Hacettepe Journal of Mathematics and Statistics
Volume 47 (2) (2018), 365 – 381

The Marshall-Olkin additive Weibull distribution with variable shapes for the hazard rate

Ahmed Z. Afify* , Gauss M. Cordeiro†, Haitham M. Yousof‡, Abdus Saboor§¶, Edwin M.M. Ortega‖

Abstract

We introduce and study the Marshall-Olkin additive Weibull distribution in order to allow a wide variation in the shape of the hazard rate, including increasing, decreasing, bathtub and unimodal shapes. The new distribution generalizes at least eleven lifetime models existing in the literature. Various of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, moments of the residual and reversed residual life functions and order statistics are derived. The parameters of the new distribution are estimated by the maximum likelihood method. We illustrate empirically the superiority of the new model over other distributions by means of a real life data set.

Keywords: Additive Weibull, Goodness of fit, Lifetime data, Maximum likelihood, Moment, Order statistic, Residual life function.

2000 AMS Classification: 60E05, 62E15

Received : 01.05.2015 *Accepted* : 22.05.2016 *Doi* : 10.15672/HJMS.201612618532

*Department of Statistics, Mathematics and Insurance, Benha University, Egypt
†Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Brazil
‡Department of Statistics, Mathematics and Insurance, Benha University, Egypt
§Department of Mathematics, Kohat University of Science & Technology, Kohat, Pakistan 26000, Email: saboorhangu@gmail.com
¶Corresponding Author
‖Departamento de Ciências Exatas, Universidade de São Paulo, 13418-900, Brazil

1. Introduction

In recent years, several lifetime models have been proposed and studied in order to improve the modeling of survival data. The Weibull distribution does not provide a good fit to data sets with bathtub shaped or upside down bathtub shaped (unimodal) failure rates, often encountered in reliability, engineering and biological studies. Other popular lifetime models are the gamma and lognormal distributions but their survival functions have no closed-form expressions.

Extensions of the Weibull distribution arise in different research areas, see, for example, Saboor *et al.* [24] and the references therein. Various extended Weibull models have an upside–down bathtub shaped hazard rate such as the extensions discussed by Bebbington *et al.* [7], Nadarajah and Cordeiro [18] and Saboor *et al.* [24], among others.

The procedure of adding one or two parameters to a family of distributions to obtain more flexibility is a well-known technique in the existing literature. Marshall and Olkin [16] pioneered a simple method of adding a positive shape parameter into a family of distributions and several authors used their method to extend well-known distributions in the last few years. If $\overline{G}(x)$, $g(x)$ and $r(x)$ denote the survival function (sf), probability density function (pdf) and hazard rate function (hrf) of a parent distribution, then the survival function $\overline{F}(x)$ of the the Marshall and Olkin (MO) family is defined by

$$\overline{F}(x;\delta) = \frac{\delta\,\overline{G}(x)}{1-\bar{\delta}\,\overline{G}(x)}, \; x\in\Re, \; \delta>0, \tag{1.1}$$

where $\bar{\delta} = 1-\delta$. Clearly, for $\delta = 1$, we obtain the baseline distribution, i.e., $\overline{F}(x) = \overline{G}(x)$. They called the shape parameter δ "tilt parameter", since the hrf $h(x;\delta)$ of the transformed distribution is shifted below ($\delta \geq 1$) or above ($0<\delta\leq 1$) from the baseline hrf, say $h_G(x)$. In fact, for all $x>0$, $h(x;\delta)\leq h_G(x)$ when $\delta\geq 1$, and $h(x;\delta)\geq h_G(x)$ when $0<\delta\leq 1$.

The main motivation for the MO family is given as follows: let $Z_1, Z_2, \ldots$ be a sequence of IID random variables from $G(x)$ and N be a random variable with probability mass function $\delta(1-\delta^{n-1})$ (for $n=1,2,\ldots$). By defining $T_N = \min\{Z_1,\ldots,Z_N\}$, we have

$$P\big[T_N \leq x\big] = 1-\sum_{n=1}^{\infty} P\big[T_N\geq x \,\big|\, N=n\big]\; P\big[N=n\big] = \frac{G(x)}{G(x)+\delta\;\overline{G}(x)},$$

which is equivalent to (1.1).

The pdf corresponding to (1.1) is given by

$$f(x;\delta) = \frac{\delta\,g(x)}{\big[1-\bar{\delta}\,\overline{G}(x)\big]^2} \tag{1.2}$$

and its hrf reduces to

$$h(x;\delta) = \frac{r(x)}{1-\bar{\delta}\,\overline{G}(x)}. \tag{1.3}$$

From equation (1.3) it follows that $h(x;\delta)/r(x)$ is increasing in x for $\delta\geq 1$ and decreasing in x for $0<\delta\leq 1$.

Xie and Lai [27] defined the four-parameter *additive Weibull* (AW) distribution with cdf given by

$$G(x;\alpha,\beta,\gamma,\theta) = 1-\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right), \qquad x>0, \tag{1.4}$$

where θ and β are positive shape parameters and α and γ are positive scale parameters. Further, $0<\theta<\beta$ or $0<\beta<\theta$.

The pdf corresponding to (1.4) is given by

$$g(x;\alpha,\beta,\gamma,\theta)=\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right). \tag{1.5}$$

A random variable X having the pdf (1.5) is denoted by $X\sim$AW$(\alpha,\beta,\gamma,\theta)$.

In this paper, the AW distribution is embedded in a larger family by adding an extra shape parameter. The model generated by applying the Marshall-Olkin transformation to the AW distribution is called the *Marshall-Olkin additive Weibull* (MOAW) distribution.

The rest of the paper is outlined as follows. In Section 2, we define the new distribution, derive a linear representation for its pdf, provide some sub-models and plots of the densities and hrfs. Some mathematical properties including quantile function (qf), ordinary, central and incomplete moments, moment generating function (mgf), mean deviations, moments of the residual life and reversed residual life are derived in Section 3. The order statistics and their moments are investigated in Section 4. In Section 5, we discuss maximum likelihood estimation of the model parameters. In Section 6, we show empirically the potentiality of the MOAW distribution by means of a real data set. Finally, some concluding remarks are offered in Section 7.

