7180**	**31.8639**	**32.9998**
FEM(SAP2000)		28.5193	29.7002	30.8617	32.0090	33.1457
TMM	3rd	**56.1067**	**56.7276**	**57.3513**	**57.9745**	**58.5981**
FEM(SAP2000)		56.6801	57.3052	57.9309	58.5571	59.1834
TMM	4th	**92.7637**	**93.1687**	**93.5612**	**93.9588**	**94.3555**
FEM(SAP2000)		94.2953	94.6981	95.1001	95.5014	95.9017
TMM	5th	**136.8811**	**137.1361**	**137.3907**	**137.6649**	**137.8986**
FEM(SAP2000)		140.0654	140.3239	140.5819	140.8396	141.0968

Tables 1-2 show that TMM results are in very good aggreement with FEM. According to results, the natural frequencies are increased by augmentation of k_s for both models. Tables 1-2 also reveal that Model-b provides higher natural frequencies in accordance with Model-a because of simpy supported end. Variation of fundamental frequencies of Model-a and Model-b is plotted in Figure 2 for different k_s values.

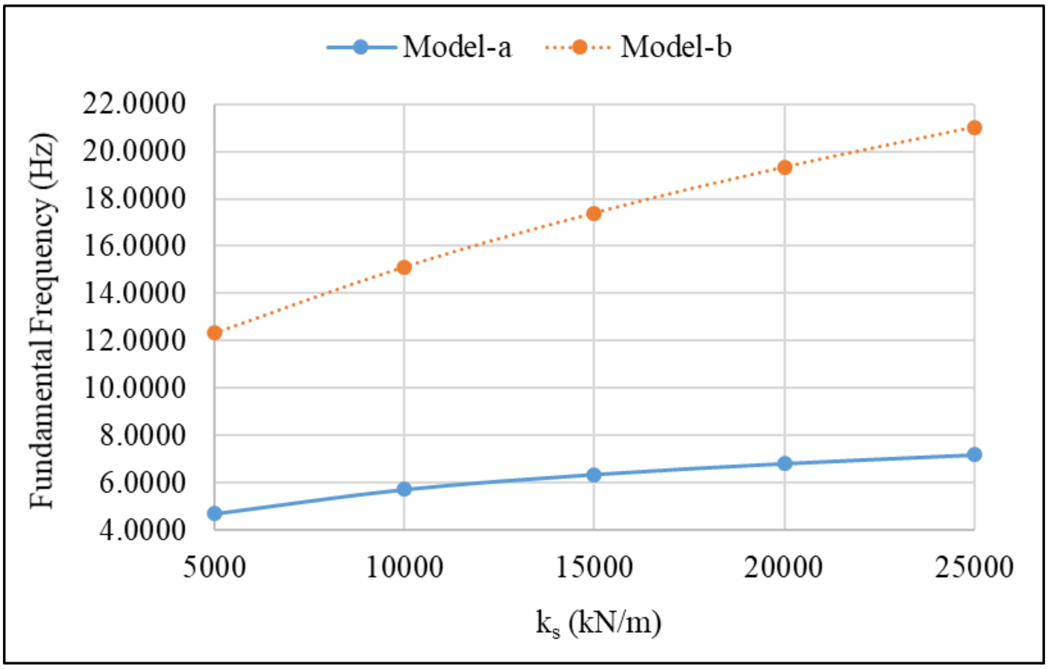

Figure 2. Variation of fundamental frequencies of partially embedded pile models for different k_s values

According to Figure 3, use of simple support instead of free end significantly effects the dynamic characteristics of partially embedded pile. Increasing k_s values decreases the effect of boundary conditions for 2nd mode and does not effect the 3rd, 4th and 5th mode significantly. In discordance with higher modes, increment of k_s increases the effect of boundary condition on fundamental frequency significantly.

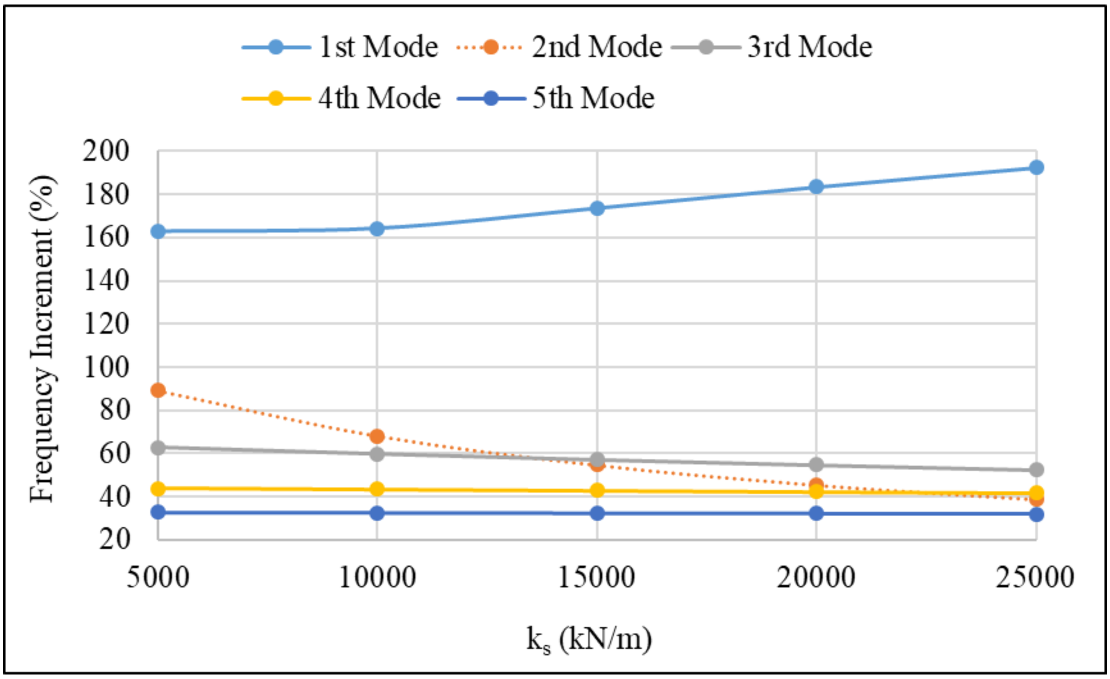

Figure 3. Increment of frequencies of partially embedded piles by using simply supported end instead of free end

CONCLUSION

This study reveals the effects of Winkler spring stiffness on natural frequencies of partially embedded Timoshenko element piles using TMM and FEM comparatively. Different support conditions are used to reflect the influences on free vibration characteristics of partially embedded piles. The results show that TMM which provides exact results can be used effectively for vibrations of partially embedded Timoshenko beam-columns. The computer programs that prepared for analyses are working fast.

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