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On classes of C3 and D3 modules

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Abstract

This paper aims to study the notions of $\mathcal{A}$-C3 and $\mathcal{A}$-D3 modules for some class $\mathcal{A}$ of right modules. Several characterizations of these modules are provided and used to describe some well-known classes of rings and modules. For example, a regular right R-module F is a V-module if and only if every F-cyclic module is an $\mathcal{A}$-C3 module, where $\mathcal{A}$ is the class of all simple right R-modules. Moreover, let R be a right artinian ring and $\mathcal{A}$, a class of right R-modules with a local ring of endomorphisms, containing all simple right R-modules and closed under isomorphisms. If all right R-modules are $\mathcal{A}$-injective, then R is a serial artinian ring with $J^2(R) = 0$ if and only if every $\mathcal{A}$-C3 right R-module is quasi-injective, if and only if every $\mathcal{A}$-C3 right R-module is C3.

Keywords: $\mathcal{A}$-C3 module, $\mathcal{A}$-D3 module, V-module.

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1. Introduction and notation.

The study of modules with summand intersection property was motivated by the following result of Kaplansky: every free module over a commutative principal ideal ring has the summand intersection property (see [14, Exercise 51(b)]). A module M is said to have the *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M. This definition is introduced by Wilson [18]. Dually, Garcia [10] considered the summand sum property. A module M is said to have the *summand sum property* if the sum of any two direct summands is a direct summand of M. These properties have been studied by several authors (see [1, 3, 11, 12, 17],...). Moreover, the classes of C3-modules and D3-modules have recently studied by Yousif et al. in [4, 20]. Some characterizations of semisimple rings and regular rings and other classes of rings are studied via C3-modules and D3-modules. On the other hand, several authors investigated some properties of generalizations of C3-modules and D3-modules in [6, 13]; namely, simple-direct-injective modules and simple-direct-projective modules. A right R-module M is called a $C3$-module if, whenever A and B are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, then $A \oplus B \subset_d M$. M is called *simple-direct-injective* in [6] if the submodules A and B in the above definition are simple. Dually, M is called a $D3$-module if, whenever M_1 and M_2 are direct summands of M and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. M is called *simple-direct-projective* in [13] if the submodules M_1 and M_2 in the above definition are maximal.

In Sect. 2, we study some properties of $\mathcal{A}$-C3 modules and $\mathcal{A}$-D3 modules. Let $\mathcal{A}$ be a class of right modules over a ring R and closed under isomorphisms. We call that a right R-module M is an $\mathcal{A}$-C3 module if, whenever $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, then $A \oplus B \subset_d M$. Dually, M is an $\mathcal{A}$-D3 module if, whenever M_1 and M_2 are direct summands of M with $M/M_1, M/M_2 \in \mathcal{A}$ and $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M. It is shown that if each factor module of M is $\mathcal{A}$-injective, then M is an $\mathcal{A}$-D3 module if and only if M satisfies D2 for the class $\mathcal{A}$, if and only if M have the summand intersection property for the class $\mathcal{A}$ in Proposition 2.7. On the other hand, if every submodule of M is $\mathcal{A}$-projective, then M is an $\mathcal{A}$-C3 module if and only if M satisfies C2 for the class $\mathcal{A}$, if and only if M have the summand sum property for the class $\mathcal{A}$ in Proposition 2.14. These results are applied to the class $\mathcal{A}$ of all simple right R-modules, and to the class $\mathcal{A}$ of all semisimple right R-modules. In the case when $\mathcal{A}$ is the class of all simple right R-modules, we obtained the known properties of the simple-direct-injective modules and simple-direct-projective modules [6, 13].

In Sect. 3, we provide some characterizations of serial artinian rings and semisimple artinian rings. The Theorem 3.2 and Theorem 3.3 are indicated that let R be a right artinian ring and $\mathcal{A}$, a class of right R-modules with a local ring of endomorphisms, containing all simple right R-modules and closed under isomorphisms:

(1) If all right R-modules are $\mathcal{A}$-injective, the following conditions are equivalent for a ring R:
   (i) R is a serial artinian ring with $J^2(R) = 0$.
   (ii) Every $\mathcal{A}$-C3 right R-module is quasi-injective.
   (iii) Every $\mathcal{A}$-C3 right R-module is $C3$.

(2) If all right R-modules are $\mathcal{A}$-projective, then the following conditions are equivalent for a ring R:
   (i) R is a serial artinian ring with $J^2(R) = 0$.
   (ii) Every $\mathcal{A}$-D3 right R-module is quasi-projective.
   (iii) Every $\mathcal{A}$-D3 right R-module is $D3$.

Moreover, we give an equivalent condition for a regular V-module. It is shown that a regular right R-module F is a V-module if and only if every F-cyclic module is simple-direct-injective in Theorem 3.9. It is an extension the result of rings to modules.

Throughout this paper R denotes an associative ring with identity, and modules will be unitary right R-modules. The Jacobson radical ideal in R is denoted by $J(R)$. The notations $N \leq M$, $N \leq_e M$, $N \trianglelefteq M$, or $N \subset_d M$ mean that N is a submodule, an essential submodule, a fully invariant submodule, and a direct summand of M, respectively. Let M and N be right R-modules. M is called N-injective if for any right R-module K and any monomorphism $f : K \to N$, the induced homomorphism $\mathrm{Hom}(N, M) \to \mathrm{Hom}(K, M)$ by f is an epimorphism. M is called N-projective if for any right R-module K and any epimorphism $f : N \to K$, the induced homomorphism $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, K)$ by f is an epimorphism. Let $\mathcal{A}$ be a class of right modules over the ring R. M is called *$\mathcal{A}$-injective* (*$\mathcal{A}$-projective*) if M is N-injective (resp., N-projective) for all $N \in \mathcal{A}$. We refer to [5], [7], [16], and [19] for all the undefined notions in this paper.

2. On $\mathcal{A}$-C3 modules and $\mathcal{A}$-D3 modules

In this section, we give some basic properties of $\mathcal{A}$-C3 modules and $\mathcal{A}$-D3 modules. They will be used for the next section. We first have the following remark.