The manipulations for the generating functions and Bell polynomials were carried out with the help of computational package *MATHEMATICA*.

2. The MOAW distribution

In this section, we define the new distribution and present eleven sub-models. By inserting (1.4) and (1.5) in equations (1.1) and (1.2), we obtain the cdf of the MOAW distribution (for $x>0$) with vector of parameters $\upsilon=(\alpha,\beta,\gamma,\theta,\delta)$ given by

$$F(x;\upsilon)=1-\frac{\delta\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}{1-(1-\delta)\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}, \tag{2.1}$$

where α and γ are the scale parameters representing the characteristic lifetime and θ,β and δ are the shape parameters representing different patterns of the MOAW distribution.

The MOAW density function is given by

$$f(x;\upsilon)=\frac{\delta\left(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1}\right)\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}{\left[1-(1-\delta)\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)\right]^{2}}. \tag{2.2}$$

Henceforth, let $X\sim$MOAW(υ) be a random variable having the pdf (2.2). The sf, hrf and cumulative hazard rate function (chrf) of X are given by

$$\overline{F}(x;\upsilon)=\frac{\delta\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}{1-(1-\delta)\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)},$$

$$h(x;\upsilon)=\frac{\alpha\,\theta x^{\theta-1}+\gamma\beta x^{\beta-1}}{1-(1-\delta)\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}$$

and

$$H(x;\upsilon)=-\log\left[\frac{\delta\,\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}{1-(1-\delta)\exp\left(-\alpha x^{\theta}-\gamma x^{\beta}\right)}\right],$$

respectively. Sometimes, we omit the dependence on υ in these equations.

Table 1. Sub-models of the MOAW$(\alpha, \beta, \gamma, \theta, \delta)$

No.	Distribution	α	β	γ	θ	δ	Author
1	AW	α	β	γ	θ	1	Xie and Lai [27]
2	MOMW	α	β	γ	1	δ	Alshangiti *et al.* [6]
3	MOLFR	α	2	γ	1	δ	New
4	MOW	0	β	γ	–	δ	–
5	MOR	0	2	γ	–	δ	–
6	MOE	α	–	0	1	δ	–
7	MW	α	β	γ	1	1	Sarhan and Zaindin [25]
8	LFR	α	2	γ	1	1	–
9	W	0	β	γ	–	1	Weibull [26]
10	R	0	2	γ	–	1	Rayleigh [23]
11	E	α	–	0	1	1	–

Abbreviations: A = Additive, M = Modified, W = Weibull, E = Exponential, MO = Marshall-Olkin, LF = Linear Failure, R = Rayleigh, AW = Additive Weibull.

2.1. Linear representation. An expansion for equation (2.2) can be derived using the power series

$$(1-z)^{-\tau} = \sum_{n=0}^{\infty} \frac{\Gamma(\tau+n)}{\Gamma(\tau)\, n!}\, z^n, \quad \tau > 0.$$

Then, the MOAW density function can be expressed as

$$f(x; \upsilon) = \sum_{j=0}^{\infty} b_j\, g_{j+1}(x), \tag{2.3}$$

where, for $j \geq 0$, $b_j = \delta\,(1-\delta)^j\; \Gamma(j+2)\,/j!$ and

$$g_{j+1}(x) = (j+1)\left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1}\right) \exp\left[-(j+1)\left(\alpha x^{\theta} + \gamma x^{\beta}\right)\right]$$

is the pdf of the random variable $Y_{j+1} \sim$AW$((j+1)\alpha, \beta, (j+1)\gamma, \theta)$.

Hence, the MOAW density function can be written as a mixture of AW densities and then some of its mathematical properties can be obtained directly from those properties of the AW distribution.

2.2. Sub-models and plots. The MOAW distribution is a very flexible model that approaches to different distributions. Its eleven sub-models are listed in Table 1 and only AW will be used in the empirical comparisons in Section 6.

Figure 1 and 2 display some plots of the MOAW density for selected values of $\alpha, \beta, \gamma, \theta$ and δ. These plots illustrate the versatility and modality of this distribution. The plots of Figure 3 reveal that the hrf of X can have bathtub, unimodal, increasing, decreasing and constant shapes.

3. Mathematical properties

The mathematical properties of the MOAW distribution including qf and random number generation, ordinary, central and incomplete moments, mean deviations, mgf and moments of the residual life and reversed residual life are investigated in this section.

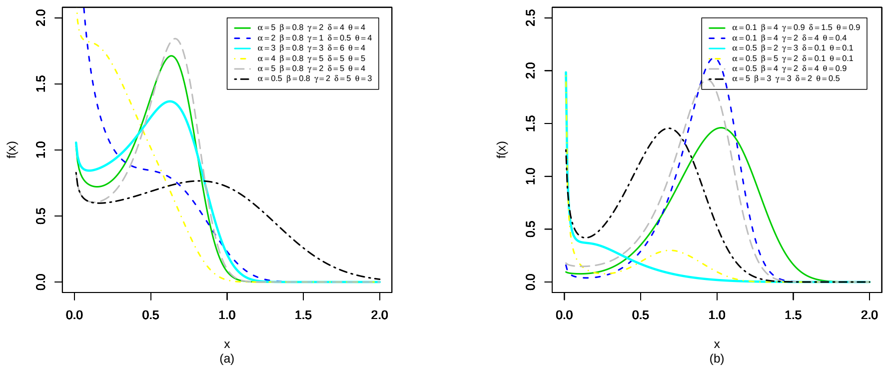

Figure 1. The MOAW density for some parameter vectors.