2.1. Remark. Let M be a right R-module and $\mathcal{A}$, a class of right R-modules.

(1) If M is a C3 (D3) module, then M is an $\mathcal{A}$-C3 (resp., $\mathcal{A}$-D3) module.
(2) If $\mathcal{A} = \mathrm{Mod} - R$, then $\mathcal{A}$-C3 modules ($\mathcal{A}$-D3 modules) modules are precisely the C3 modules (resp., D3) modules.
(3) If $\mathcal{A}$ is the class of simple right R-modules, then $\mathcal{A}$-C3 modules ($\mathcal{A}$-D3 modules) modules are precisely the simple-direct-injective (resp., simple-direct-projective) modules that studied in [6, 13].
(4) If $\mathcal{A}$ is the class of injective right R-modules, then M is always an $\mathcal{A}$-C3 module.
(5) If $\mathcal{A}$ is the class of projective right R-modules, then M is always an $\mathcal{A}$-D3 module.

2.2. Lemma. Let $\mathcal{A}$ be a class of right R-modules and closed under isomorphisms. Then every direct summand of an $\mathcal{A}$-C3 module ($\mathcal{A}$-D3 module) is also an $\mathcal{A}$-C3 module (resp., $\mathcal{A}$-D3 module).

Proof. The proof is straightforward. □

2.3. Proposition. Let $\mathcal{A}$ be a class of right R-modules and closed under direct summands. Then the following conditions are equivalent for a module M:

(1) M is an $\mathcal{A}$-C3 module.
(2) If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B = 0$, there exist submodules A_1 and B_1 of M such that $M = A \oplus B_1 = A_1 \oplus B$ with $A \leq A_1$ and $B \leq B_1$.
(3) If $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are submodules of M with $A \subset_d M$, $B \subset_d M$ and $A \cap B \subset_d M$, then $A + B \subset_d M$.

Proof. It is similar to the proof of Proposition 2.2 in [4]. □

Dually Proposition 2.4, we have the following proposition.

2.4. Proposition. Let $\mathcal{A}$ be a class of right R-modules and closed under isomorphisms. Then the following conditions are equivalent for a module M:

(1) M is an $\mathcal{A}$-D3 module.
(2) If $M/A, M/B \in \mathcal{A}$ with $A \subset_d M$, $B \subset_d M$ and $M = A + B$, then $M = A \oplus B_1 = A_1 \oplus B$ with $A_1 \leq A$ and $B_1 \leq B$.

(3) If $M/A, M/B \in \mathcal{A}$ with $A \subset_d M$, $B \subset_d M$ and $A+B \subset_d M$, then $A \cap B \subset_d M$.

Let $f : A \to B$ be a homomorphism. We denote by $\langle f \rangle$ the submodule of $A \oplus B$ as follows:

$$\langle f \rangle = \{a + f(a) \mid a \in A\}.$$

The following result is proved in Lemma 2.6 of [15].

2.5. Lemma. Let $M = X \oplus Y$ and $f : A \to Y$, a homomorphism with $A \leq X$. Then the following conditions hold

(1) $A \oplus Y = \langle f \rangle \oplus Y$.
(2) $\mathrm{Ker}(f) = X \cap \langle f \rangle$.

2.6. Proposition. Let M be an $\mathcal{A}$-D3 module with $\mathcal{A}$ a class of right R-modules and closed under isomorphisms and direct summands. If $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ is a homomorphism with $\mathrm{Im}(f) \subset_d M_2$ and $\mathrm{Im}(f) \in \mathcal{A}$, then $\mathrm{Ker}(f)$ is a direct summand of M_1.

Proof. Assume that $M = M_1 \oplus M_2$ and $f : M_1 \to M_2$ is a homomorphism with $\mathrm{Im}(f) \subset_d M_2$ and $\mathrm{Im}(f) \in \mathcal{A}$. Call $M' := M_1 \oplus \mathrm{Im}(f)$. Then M' is a direct summand of M and so it is an $\mathcal{A}$-D3 module. It follows that $M' = M_1 \oplus \mathrm{Im}(f) = \langle f \rangle \oplus \mathrm{Im}(f)$ by Lemma 2.5. It is easily to check $M'/M_1, M'/\langle f \rangle \in \mathcal{A}$ and $M' = M_1 + \langle f \rangle$. As M' is an $\mathcal{A}$-D3 module and again by Lemma 2.5, $\langle f \rangle \cap M_1 = \mathrm{Ker}(f)$ is a direct summand of M'. Thus $\mathrm{Ker}(f)$ is a direct summand of M_1. □

2.7. Proposition. Let M be a right R-module and $\mathcal{A}$, a class of right R-modules and closed under isomorphisms and direct summands. If each factor module of M is $\mathcal{A}$-injective, then the following conditions are equivalent:

(1) If M_1 and M_2 are direct summands of M with $M/M_1, M/M_2 \in \mathcal{A}$, then $M_1 \cap M_2$ is a direct summand of M.
(2) M is an $\mathcal{A}$-D3 module.
(3) If $N \leq M$ such that $M/N \in \mathcal{A}$ is isomorphic to a direct summand of M, then N is a direct summand of M.
(4) For any decomposition $M = M_1 \oplus M_2$ with $M_2 \in \mathcal{A}$, every homomorphism $f : M_1 \to M_2$ has the kernel a direct summand of M_1.
(5) Whenever $X_1, \ldots, X_n$ are direct summands of M and $M/X_1, \ldots, M/X_n \in \mathcal{A}$, then $\cap_{i=1}^{n} X_i$ is a direct summand of M.

Proof. (2) $\Rightarrow$ (1). Let M_1, M_2 be direct summands of M with $M/M_1, M/M_2 \in \mathcal{A}$. Then $M = M_1 \oplus M_1'$. Without loss of generality we can assume that $M_2 \nsubseteq M_1, M_2 \nsubseteq M_1'$. From our assumption, $\pi(M_2)$ is a direct summand of M_1'. Then we can write $M_1' = \pi(M_2) \oplus M_1''$ for some $M_1'' \leq M_1'$. Since the class $\mathcal{A}$ is closed under direct summands, $M_1'' \in \mathcal{A}$. It is easy to see that $M_1 + M_1''$ is a direct summand of M. We have $M/(M_1 + M_1'') \in \mathcal{A}$ and $M_1 + M_1'' + M_2 = M$. It follows that $M_1 \cap M_2 = (M_1 + M_1'') \cap M_2$ is a direct summand of M.