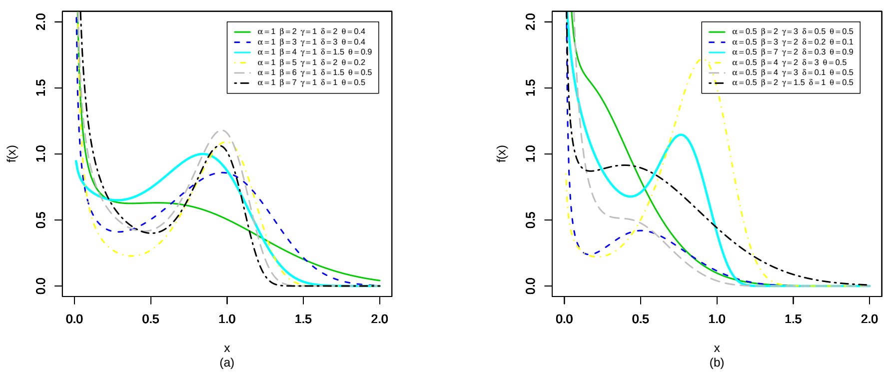

Figure 2. The MOAW density for some parameter vectors.

3.1. Quantile function. The qf of X follows by inverting $F(x_p; \upsilon) = p$ in (2.1). We obtain

$$\alpha\, x_p^{\theta} + \gamma\, x_p^{\beta} + \log\left[\frac{1-p}{1-p(1-\delta)}\right] = 0. \tag{3.1}$$

Since equation (3.1) has no closed-form solution in x_q, we require numerical methods to obtain the quantiles.

3.2. Moments. The kth moment of X, say μ'_k, is given by the following theorem:

Theorem 1. *If X is a continuous random variable having the MOAW$(\alpha, \beta, \gamma, \theta, \delta)$ distribution, the $k(\geq 1)$th non-central moment of X is given by*

$$\mu'_k = E(X^k) = \sum_{j=0}^{\infty} (j+1)\; b_j\; \left(\alpha\theta\, I_{k+\theta-1,j} + \gamma\, \beta I_{k+\beta-1,j}\right). \tag{3.2}$$

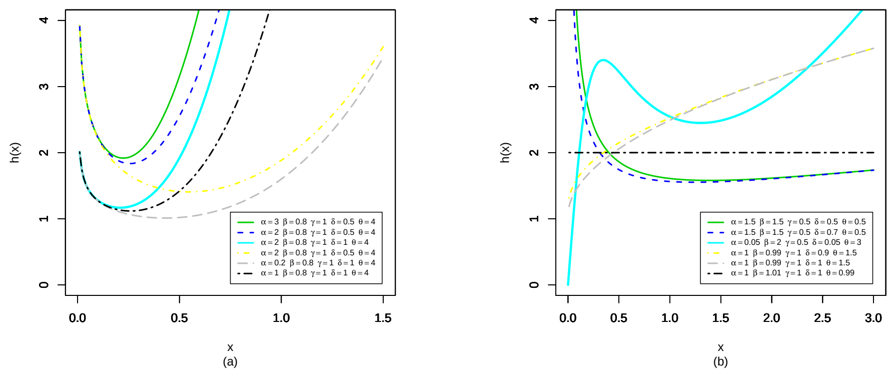

Figure 3. The MOAW hrf.

Proof:

We can determine μ'_k from (2.3) and an integral of the type (for $\delta > 0$)

$$I_{\delta,j} = I(\delta;(j+1)\,\alpha,\beta,(j+1)\,\gamma,\theta) = \int_0^\infty x^\delta \, \exp\left[-\,(j+1)\,(\alpha x^\theta + \gamma x^\beta)\right]\,dx. \tag{3.3}$$

By expanding $\exp\left[-\alpha(j+1)\,x^\theta\right]$ in power series, equation (3.3) reduces to

$$\begin{aligned} I_{\delta,j} &= \sum_{m=0}^\infty \frac{(-1)^m\,[\alpha\,(j+1)]^m}{m!} \int_0^\infty x^{\delta+m\theta}\,\exp\left[-\,(j+1)\,\gamma x^\beta\right] dx \\ &= \frac{1}{\beta\gamma^{(\delta+1)/\beta}} \sum_{m=0}^\infty \frac{(-1)^m}{m!}\left[\frac{\alpha\,(j+1)}{\gamma^{\theta/\beta}}\right]^m \Gamma\left(\frac{\delta+1+\theta m}{\beta}\right). \end{aligned} \tag{3.4}$$

Then, equation (3.4) can be expressed in a simple form, provided that $\beta > 1$, using the confluent hypergeometric function ${}_1F_0$. We obtain

$$I_{\delta,j} = \frac{1}{\beta\gamma^{(\delta+1)/\beta}}\,{}_1F_0\left[\begin{array}{c}\left(\frac{\delta+1}{\beta},\frac{\theta}{\beta}\right)\\ -\end{array};-\frac{\alpha\,(j+1)}{\gamma^{\theta/\beta}}\right], \tag{3.5}$$

where ${}_1F_0(a;-;z)$ has the series expansion $\sum_{k=0}^\infty (a)_k\,\frac{z^k}{k!}$ (see Erdelyi *et al.* [9]). Applying (3.5) to equation (2.3) gives (for $k \geq 1$)

$$\mu'_k = \sum_{j=0}^\infty (j+1)\;b_j\;\left(\alpha\theta\,I_{k+\theta-1,j} + \gamma\,\beta I_{k+\beta-1,j}\right). \tag{3.6}$$

The characteristic function (chf) has many useful and important properties, which gives it a central role in statistical theory. It is particularly useful in analysis of linear combination of independent random variables. The chf of X is given by $\phi(t) = E(\mathrm{e}^{\mathrm{itX}})$, where $\mathrm{i} = \sqrt{-1}$. We can write

$$\phi(t) = \int_0^\infty \cos(tx)\,f\,(x;\upsilon)\;dx + i\int_0^\infty \sin(tx)\,f\,(x;\upsilon)\,dx.$$

The nth central moment $\mu_n = E(X-\mu'_1)^n$ (for $n \geq 1$) of X is given by

$$\mu_n = \sum_{k,j=0}^n \binom{n}{k}\;\left(-\mu'_1\right)^{n-k}\;(j+1)\;b_j\;\left(\alpha\,\theta\,I_{k+\theta-1,j} + \gamma\,\beta I_{k+\beta-1,j}\right).$$

The variance, skewness, kurtosis and cumulants of higher-order of X can be determined from the central moments using well-known relationships.

3.3. Incomplete moments. The sth incomplete moment of X is $\varphi_s(t) = \int_0^t x^s\, f(x)dx$. Henceforth, let $J(t;s,j) = \int_0^t x^s \, \exp\left[-(j+1)\,\gamma\, x^{\beta}\right] dx$. We obtain from equation (2.3)

$$\varphi_s(t) = \sum_{j=0}^{\infty} b_j \int_0^t x^s\,(\alpha\,\theta x^{\theta-1} + \gamma\,\beta\, x^{\beta-1})\, \exp\left[-(j+1)\,\left(\alpha\, x^{\theta} + \gamma\, x^{\beta}\right)\right] dx.$$

By expanding $\exp\left[-(j+1)\,\alpha\, x^{\theta}\right]$, we have

(3.7)

$$\varphi_s(t) = \sum_{j,k=0}^{\infty} \frac{b_j\,[(j+1)\,\alpha]^k}{(-1)^k\, k!}\left[\alpha\,\theta J\,(t; s+\theta k+\theta-1, j) + \beta\,\gamma\, J\,(t; s+\theta k+\beta-1, j)\right],$$

where

$$J(t;s,j) = \beta^{-1}\,\left[(j+1)\gamma\right]^{(s+1)/\beta}\,\gamma\left(\frac{s+1}{\beta}, t\right)$$

and $\gamma(a,z) = \int_0^z y^{a-1}\,\mathrm{e}^{-y}\mathrm{d}y$ is the the lower incomplete gamma function.

The amount of scatter in a population is evidently measured to some extent by the totality of the deviations from the mean and median. The mean deviations about the mean $\delta_1 = E(|X-\mu_1'|)$ and median $\delta_2 = E(|X-M|)$ of X can be used as measures of spread in a population. They are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1\,(M)$, respectively, where $\mu_1' = E\,(X)$ comes from (3.6), $F(\mu_1')$ is evaluated from (2.1), $\varphi_1(\mu_1')$ is the first incomplete moment given by (3.7) and M is the median of X obtained from (3.1) with $p = 0.5$. The Lorenz and Bonferroni curves are defined by $L(p) = \varphi_1(x_p)/\mu_1'$ and $B(p) = \varphi_1(x_p)/(p\mu_1')$, respectively, where $x_p = F^{-1}(p;\upsilon)$ can be computed numerically by (3.1) for a given probability p. These curves have significant role in economics, reliability, demography, insurance and medicine. Readers are referred to Pescim *et al.* [21] and Pundir *et al.* [22].

3.4. Moments of the residual life. Some functions related to the residual life are now defined. For instance, the hazard rate, mean residual life and left censored mean function. These three functions uniquely determine $F(x)$. See, for instance, Gupta [10], Kotz and Shanbhag [13] and Zoroa *et al.* [28].

3.1. Definition. *Let X be a random variable representing the life length for a certain unit at age t (where this unit can have multiple interpretations). Then, $X_t = X - t \mid X > t$ represents the remaining lifetime beyond that age t.*

The cdf $F(x)$ is uniquely determined by the nth moment of the residual life of X (for $n = 1, 2, \ldots$) (Navarro *et al.* [19]), and it is given by

$$m_n\,(t) = \frac{1}{1-F\,(t)} \int_t^{\infty} (x-t)^n \; dF\,(x)\,.$$

Further, it can be expressed from (2.3) as

$$m_n\,(t) = \frac{1}{R\,(t)} \sum_{r=0}^{n} \sum_{j,k=0}^{\infty} \frac{(-1)^{n+k-r}\; n!\, t^{n-r}}{r!\, k!\, (n-r)!}\, b_j\, \left[\alpha\,(j+1)\right]^k$$
$$\times \left[\alpha\,\theta J\,(t; n+\theta k+\theta-1, j) + \beta\gamma\, J\,(t; n+\theta k+\beta-1, j)\right].$$

Another interesting function is the mean residual life (MRL) function given by $m_1(t)$. It represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n=1$ in the last equation.

Guess and Proschan [11] gave an extensive coverage of applications of the MRL in survival analysis, biomedical sciences, life insurance, social studies, economics, demography, maintenance and product quality control and product technology (Lai and Xie [15]).

3.5. Moments of the reversed residual life. The cdf $F(x)$ is uniquely determined by the mean reversed residual life of X (for $n = 1, 2 \ldots$) (Navarro *et al.* [19]), and it is given by

$$M_n\left(t\right)=\frac{1}{F\left(t\right)}\int_0^t\left(t-x\right)^n dF\left(x\right).$$

In a similar manner, it can be expressed as

$$M_n\left(t\right)=\frac{1}{F\left(t\right)}\sum_{r=0}^{n}\sum_{j,k=0}^{\infty}\frac{\left(-1\right)^{r+k}\,n!\,t^{n-r}}{r!\,k!\,(n-r)!}b_j\,\left[\alpha\left(j+1\right)\right]^k$$
$$\times\left[\alpha\,\theta J\left(t;n+\theta k+\theta-1,j\right)+\beta\gamma\,J\left(t;n+\theta k+\beta-1,j\right)\right].$$

The mean inactivity time (MIT) of X is obtained by setting $n=1$ in the above equation. Some properties of MIT have been explored by Kayid and Ahmad [14] and Ahmad *et al.* [4], among several others.