(3) $\Rightarrow$ (2). It is obvious.

(1) $\Rightarrow$ (4). Assume that $M = M_1 \oplus M_2$ with $M_2 \in \mathcal{A}$ and a homomorphism $f : M_1 \to M_2$. It follows that $M = M_1 \oplus M_2 = \langle f \rangle \oplus M_2$ by Lemma 2.5. Note that $M/M_1, M/\langle f \rangle \in \mathcal{A}$. By (1) and Lemma 2.5, $\langle f \rangle \cap M_1 = \mathrm{Ker}(f)$ is a direct summand of M. Thus $\mathrm{Ker}(f)$ is a direct summand of M_1.

(4) $\Rightarrow$ (3). Let M_1, M_2 be submodules of M such that $M = M_1 \oplus A$, $M/M_2 \cong A$ and $A \in \mathcal{A}$. Call $\pi_1 : M \to M_1$ and $\pi_2 : M \to A$ the canonical projections. By the hypothesis, $\pi_2(M_2)$ is a direct summand of A and hence $A = \pi_2(M_2) \oplus B$ for some submodule B of A. Call $p : M \to M/M_2$ the canonical projection and isomorphism $\phi : M/M_2 \to A$. Take

the homomorphism $f = \phi \circ (p|_{M_1}) : M_1 \to A$. It follows that $\mathrm{Ker}(f) = M_1 \cap M_2$. By (4), $\mathrm{Ker}(f) = M_1 \cap M_2$ is a direct summand of M_1. Take N_1 a submodule of M_1 with $M_1 = N_1 \oplus (M_1 \cap M_2)$. Note that $M_1 + M_2 = M_1 \oplus \pi_2(M_2)$ and $N_1 \cap M_2 = 0$. This gives that

$$\begin{aligned} M &= M_1 \oplus \pi_2(M_2) \oplus B \\ &= (M_1 + M_2) \oplus B \\ &= [N_1 \oplus (M_1 \cap M_2) + M_2] \oplus B = (N_1 + M_2) \oplus B \\ &= (N_1 \oplus M_2) \oplus B. \end{aligned}$$

(1) $\Rightarrow$ (5). We prove this by induction on n. When $n = 2$, the assertion is true from (1). Suppose that the assertion is true for $n = k$. Let $X_1, X_2, \ldots, X_{k+1}$ be direct summands of M and $M/X_1, M/X_2, \ldots, M/X_{k+1} \in \mathcal{A}$. We can write $M = \cap_{i=1}^{k} X_i \oplus N$ for some submodule N of M. Without loss of generality we can assume that $\cap_{i=1}^{k} X_i \nsubseteq X_{k+1}$. Let $f : M \to M/X_{k+1}$ be the natural projection. Then $(\cap_{i=1}^{k} X_i)/[(\cap_{i=1}^{k} X_i) \cap X_{k+1}]$ is $\mathcal{A}$-injective, and therefore, it is isomorphic to a direct summand of $M/X_{k+1} \in \mathcal{A}$. This gives that $\cap_{i=1}^{k} X_i / \cap_{i=1}^{k+1} X_i$ is isomorphic to a direct summand of M and

$$M/(\cap_{i=1}^{k+1} X_i \oplus N) = (\cap_{i=1}^{k} X_i \oplus N)/(\cap_{i=1}^{k+1} X_i \oplus N) \in \mathcal{A}.$$

Since the equivalence of (1) and (3), $(\bigcap_{i=1}^{k+1} X_i) \oplus N$ is a direct summand of M. Thus $\bigcap_{i=1}^{k+1} X_i$ is a direct summand of M. □

A right R-module M is called a *D2-module* if, for every submodule A of M with M/A isomorphic to a direct summand of M, then A is a direct summand of M. Assume that M is an injective right R-module over a right hereditary ring R. Then every factor module of M is injective. From Proposition 2.7, we have the following corollary.

2.8. Corollary. Let M be an injective right R-module over a right hereditary ring R. The following conditions are equivalent:

(1) M is a D3-module.
(2) M is a D2-module.
(3) M has the summand intersection property.

2.9. Corollary. The following conditions are equivalent for a module M:

(1) If M/A is a semisimple module and B, a submodule of M with $M/A \cong B \subset_d M$, then $A \subset_d M$.
(2) If A and B are any two direct summands of M such that M/A and M/B are semisimple modules, then $A \cap B \subset_d M$.
(3) If A and B are any two direct summands of M such that $M/A, M/B$ are semisimple modules and $A + B = M$, then $A \cap B$ is a direct summand of M.
(4) Whenever $X_1, X_2, \ldots, X_n$ are direct summands of M and $M/X_1, M/X_2, \ldots, M/X_n$ are semisimple modules, then $\cap_{i=1}^{n} X_i$ is a direct summand of M.

2.10. Corollary. Let P be a quasi-projective module. If $X_1, \ldots, X_n$ are direct summands of P and $P/X_1, \ldots, P/X_n$ are semisimple modules, then $\cap_{i=1}^{n} X_i$ is a direct summand of P.

2.11. Corollary. The following conditions are equivalent for a module M:

(1) For any maximal submodule A of M and any submodule B of M such that $M/A \cong B \subset_d M$, $A \subset_d M$.
(2) For any two maximal direct summands A, B of M, $A \cap B \subset_d M$.
(3) If M/A is a finitely generated semisimple module with $M/A \cong B \subset_d M$, then $A \subset_d M$.

(4) Whenever $X_1, X_2, \ldots, X_n$ are maximal direct summands of M, then $\cap_{i=1}^n X_i$ is a direct summand of M.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). Follow from Proposition 2.7.

(3) $\Rightarrow$ (1). Clearly.