3.6. Generating function. In this section, we obtain the mgf of X using a power series for its qf. The nonlinear equation (3.1) can be expressed as $\alpha\,x^{\theta}+\gamma\,x^{\beta}=z$, where

$$z=z(\delta,p)=-\log\left[\frac{1-p}{1-p(1-\delta)}\right].$$

By expanding x^{θ} in Taylor series, we obtain $x^{\theta}=\sum_{k=0}^{\infty}(\theta)^{(k)}\,(x-1)^k/k!=\sum_{j=0}^{\infty}f_j\,x^j$, where $f_j=\sum_{k=j}^{\infty}(-1)^{k-j}\,\binom{k}{j}\,(\theta)^{(k)}/k!$ and $(\theta)^{(k)}=\theta(\theta-1)\ldots(\theta-k+1)$ is the descending factorial. Analogously, by expanding x^{β}, we can write

$$z=H(x)=\sum_{j=0}^{\infty}h_j\,x^j, \tag{3.8}$$

where $h_j=\alpha\,f_j+\gamma\,g_j$ and $g_j=\sum_{k=j}^{\infty}(-1)^{k-j}\,\binom{k}{j}\,(\beta)^{(k)}/k!$. By using *MATHEMATICA*

$$H(x)=(\alpha+\gamma)+(\alpha\,\theta+\gamma\,\beta)\,(x-1)+\left[\alpha\,(\theta)^{(2)}+\gamma\,(\beta)^{(2)}\right]\,\frac{(x-1)^2}{2}$$
$$+\left[\alpha\,(\theta)^{(3)}+\gamma\,(\beta)^{(3)}\right]\,\frac{(x-1)^3}{3!}$$
$$+\left[\alpha\,(\theta)^{(4)}+\gamma\,(\beta)^{(4)}\right]\,\frac{(x-1)^4}{4!}+O\left((x-1)^5\right).$$

Then, we obtain $H(x)$ in the form (3.8) by expanding the powers of $(x-1)$.

Further, we use the Lagrange theorem to obtain an expansion for the qf $Q(p;\upsilon)=F^{-1}(p;\upsilon)$. We assume that the power series holds

$$z=H(x)=h_0+\sum_{j=1}^{\infty}h_j\,t^j,\qquad h_1=H'(x)\neq 0,$$

where $H(x)$ is analytic at zero. Then, the inverse power series $x = H^{-1}(z)$ exists, it is single-valued in the neighborhood of the point $z = 0$, and it reduces to

$$x = H^{-1}(z) = \sum_{j=1}^{\infty} v_j \, z^j,$$

where the coefficient v_j (for $j \geq 1$) is given by

$$v_j = \frac{1}{n!} \left. \frac{d^{j-1} \left[\psi(x)\right]^j}{dx^{j-1}} \right|_{t=0}, \qquad \psi(x) = \frac{x}{H(x) - h_0}.$$

Then, the qf X becomes

$$Q(p) = \sum_{j=1}^{\infty} v_j \left[-\log \left(\frac{1-p}{1-p(1-\delta)} \right) \right]^j. \tag{3.9}$$

Hence, the mgf of X, say $\rho_X(s)$, can be expressed as

$$\rho_X(s) = \int_0^1 \exp \left\{ s \sum_{j=1}^{\infty} v_j \left[-\log \left(\frac{1-p}{1-p(1-\delta)} \right) \right]^j \right\} dp. \tag{3.10}$$

The exponential partial Bell polynomials are defined by Abramowitz and Stegun [1].

$$\exp \left(u \sum_{m \geq 1} x_m \frac{t^m}{m!} \right) = \sum_{n,k \geq 0} \frac{B_{n,k}}{n!} \, t^n \, u^k, \tag{3.11}$$

where

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{c_1! \, c_2! \ldots (1!)^{c_1} (2!)^{c_2} \ldots} \, x_1^{c_1} \, x_2^{c_2}, \ldots,$$

and the summation takes place over all integers $c_1, c_2, \ldots \geq 0$, which satisfy $c_1 + 2c_2 + 3c_3 + \cdots = n$ and $c_1 + c_2 + c_3 + \cdots = k$.

These polynomials can be evaluated as **IncompleteBellB(n, k, x[1], x[2], . . . , x[n = k+1])** in MAPLE.

Applying (3.11) in equation (3.10) gives

$$\rho_X(s) = \sum_{n,k \geq 0} \frac{s^k \, B_{n,k}}{n!} \int_0^1 \left[-\log \left(\frac{1-p}{1-p(1-\delta)} \right) \right]^n dp, \tag{3.12}$$

where $B_{n,k} = B_{n,k}(1!v_1, 2!v_2, \ldots, (n-k+1)!v_{n-k+1})$ and the integral can be evaluated in the software before. The final expression (3.12) is a polynomial in s up to a desired order if we evaluate numerically the integral for every n.

4. Order statistics

The order statistics have great importance in some statistical problems and many applications in reliability analysis and life testing. They can represent the lifetimes of units or components of a reliability system. Let $X_1, \ldots, X_n$ be a random sample of size n from the MOAW$(\alpha, \beta, \gamma, \theta, \delta)$ model with cdf and pdf given by (2.1) and (2.2), respectively. The pdf of the ith order statistic, say $X_{i:n}$, $1 \leq i \leq n$, is given by

$$f_{i:n}(x) = \frac{\delta^{n-i+1} \left(\alpha\theta x^{\theta-1} + \gamma\beta x^{\beta-1} \right) \exp\left[-(n-i+1) \left(\alpha x^{\theta} + \gamma x^{\beta} \right) \right]}{B(i, n-i+1) \left[1 - (1-\delta) \exp\left(-\alpha x^{\theta} - \gamma \, x^{\beta} \right) \right]^{n-i+2}} \times \left[1 - \frac{\delta \exp\left(-\alpha x^{\theta} - \gamma x^{\beta} \right)}{1 - (1-\delta) \exp\left(-\alpha x^{\theta} - \gamma \, x^{\beta} \right)} \right]^{i-1},$$

where $B(\cdot,\cdot)$ is the beta function.

The pdf of $X_{i:n}$ can be expressed as a linear combination of AW densities

$$f_{i:n}\,(x) = \sum_{j,w=0}^{\infty} \Im_{j,w}\, g\,(x;\alpha^*,\beta,\gamma^*,\theta)\,. \tag{4.1}$$

Here,

$$\Im_{j,w} = \frac{(-1)^{j+w}\,\Gamma\,(i)\,\Gamma\,(i-j-n-1)\,\delta^{n+j-i+1}\,(1-\delta)^w}{B\,(i,n-i+1)\,j!w!\Gamma\,(i-j)\,\Gamma\,(i-j-n-w-1)\,(n-i+j+w+1)},$$

and

$$g\,(x;\alpha^*,\beta,\gamma^*,\theta) = \left(\alpha^*\,\theta x^{\theta-1} + \gamma^*\,\beta x^{\beta-1}\right)\exp\left[-\left(\alpha^* x^{\theta} + \gamma^* x^{\beta}\right)\right]$$

is the AW density function with parameters $\alpha^*,\beta,\gamma^*,\theta$, where $\alpha^* = (n-i+j+w+1)\,\alpha$ and $\gamma^* = (n-i+j+w+1)\,\gamma$.

Thus, the density function of the MOAW order statistics is a linear combination of AW densities. Based on equation (4.1), we can obtain some structural properties of $X_{i:n}$ from those AW properties. For example,

$$E\left(X_{i:n}^q\right) = \sum_{j,w=0}^{\infty} \Im_{j,w} E\left(Y_*^q\right), \tag{4.2}$$

where $Y_* \sim \mathrm{AW}(\alpha^*,\beta,\gamma^*,\theta)\,.$

The L-moments defined by linear combinations of expected order statistics are analogous to the ordinary moments. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. They are robust to outliers and virtually unbiased for small samples, making them suitable for flood frequency analysis, including identification of distribution and parameter estimation. The rth L-moment is given by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E\left(X_{r-d:r}\right).$$

Then, we can obtain the L-moments of X from equation (4.2) with $q=1$ as infinite weighted linear combinations of suitable AW means. The L-mean, λ_1, is a measure of central tendency and the L-standard deviation, λ_2, is a measure of dispersion. Their ratio, λ_2/λ_1, is called the L-coefficient of variation, the ratio λ_3/λ_2 is called the L-skewness, while the ratio λ_4/λ_2 is referred to the L-kurtosis. For further details of L-moments, readers are referred to as Hosking [12].

5. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for the new distribution from complete samples only by maximum likelihood. Let $x_1,\ldots,x_n$ be a random sample of this distribution with unknown parameter vector $\boldsymbol{\upsilon} = (\alpha,\beta,\gamma,\theta,\delta)^T$.

The log-likelihood function, say $\ell(\boldsymbol{\upsilon})$ obtained from Equation (2.2), is given by

$$\ell(\boldsymbol{\upsilon}) = n\log(\delta) + \sum_{i=1}^{n}\log(s_i) + \sum_{i=1}^{n}\log(z_i) - 2\sum_{i=1}^{n}\log(t_i)\,, \tag{5.1}$$

where $z_i = \alpha\theta x_i^{\theta-1} + \gamma\beta x_i^{\beta-1}$, $s_i = \exp(-\alpha x_i^{\theta} - \gamma x_i^{\beta})$ and $t_i = 1 - (1-\delta)\, s_i$.

The components of the score vector $\mathbf{U}(\boldsymbol{\upsilon}) = \frac{\partial \ell}{\partial \boldsymbol{\upsilon}} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \gamma}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \delta}\right)^T$ are:

$$\frac{\partial \ell(\boldsymbol{\upsilon})}{\partial \alpha} = \theta \sum\nolimits_{i=1}^{n} \frac{x_i^{\theta-1}}{z_i} - \sum\nolimits_{i=1}^{n} x_i^{\theta} - 2(1-\delta) \sum\nolimits_{i=1}^{n} \frac{s_i x_i^{\theta}}{t_i},$$

$$\begin{aligned}\frac{\partial \ell(\boldsymbol{\upsilon})}{\partial \beta} &= \gamma \sum\nolimits_{i=1}^{n} \frac{x_i^{\beta-1}(\beta \log(x_i) + 1)}{z_i} - \gamma \sum\nolimits_{i=1}^{n} x_i^{\beta} \log(x_i) \\ &\quad - 2\gamma(1-\delta) \sum\nolimits_{i=1}^{n} \frac{s_i x_i^{\beta} \log(x_i)}{t_i},\end{aligned}$$

$$\frac{\partial \ell(\boldsymbol{\upsilon})}{\partial \gamma} = \beta \sum\nolimits_{i=1}^{n} \frac{x_i^{\beta-1}}{z_i} - \sum\nolimits_{i=1}^{n} x_i^{\beta} - 2(1-\delta) \sum\nolimits_{i=1}^{n} \frac{s_i x_i^{\beta}}{t_i},$$

$$\begin{aligned}\frac{\partial \ell(\boldsymbol{\upsilon})}{\partial \theta} &= \alpha \sum\nolimits_{i=1}^{n} \frac{x_i^{\theta-1}(\theta \log(x_i) + 1)}{z_i} - \alpha \sum\nolimits_{i=1}^{n} x_i^{\theta} \log(x_i) \\ &\quad - 2\alpha(1-\delta) \sum\nolimits_{i=1}^{n} \frac{s_i x_i^{\theta} \log(x_i)}{t_i}\end{aligned}$$

and

$$\frac{\partial \ell(\boldsymbol{\upsilon})}{\partial \delta} = \frac{n}{\delta} - 2 \sum\nolimits_{i=1}^{n} \frac{s_i}{t_i}.$$

We require iterative techniques such as the Newton-Raphson algorithm to solve these equations numerically. For the proposed distribution, all the second order log-likelihood derivatives exist.

For interval estimation and hypothesis tests on the model parameters, we require the 5×5 observed ($\ddot{\mathbf{L}}(\boldsymbol{\upsilon})$) and expected ($\mathbf{J}(\boldsymbol{\upsilon})$) information matrices. Under general regularity conditions, we can construct approximate confidence intervals for the individual parameters based on the multivariate normal $N_5(0, \mathbf{J}(\widehat{\boldsymbol{\upsilon}})^{-1})$ distribution, where $\widehat{\boldsymbol{\upsilon}}$ is the MLE of $\boldsymbol{\upsilon}$.