(1) $\Rightarrow$ (3). Assume that M/A is a finitely generated semisimple module and isomorphic to a direct summand of M. Write $M/A = M_1/A \oplus \cdots \oplus M_n/A$ with simple submodules M_i/A of M/A. Then $M_i \cap (\sum_{j \neq i} M_j) = A$ for all $i = 1, 2 \ldots, n$. For any subset $\{i_1, i_2, \ldots, i_{n-1}\}$ of the set $I := \{1, 2, \ldots, n\}$, it is easily to see that

$$M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}) \simeq M_k/A$$

for some $k \in I \setminus \{i_1, i_2, \ldots, i_{n-1}\}$. It follows that $M/(M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}})$ is isomorphic to a simple direct summand of M. By (1), $M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}$ is a maximal direct summand of M. On the other hand, we can check that

$$A = \bigcap_{\{i_1, i_2, \ldots, i_{n-1}\} \subset I} (M_{i_1} + M_{i_2} + \cdots + M_{i_{n-1}}).$$

So, by (4), A is a direct summand of M. □

2.12. Proposition. Let M be an $\mathcal{A}$-C3 module with $\mathcal{A}$ a class of right R-modules and closed under isomorphisms and direct summands. If $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is a homomorphism with $\mathrm{Ker}(f) \in \mathcal{A}$ and $\mathrm{Ker}(f) \subset_d A_1$, then $\mathrm{Im}(f)$ is a direct summand of A_2.

Proof. Let $f : A_1 \to A_2$ be an R-homomorphism with $\mathrm{Ker}(f) \in \mathcal{A}$. By the hypothesis, there exists a decomposition $A_1 = \mathrm{Ker}(f) \oplus B$ for some submodule B of A_1. Then $B \oplus A_2$ is a direct summand of M. Note that every direct summand of an $\mathcal{A}$-C3 module is also an $\mathcal{A}$-C3 module. Hence $B \oplus A_2$ is an $\mathcal{A}$-C3 module. Let $g = f|_B : B \to A_2$. Then g is a monomorphism and $\mathrm{Im}(g) = \mathrm{Im}(f)$. It is easy to see that $B \oplus A_2 = \langle g \rangle \oplus A_2$, $\langle g \rangle \cap B = 0$ and $\langle g \rangle \simeq B$. Note that $B, \langle g \rangle \in \mathcal{A}$. As $B \oplus A_2$ is an $\mathcal{A}$-C3 module, $B \oplus \langle g \rangle$ is a direct summand of $B \oplus A_2$. Thus $B \oplus \langle g \rangle = B \oplus \mathrm{Im}(g)$, which implies that $\mathrm{Im}(g)$ or $\mathrm{Im}(f)$ is a direct summand of A_2. □

2.13. Proposition. Let M be a right R-module and $\mathcal{A}$, a class of right R-modules and closed under isomorphisms and direct summands. If every submodule of M is $\mathcal{A}$-projective, the following conditions are equivalent:

(1) For any two direct summands M_1, M_2 of M such that $M_1, M_2 \in \mathcal{A}$, $M_1 + M_2$ is a direct summand of M.
(2) M is an $\mathcal{A}$-C3 module.
(3) For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \to A_2$ has the image a direct summand of A_2.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $f : A_1 \to A_2$ be an R-homomorphism with $A_1 \in \mathcal{A}$. By the hypothesis, $\mathrm{Ker}(f)$ is a direct summand of A_1. The rest of proof is followed from Proposition 2.12.

(3) $\Rightarrow$ (1) Let N and K be direct summands of M such that $N, K \in \mathcal{A}$. Write $M = N \oplus N'$ and $M = K \oplus K'$ for some submodules N', K' of M. Consider the canonical projections $\pi_K : M \to K$ and $\pi_{N'} : M \to N'$. Let $A = \pi_{N'}(\pi_K(N))$. Then $A = (N + K) \cap (N + K') \cap N'$ is a direct summand of M by (3). Write $M = A \oplus L$ for some submodule L of M. Clearly,

$$(N + K) \cap [(N + K') \cap (N' \cap L)] = 0.$$

Hence, $N' = A \oplus (N' \cap L)$ and $M = (N \oplus A) \oplus (N' \cap L)$. Since $A \leq N + K$ and $A \leq N + K'$, we get

$$N + K = (N \oplus A) \cap [(N + K) \cap (N' \cap L)]$$

and

$$N + K' = (N \oplus A) \cap [(N + K') \cap (N' \cap L)].$$

They imply

$$\begin{aligned} M &= N + K' + K \\ &= (N \oplus A) + [(N + K) \cap (N' \cap L)] + [(N + K') \cap (N' \cap L)] \\ &\leq (N + K) + [(N + K') \cap (N' \cap L)]. \end{aligned}$$

Thus $M = (N + K) \oplus [(N + K') \cap (N' \cap L)$. □

2.14. Proposition. Let M be a right R-module and $\mathcal{A}$, a class of artinian right R-modules and closed under isomorphisms and direct summands. If every submodule of M is $\mathcal{A}$-projective, then the following conditions are equivalent:

(1) M is an $\mathcal{A}$-C3 module.
(2) If a submodule $N \in \mathcal{A}$ of M is isomorphic to a direct summand of M, then N is a direct summand of M.
(3) Whenever $X_1, X_2, \ldots, X_n$ are direct summands of M and $X_1, X_2, \ldots, X_n \in \mathcal{A}$, then $\sum_{i=1}^{n} X_i$ is a direct summand of M.

Proof. (1) ⇒ (2). Let M_1 be a submodule of M and isomorphic to a direct summand M_2 of M and $M_1 \in \mathcal{A}$. Then $M = M_2 \oplus M_2'$. Suppose that $M_1 \subset M_2$. Since M_2 is artinian and $M_1 \cong M_2$, then $M_1 = M_2$. If $M_1 \nsubseteq M_2$ and denote $\pi : M_2 \oplus M_2' \to M_2'$ the canonical projection, then by the hypothesis we have $\mathrm{Ker}(\pi_{|M_1})$ is a direct summand of M_1. It follows that $M_1 = (M_1 \cap M_2) \oplus N_1$. Since $N_1 \cong \pi(M_1)$ and $M_1 \cong M_2$, then there is an isomorphism $\phi : N' \to \pi(M_1)$, where N' is a direct summand of M_1. Since $\langle \phi \rangle \in \mathcal{A}$ and $\langle \phi \rangle \cap M_2 = 0$, $M_2 + \langle \phi \rangle = M_2 \oplus N_1$ is a direct summand of M. Therefore, N_1 is a non-zero direct summand of M. It is clear that $M_1 \cap M_2 \in \mathcal{A}$ and $M_1 \cap M_2$ is isomorphic to a direct summand of M. If $M_1 \cap M_2$ is not a direct summand of M, by using an argument that are similar to the argument presented above, we can show that $M_1 \cap M_2 = N_2 \oplus N_2'$, where $N_2 \in \mathcal{A}$ is a non-zero direct summand of M and $N_2' \in \mathcal{A}$ is a submodule of M isomorphic to a direct summand of M. Since each module of the class $\mathcal{A}$ is artinian, by conducting similar constructions continue for some k, we obtain a decomposition $M_1 = N_1 \oplus \ldots \oplus N_k$, where N_i is a direct summand of M and $N_i \in \mathcal{A}$ for each i. Since M is an $\mathcal{A}$-C3 module, $N_1 \oplus N_2 \oplus \ldots \oplus N_k$ is a direct summand of M.

(2) ⇒ (1). It is obvious.

(1) ⇒ (3). We prove this by induction on n. When $n = 2$, the assertion follows from Proposition 2.13. Suppose that the assertion is true for $n = k$. Let $X_1, X_2, \ldots, X_{k+1}$ be direct summands of M and $X_1, X_2, \ldots, X_{k+1} \in \mathcal{A}$. Then there exists a submodule N of M such that $M = (\sum_{i=1}^{k} X_i) \oplus N$. Let $\pi : (\sum_{i=1}^{k} X_i) \oplus N \to N$ be the canonical projection. As $\pi(X_{k+1})$ is $\mathcal{A}$-projective, then $X_{k+1} = ((\sum_{i=1}^{k} X_i) \cap X_{k+1}) \oplus S$ for some submodule S of M. Since the equivalence of (1) and (2), $\pi(X_{k+1})$ is a direct summand of M and, therefore, $N = \pi(X_{k+1}) \oplus T$ with T a submodule M. It follows that $\sum_{i=1}^{k+1} X_i = (\sum_{i=1}^{k} X_i) \oplus \pi(X_{k+1})$ and $M = (\sum_{i=1}^{k} X_i) \oplus \pi(X_{k+1}) \oplus T$. Thus, $\sum_{i=1}^{k+1} X_i$ is a direct summand of M. □

2.15. Remark. Let F be a nonzero free module over $\mathbb{Z}$ and $\mathcal{A}$, a class of all free $\mathbb{Z}$-modules. It is well known that F is a quasi-continuous module and not a continuous module. Thus, F is an $\mathcal{A}$-C3 module and satisfies the following property: there exists a submodule $N \in \mathcal{A}$ of F such that N is isomorphic to a direct summand of F and not a direct summand of F.

A right R-module M is said to be a *C2-module* if, whenever A and B are submodules of M with $A \cong B$ and $B \subset_d M$, then $A \subset_d M$. If M is a hereditary module, then all submodules of M is projective. Then we get the following result.

2.16. Corollary. Let M be a hereditary artinian module. The following conditions are equivalent:

(1) M is a C3-module.
(2) M is a C2-module.
(3) M has the summand sum property.

2.17. Proposition. Let M be a right R-module and $\mathcal{A}$, a class of right R-modules and closed under isomorphisms and direct summands. If every factor module of M is $\mathcal{A}$-projective, then the following conditions are equivalent:

(1) For any two direct summands M_1, M_2 of M such that $M_1, M_2 \in \mathcal{A}$, $M_1 + M_2$ is a direct summand of M.
(2) M is an $\mathcal{A}$-C3 module.
(3) For any decomposition $M = A_1 \oplus A_2$ with $A_1 \in \mathcal{A}$, then every homomorphism $f : A_1 \to A_2$ has the image a direct summand of A_2.
(4) Every submodule $N \in \mathcal{A}$ of M that is isomorphic to a direct summand of M is itself a direct summand.
(5) Whenever $X_1, X_2, \dots, X_n$ are direct summands of M and $X_1, X_2, \dots, X_n \in \mathcal{A}$, then $\sum_{i=1}^{n} X_i$ is a direct summand of M.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1) are proved similarly to the argument proof of Proposition 2.13.

(4) $\Rightarrow$ (2) is obvious.

(3) $\Rightarrow$ (4). Let $\sigma : A \to B$ be an isomorphism with $A \in \mathcal{A}$ a direct summand of M and $B \leq M$. We need to show that B is a direct summand of M. Write $M = A \oplus T$ for some submodule T of M. We have $A/A \cap B$ is an image of M and obtain that $A \cap B$ is a direct summand of A. Take $A = (A \cap B) \oplus C$ for some submodule C of A. Now $M = (A \cap B) \oplus (C \oplus T)$. Clearly, $A \cap [(C \oplus T) \cap B] = 0$ and $B = (A \cap B) \oplus [(C \oplus T) \cap B]$. Let $H := \sigma^{-1}((C \oplus T) \cap B)$. Then H is a submodule of A, $H \cap [(C \oplus T) \cap B] = 0$ and $A = H \oplus H'$ for some submodule H' of H. Note that $M = H \oplus (H' \oplus T)$. Consider the projection $\pi : M \to H' \oplus T$. Then

$$H \oplus [(C \oplus T) \cap B] = H \oplus \pi((C \oplus T) \cap B).$$

By (3), the image of the homomorphism $\pi|_{(C \oplus T) \cap B} \circ \sigma|_H : H \to H' \oplus T$ is a direct summand of $H' \oplus T$ since H is contained in $\mathcal{A}$. Write $H' \oplus T = \pi|_{(C \oplus T) \cap B} \sigma(H) \oplus K$ for some submodule K of $H' \oplus T$. Then $H' \oplus T = \pi((C \oplus T) \cap B) \oplus K$. It follows that

$$M = H \oplus \pi((C \oplus T) \cap B) \oplus K = H \oplus [(C \oplus T) \cap B] \oplus K.$$