Approximate $100(1-\phi)\%$ confidence intervals for $\alpha, \beta, \gamma, \theta$ and δ can be determined as:

$$\widehat{\alpha} \pm z_{\phi/2}\sqrt{\widehat{J}_{\alpha\alpha}}, \qquad \widehat{\beta} \pm z_{\phi/2}\sqrt{\widehat{J}_{\beta\beta}}, \qquad \widehat{\gamma} \pm z_{\phi/2}\sqrt{\widehat{J}_{\gamma\gamma}},$$

$$\widehat{\theta} \pm z_{\phi/2}\sqrt{\widehat{J}_{\theta\theta}} \qquad \text{and} \qquad \widehat{\delta} \pm z_{\phi/2}\sqrt{\widehat{J}_{\delta\delta}},$$

where $z_{\phi/2}$ is the upper ϕ-th percentile of the standard normal distribution.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct likelihood ratio (LR) statistics for testing some sub-models of the MOAW model. For example, the test of $H_0 : \delta = 1$ versus $H_1 : H_0$ *is not true* is equivalent to compare the MOAW and AW and the LR statistic reduces to

$$w = 2[\ell(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}, \hat{\delta}) - \ell(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\theta}, 1)],$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\theta}$ and $\hat{\delta}$ are the MLEs under H_1 and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\theta}$ are the estimates under H_0. Statistical w is approximately Chi-square distribution with degree of freedom corresponding to the difference between the number of parameters of the two models.

6. Application

In this section, we provide an application of the new distribution to show empirically its potentiality. We shall compare the fits of the MOAW and AW models and the following competing non-nested distributions, whose pdfs (for $x > 0$) are given below:

- The transmuted additive Weibull (TAW) distribution introduced by Elbatal and Aryal [8], whose pdf is

$$f\left(x;\alpha,\beta,\gamma,\theta,\lambda\right) = \left(\alpha\theta x^{\theta-1} + \gamma\,\beta x^{\beta-1}\right)\exp\left(-\alpha\,x^{\theta} - \gamma x^{\beta}\right) \times \left[1-\lambda+2\lambda\exp\left(-\alpha\,x^{\theta}-\gamma x^{\beta}\right)\right].$$

- The exponentiated transmuted generalized Rayleigh (ETGR) distribution defined by Afify *et al.* [2], whose pdf is

$$f\left(x;\alpha,\beta,\lambda,\delta\right) = 2\,\alpha\,\delta\,\beta^{2}x\exp\left[-\left(\beta x\right)^{2}\right]\left\{1-\exp\left[-\left(\beta\,x\right)^{2}\right]\right\}^{\alpha\delta-1} \times\left\{1+\lambda-2\lambda\left(1-\exp\left[-\left(\beta x\right)^{2}\right]\right)^{\alpha}\right\} \times\left\{1+\lambda-\lambda\left(1-\exp\left[-\left(\beta x\right)^{2}\right]\right)^{\alpha}\right\}^{\delta-1}.$$

- The Kumaraswamy linear exponential (KLE) distribution proposed by Merovci and Elbatal [17], whose pdf is

$$f(x;\alpha,\gamma,\delta,\theta) = \delta\,\theta\,\left(\alpha+\gamma x\right)\exp\left(-\alpha\,x-\frac{\gamma}{2}x^{2}\right)\times\left[1-\exp\left(-\alpha\,x-\frac{\gamma}{2}x^{2}\right)\right]^{\delta-1} \times\left\{1-\left[1-\exp\left(-\alpha\,x-\frac{\gamma}{2}x^{2}\right)\right]^{\delta}\right\}^{\theta-1}.$$

- The new modified Weibull (NMW) distribution defined by Almalki and Yuan [5], whose pdf is

$$f(x;\alpha,\beta,\gamma,\delta,\theta) = \left(\alpha\,\theta x^{\theta-1} + \gamma\left(\beta+\delta\,x\right)x^{\beta-1}\exp\left(\delta\,x\right)\right)\exp\left(-\alpha\,x^{\theta}-\gamma\,x^{\beta}\right).$$

- The transmuted Weibull Lomax (TWL) distribution introduced by Afify *et al.* [3], whose pdf is

$$f(x;\alpha,\beta,\lambda,a,b) = \frac{a\,b\,\alpha}{\beta}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{b-1}\exp\left\{-a\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\} \times\left(1+\frac{x}{\beta}\right)^{b\alpha-1}\left\{1-\lambda+2\lambda\exp\left[-a\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right]\right\}.$$

  The parameters of the above densities are all positive real numbers except $|\lambda| \le 1$, and the parameters θ and β for the TAW model, where $0 < \theta < \beta$ or $0 < \beta < \theta$.

We consider a data set referring to nicotine measurements made from several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission, which is an independent agency of the US government, whose main mission is the promotion of consumer protection. The report entitled tar, nicotine, and carbon monoxide of the smoke of 1,206 varieties of domestic cigarettes for the year of 1998 is available at *http://www.ftc.gov/reports/tobacco* and consists of the data sets and some information about the source, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. The data set is at *http://pw1.netcom.com/rdavis2/smoke.html*. The site *http://home.att.net/rdavis2/cigra.html* contains $n = 346$ observations. These

Table 2. MLEs of the parameters (standard errors in parentheses)