By the modular law, $C \oplus T = [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)]$. Thus

$$\begin{aligned} M &= (A \cap B) \oplus [(C \oplus T) \cap B] \oplus [(H \oplus K) \cap (C \oplus T)] \\ &= B \oplus [(H \oplus K) \cap (C \oplus T)]. \end{aligned}$$

The implication (1) $\Rightarrow$ (5) is proved similarly to the argument proof of Proposition 2.14. □

Call $\mathcal{A}$ the class of all semisimple right R-modules. Then by Proposition 2.17, we have the following result:

2.18. Corollary. The following conditions are equivalent for a module M:

(1) If A, B are semisimple submodules of M such that $A \cong B \subset_d M$, then $A \subset_d M$.
(2) If A, B are semisimple summands of M, then $A + B \subset_d M$.
(3) If A, B are semisimple summands of M with $A \cap B = 0$, then $A + B \subset_d M$.
(4) Whenever $X_1, \ldots, X_n$ are semisimple direct summands of M and $X_1, \ldots, X_n \in \mathcal{A}$, then $\sum_{i=1}^{n} X_i$ is a direct summand of M.

2.19. Corollary. Let Q be a quasi-injective module. If $X_1, \ldots, X_n$ are semisimple direct summands of Q, then $\sum_{i=1}^{n} X_i$ is a direct summand of Q.

2.20. Corollary ([6, Proposition 2.1])**.** The following conditions are equivalent for a module M:

(1) For any simple submodules A, B of M with $A \cong B \subset_d M$, $A \subset_d M$.
(2) For any simple direct summands A, B of M with $A \cap B = 0$, $A \oplus B \subset_d M$.
(3) For any finitely generated semisimple submodules A, B of M with $A \cong B \subset_d M$, $A \subset_d M$.
(4) For any finitely generated semisimple direct summands A, B of M with $A \cap B = 0$, $A \oplus B \subset_d M$.

3. Characterizations of rings

In this section, we will characterize some classes of rings and modules via $\mathcal{A}$-C3 modules and $\mathcal{A}$-D3 modules. We first get the following lemma.

3.1. Lemma. Let $\mathcal{A}$ be a class of right R-modules with a local ring of endomorphisms and closed under isomorphisms. Assume that K and M are indecomposable right R-modules and not contained in $\mathcal{A}$. Then

(1) $N = M \oplus P$ is an $\mathcal{A}$-D3 module for all projective modules P.
(2) $N = M \oplus E$ is an $\mathcal{A}$-C3 module for all injective modules E.
(3) $N = M \oplus K$ is an $\mathcal{A}$-D3 module and an $\mathcal{A}$-C3 module.

Proof. (1) Let $N/A \cong S \subset_d N$ with $S \in \mathcal{A}$. By [5, Lemma 26.4], there exist a direct summand M_1 of M and a direct summand P_1 of P such that $N = S \oplus M_1 \oplus P_1$. Write $P = P_1 \oplus P_2$ for some submodule P_2 of P. Since M is an indecomposable module, we have either $M_1 = 0$ or $M = M_1$. If $M_1 = 0$, then $N = S \oplus P_1 = (M \oplus P_2) \oplus P_1$ and it follows that $M \oplus P_2 \cong S$, and hence $M \in \mathcal{A}$ contradicting. So $M_1 = M$. Then $N = S \oplus (M \oplus P_1) = (M \oplus P_1) \oplus P_2$. This gives $S \cong P_2$, and consequently $N/A \cong S$ is projective. Hence, A is a direct summand of N and (1) holds.

(2) Suppose that A is a submodule of N such that $A \simeq S$ with S a submodule of N and $S \in \mathcal{A}$. As in (1), we see that $N = S \oplus M_1 \oplus E_1$ with $M = M_1 \oplus M_2$ and $E = E_1 \oplus E_2$. Also, as in (1), $M_1 = M$. Therefore,

$$N = S \oplus M \oplus E_1 = M \oplus E = (M \oplus E_1) \oplus E_2.$$

It follows that $S \simeq E_2$ is an injective module. Thus A is a direct summand of N.

(3) We show that N has no a nonzero direct summand S with $S \in \mathcal{A}$. Assume on the contrary that there exists a non-zero direct summand $S \subset_d N$ with $S \in \mathcal{A}$. As, in (1), $N = S \oplus M_1 \oplus K_1$ with $M = M_1 \oplus M_2$ and $K = K_1 \oplus K_2$. Also, as in (1), $M_1 = M$. Therefore,

$$N = S \oplus M \oplus K_1 = M \oplus K.$$

Since K is indecomposable, $K = K_1$ or $K = K_2$. If $K = K_1$, then $S \oplus M \oplus K = M \oplus K$ and consequently $S = 0$, a contradiction. If $K = K_2$, then $K_1 = 0$ and so $S \oplus M = M \oplus K$. Therefore, $K \cong S$ and hence $K \in \mathcal{A}$, a contradiction. □

Recall that a module is *uniserial* if the lattice of its submodules is totally ordered under inclusion. A ring R is called right *uniserial* if R_R is a uniserial module. A ring R is called *serial* if both modules ${}_RR$ and R_R are direct sums of uniserial modules.

3.2. Theorem. Let R be a right artinian ring and $\mathcal{A}$, a class of right R-modules with a local ring of endomorphisms, containing all right simple right R-modules and closed under isomorphisms. If all right R-modules are $\mathcal{A}$-injective, then the following conditions are equivalent for a ring R:

(1) R is a serial artinian ring with $J^2(R) = 0$.
(2) Every $\mathcal{A}$-C3 module is quasi-injective.
(3) Every $\mathcal{A}$-C3 module is $C3$.

Proof. (1) $\Rightarrow$ (2) Assume that R is an artinian serial ring with $J^2(R) = 0$. Then every right R-module is a direct sum of a semisimple module and an injective module. Furthermore, every injective module is a direct sum of cyclic uniserial modules. Let M be an $\mathcal{A}$-C3 module. We can write $M = (\oplus_{\mathfrak{I}} S_i) \oplus (\oplus_{\mathfrak{J}} E_j)$ where each S_i is simple if $i \in \mathfrak{I}$ and $\oplus_{\mathfrak{J}} E_j$ is injective where each E_j is cyclic uniserial non-simple if $j \in J$. Note that any E_j has length at 2 by [7, 13.3]. We show that M is a quasi-injective module. To show that M is quasi-injective, by [16, Proposition 1.17] it suffices to show that $\oplus_{\mathfrak{I}} S_i$ is $\oplus_{\mathfrak{J}} E_j$-injective. By [16, Theorem 1.7], $\oplus_{\mathfrak{I}} S_i$ is $\oplus_{\mathfrak{J}} E_j$-injective if and only if S_i is $\oplus_{\mathfrak{J}} E_j$-injective for all $i \in \mathfrak{I}$. Furthermore, for any $i \in \mathfrak{I}$, if S_i is E_j-injective for all $j \in \mathfrak{J}$, then S_i is $\oplus_{\mathfrak{J}} E_j$-injective by [16, Proposition 1.5]. So, it suffices to show that S_i is E_j-injective for each $i \in \mathfrak{I}$ and $j \in \mathfrak{J}$. Suppose that E_j has a series $0 \subset X \subset E_j$. Let $f : A \to S_i$ be a homomorphism with $A \leq E_j$. If $A = 0$ or $A = E_j$ then it is obvious that f is extended to a homomorphism from E_j to S_i. Assume that $A = X$. If f is non-zero, then $X \simeq S_i$. As M is an $\mathcal{A}$-C3 module, X is a direct summand of M. It follows that $X = E_j$, a contradiction. Hence S_i is E_j-injective and so M is quasi-injective.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Let M be an indecomposable module. If $M \in \mathcal{A}$, then it is quasi-injective. Now, suppose that $M \notin \mathcal{A}$ and let $\iota : M \to E(M)$ be the inclusion. Then, by Lemma 3.1, $M \oplus E(M)$ is $\mathcal{A}$-C3 and by assumption, $M \oplus E(M)$ is a $C3$-module. It follows that $\mathrm{Im}(\iota)$ is a direct summand of $E(M)$ by [4, Proposition 2.3]. Hence M is injective. Inasmuch as every indecomposable right R-module is quasi-injective, we infer from [9, Theorem 5.3] that R is an artinian serial ring. By [8, Theorem 25.4.2], every right R-module is a direct sum of uniserial modules. Now, by [7, 13.3], we only need to show that each uniserial module, say M, has length at most 2. Suppose that M has a series $0 \subset X \subset Y \subset M$ of length 3. Assume that $Y \in \mathcal{A}$. Then X is Y-injective and hence X is a direct summand of Y, a contradiction. It follows that $Y \notin \mathcal{A}$. By Lemma 3.1, $M \oplus Y$ is an $\mathcal{A}$-C3 module and then, by hypothesis, is a C3-module. Consequently, the natural inclusion, $\eta : Y \longrightarrow M$ splits; i.e. $Y \subset_d M$ and so $Y = M$, a contradiction. Hence, R is an artinian ring with $J^2(R) = 0$. □

3.3. Theorem. Let R be a right artinian ring and $\mathcal{A}$, a class of right R-modules with a local ring of endomorphisms, containing all right simple right R-modules and closed under isomorphisms. If all right R-modules are $\mathcal{A}$-projective, then the following conditions are equivalent for a ring R:

(1) R is a serial artinian ring with $J^2(R) = 0$.
(2) Every $\mathcal{A}$-D3 module is quasi-projective.
(3) Every $\mathcal{A}$-D3 module is $D3$.

Proof. By Lemma 3.1 and [13, Theorem 4.4]. □

3.4. Proposition. Let $\mathcal{A}$ be a class of right R-modules and closed under isomorphisms and direct summands. Then the following conditions are equivalent:

(1) All modules $A \in \mathcal{A}$ are injective.
(2) Every right R-module is $\mathcal{A}$-C3.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Suppose that $A \in \mathcal{A}$. Then by (2), $A \oplus E(A)$ is an $\mathcal{A}$-C3 module. Call $\iota : A \to E(A)$ the inclusion map. By Proposition 2.12, $\mathrm{Im}(\iota) = A$ is a direct summand of $E(A)$. Thus $A = E(A)$ is an injective module. □

3.5. Corollary ([6])**.** The following conditions are equivalent for a ring R:

(1) R is a right V-ring.
(2) Every right R-module is simple-direct-injective.

3.6. Proposition. Let $\mathcal{A}$ be a class of right R-modules and closed under isomorphisms and direct summands. Then the following conditions are equivalent:

(1) All modules $A \in \mathcal{A}$ are projective.
(2) Every right R-module is $\mathcal{A}$-D3.

Proof. (1) $\Rightarrow$ (2). Assume that M is a right R-module. Let M_1, M_2 be submodules of M with $M/M_1, M/M_2 \in \mathcal{A}$ and $M = M_1 + M_2$. It follows that $M/M_1, M/M_2N$ are projective modules and the following isomorphism

$$M/(M_1 \cap M_2) = (M_1 + M_2)/(M_1 \cap M_2) \simeq M/M_1 \times M/M_2.$$

Then $M/(M_1 \cap M_2)$ is a projective module. We deduce that $M_1 \cap M_2$ is a direct summand of M. It shown that M is an $\mathcal{A}$-D3 module.

(2) $\Rightarrow$ (1). Suppose that $A \in \mathcal{A}$. Call $\varphi : R^{(I)} \to A$ an epimorphism. Then $R^{(I)} \oplus A$ is an $\mathcal{A}$-D3 module. By Proposition 2.6, A is isomorphic to a direct summand of $R^{(I)}$. Thus A is a projective module. □

3.7. Corollary ([13])**.** The following conditions are equivalent for a ring R:

(1) R is a semisimple artinian ring.
(2) Every right R-module is simple-direct-projective.

Let M be a right R-module. M is called *regular* if every cyclic submodule of M is a direct summand. A right R-module is called M*-cyclic* if it is isomorphic to a factor module of M.

3.8. Lemma. Let F be a regular module. Assume that $A \neq 0$ is a small finitely generated submodule of the factor module F/F_0 for some submodule F_0 of F. Then there exists a F-cyclic module M and satisfies the property: there is a submodule N of M such that N is isomorphic to a direct summand of M, not a direct summand of M and $N \simeq A$.

Proof. By the hypothesis we have $((x_1R + x_2R + \cdots + x_mR) + F_0)/F_0 = A$ for some $x_1, x_2, \ldots, x_m$ of F. Since F is a regular module, $x_1R + x_2R + \cdots + x_mR = \pi(F)$, where $\pi \in \mathrm{End}(F)$ and $\pi^2 = \pi$. Since A is a small submodule of F/F_0, we have $F/F_0 = ((1 - \pi)F + F_0)/F_0$. It follows that there exist epimorphisms $f_1 : \pi(F) \to A$, $f_2 : (1-\pi)(F) \to F/F_0$. It is easy to check $A \oplus (F/F_0)$ is a F-cyclic module. Call $M = A \oplus (F/F_0)$. Thus, the module $N := 0 \oplus A \simeq A$ is not a direct summand of M and isomorphic to a direct summand $A \oplus 0$ of M. □

A module M is called a *V-module* if every simple module in $\sigma[M]$ is M-injective (see [19]). R is called a right *V-ring* if the right module R_R is a V-module.

3.9. Theorem. The following conditions are equivalent for a regular module F:

(1) F is a V-module.
(2) Every F-cyclic module M is an $\mathcal{A}$-C3 module, where $\mathcal{A}$ is the class of all simple right R-modules (i.e., M is a simple-direct-injective module).

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let $S \in \sigma[F]$ is a simple module and $E_F(S)$ is the injective hull of S in the category $\sigma[F]$. Assume that $E_F(S) \neq S$. As $E_F(S)$ is generated by F, there exists a homomorphism $f : F \to E_F(S)$ such that $f(F) \neq S$. Then S is a small submodule of $f(F)$. Take $\varphi : f(F) \to F/\operatorname{Ker}(f)$ the isomorphism. By Lemma 3.8, there exists a F-cyclic module M and satisfies the property: there is a submodule N of M such that N is isomorphic to a direct summand of M, not a direct summand of M and $N \simeq \varphi(S)$. Note that N is a simple submodule of M. We infer from Proposition 2.17 that M is not an $\mathcal{A}$-C3 module, where $\mathcal{A}$ is the class of all simple right R-modules. This contradicts the condition of (2). □

3.10. Corollary ([6, Theorem 4.4.]). A regular ring R is a right V-ring if and only if every cyclic right R-module is simple-direct-injective.

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References

[1] Abyzov, A. N. and Tuganbaev A. A. *Modules in which sums or intersections of two direct summands are direct summands*, Fundam. Prikl. Mat. **19**, 3-11, 2014.
[2] Abyzov A. N. and Nhan T. H. N. *CS-Rickart Modules*, Lobachevskii Journal of Mathematics, **35**, 317-326, 2014.
[3] Alkan, M. and Harmanci, A. *On Summand Sum and Summand Intersection Property of Modules*, Turkish J. Math, **26**, 131-147, 2002.
[4] Amin, I. Ibrahim, Y. and Yousif, M. F. *C3-modules*, Algebra Colloq. **22**, 655-670, 2015.
[5] Anderson, F. W. and Fuller, K. R. *Rings and Categories of Modules*, Springer-Verlag, New York, 1974.
[6] Camillo, V. Ibrahim, Y. Yousif, M. and Zhou, Y. *Simple-direct-injective modules*, J. Algebra **420**, 39-53, 2014.
[7] Dung, N. V. Huynh, D. V. Smith, P. F. and Wisbauer, R. *Extending modules*, Pitman Research Notes in Math. **313**, Longman, Harlow, New York, 1994.
[8] Faith, C. *Algebra II. Ring Theory*, Springer-Verlag, New York, 1967.
[9] Fuller, K. R. *On indecomposable injectives over artinian rings*, Pacific J. Math **29**, 115-135, 1968.
[10] Garcia, J. L. *Properties of Direct Summands of Modules*, Comm. Algebra, **17**, 73-92, 1989.
[11] Hamdouni, A. Harmanci, A. and Ç. Özcan, A. *Characterization of modules and rings by the summand intersection property and the summand sum property*, JP Jour.Algebra, Number Theory & Appl. **5**, 469-490, 2005.
[12] Hausen, J. *Modules with the Summand Intersection Property*, Comm. Algebra **17**, 135-148, 1989.
[13] Ibrahim, Y. Kossan, M. T. Quynh, T.C. and Yousif, M. *Simple-direct-projective modules*, Comm. Algebra **44**, 5163-5178, 2016
[14] Kaplansky, I. *Infinite Abelian Groups*, Univ. of Michigan Press, Ann Arbor, 1969.
[15] Keskin T*ütüncü*, D. Mohamed, S.H. and Orhan Ertas, N. *Mixed injective modules*, Glasg. Math. J. **52**, 111-120, 2010.
[16] Mohammed, S. H. and Müller, B. J. *Continous and Discrete Modules*, London Math. Soc. LN **147**: Cambridge Univ. Press., 1990.
[17] Quynh, T. C. Kosan, M. T. and Thuyet, L. V. *On (semi)regular morphisms*, Comm. Algebra **41**, 2933-2947, 2013.

[18] Wilson, G. V. *Modules with the Direct Summand Intersection Property*, Comm. Algebra **14**, 21-38, 1986.

[19] Wisbauer, R. *Foundations of Module and Ring Theory*, Gordon and Breach. Reading, 1991.

[20] Yousif, M.F. Amin, I. and Ibrahim, Y.*D3-modules.* Commun. Algebra **42**, 578-592, 2014.