Model	Estimates				
MOAW	$\widehat{\alpha}$= 0.03177 (0.01212)	$\widehat{\beta}$= 1.8840 (0.3396)	$\widehat{\gamma}$= 0.01146 (0.00348)	$\widehat{\theta}$= 6.5173 (1.0066)	$\widehat{\delta}$= 0.02016 (0.00733)
TAW	$\widehat{\alpha}$= 1.2252 (0.239)	$\widehat{\beta}$= 0.8994 (0.091)	$\widehat{\gamma}$= 0.433 (0.229)	$\widehat{\theta}$= 2.6404 (0.267)	$\widehat{\lambda}$= −0.8831 (0.147)
NMW	$\widehat{\alpha}$= 0.0012 (0.036)	$\widehat{\beta}$= 2.3518 (0.337)	$\widehat{\gamma}$= 0.7453 (0.276)	$\widehat{\delta}$= 0.3956 (0.344)	$\widehat{\theta}$= 2.083 (0.584)
TWL	$\widehat{\alpha}$= 10.6275 (25.423)	$\widehat{\beta}$= 24.7718 (72.961)	$\widehat{a}$= 4.6163 (5.992)	$\widehat{b}$= 2.3601 (0.293)	$\widehat{\lambda}$= 0.1625 (0.217)
AW	$\widehat{\alpha}$= 1.135 (0.062)	$\widehat{\beta}$= 0.3084 (0.1)	$\widehat{\gamma}$= 0.0002 (0.001369)	$\widehat{\theta}$= 2.7219 (0.114)	
ETGR	$\widehat{\alpha}$= 0.2879 (0.071)	$\widehat{\beta}$= 0.9481 (0.04)	$\widehat{\delta}$= 8.8551 (3.2)	$\widehat{\lambda}$= 0.8266 (0.063)	
KLE	$\widehat{\alpha}$= 0.1526 (0.123)	$\widehat{\gamma}$= 0.6362 (0.325)	$\widehat{\delta}$= 1.9285 (0.309)	$\widehat{\theta}$= 6.957 (4.792)	

Table 3. Goodness-of-fit statistics

Models	A_0^*	W_0^*	KS	p-value	AIC	AICC	BIC	HQIC	CAIC
MOAW	**1.808**	**0.327**	**0.083**	**0.0157**	**222.03**	**222.21**	**241.3**	**229.69**	**222.21**
AW	2.313	0.406	0.099	0.002	229.60	229.72	244.99	235.73	229.72
TAW	1.989	0.356	0.095	0.004	231.60	231.78	250.83	239.26	231.78
KLE	2.487	0.431	0.108	0.0006	230.71	230.92	246.09	236.83	230.82
NMW	2.313	0.406	0.099	0.0022	231.60	231.78	250.83	239.26	231.78
ETGR	4.289	0.768	0.141	0.0	246.91	247.03	262.30	253.04	247.03
TWL	2.932	0.502	0.143	0.0	491.82	491.99	501.38	495.46	493.18

Table 4. Confidence interval for the model parameters

CI	α	β	γ	θ	δ
95%	(0.0080, 0.0555)	(1.2161, 2.5519)	(0.0046, 0.01828)	(4.5376, 8.4971)	(0.0058, 0.0345)

data have been used by Nofal *et al.* [20] to fit the generalized transmuted Weibull distribution. We analyze these data on nicotine, measured in milligrams per cigarette, from several cigarette brands.

We use the the procedure NLMixed in SAS to compute the MLE $\widehat{\boldsymbol{v}}$. Table 2 lists the MLEs (standard errors in parentheses) of the model parameters for the fitted seven models to the current data. The covariance matrix of the MLEs for the fitted MOAW distribution is given by

$$\begin{pmatrix} 0.000147 & -0.00762 & 0.000382 & -0.03378 & 0.000879 \\ -0.00762 & 0.1153 & -0.00098 & 0.2403 & -0.00402 \\ 0.000382 & -0.00098 & 0.0000121 & -0.00737 & 0.000242 \\ -0.03378 & 0.2403 & -0.00737 & 1.0132 & -0.01943 \\ 0.000879 & -0.00402 & 0.000242 & -0.01943 & 0.0000538 \end{pmatrix}.$$

The diagonal entries of this matrix represent the variances of the MLEs of $\alpha, \beta, \gamma, \theta$ and δ. Then, the 95% confidence intervals for these parameters are given in Table 4.

In order to compare the distributions, we consider the following statistics: Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) given by

$$
\begin{aligned}
AIC &= -2\widehat{\ell} + 2k, \quad BIC = -2\widehat{\ell} + k\log(n),\\
HQIC &= -2\widehat{\ell} + 2k\log[\log(n)],\\
CAIC &= -2\widehat{\ell} + 2kn/(n-k-1),\\
AICC &= AIC + \frac{2\,q\,(q+1)}{n-q-1}\,,
\end{aligned}
$$

where $\widehat{\ell}$ denotes the maximized log-likelihood function, k is the number of estimated parameters and n is the sample size.

Some of the most widely used test statistics like the modified Anderson–Darling (A_0^*), modified Cramér Von–Mises (W_0^*) and Kolmogorov Smirnov (K-S) statistics are given by

$$
\begin{aligned}
A_0^* &= \left(\frac{2.25}{n^2} + \frac{0.75}{n} + 1\right)\left[-n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(z_i\left(1 - z_{n-i+1}\right)\right)\right],\\
W_0^* &= \left(\frac{0.5}{n} + 1\right)\left[\sum_{i=1}^{n}\left(z_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}\right],\\
\text{K-S} &= \text{Max}\left[\frac{i}{n} - z_i, z_i - \frac{i-1}{n}\right],\\
\text{p-value} &= 1 - \frac{\sqrt{2\pi}}{\text{K-S}\sqrt{n}}\sum_{i=1}^{n} e^{-\frac{\pi^2(2i-1)^2}{8(\text{K-S})^2 n}},
\end{aligned}
$$

respectively, where $z_i = \text{cdf}(y_{(i)})$ and the $y_{(i)}^{'}s$ are the ordered observations. These statistics are used to assess the adequacy of the fit of the distributions considered in the current data set. The model with minimum AIC, BIC, $HQIC$, $CAIC$, A_0^*, W_0^* and K-S values (the last one with the p-value) can be chosen as the best model to fit the data.

Table 3 lists the values of above statistics for seven fitted models. The figures in Tables 3 reveal that the MOAW distribution yields the lowest values of these statistics and then provides the best fit to these data. It is also seen that the p–value test for the proposed model has the largest value among all models.

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 5. The numbers in this table, specially the p-value, suggest that the new MOAW model yields a better fit to these data than the other distribution.

Table 5. LR statistics.

Model	Hypotheses	Statistics w	P-value
MOAW vs AW	$H_0: \delta = 1$ vs $H_1: H_0$ is false	9.0	0.0027

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plot of the fitted MOAW, AW ant TAW density functions are displayed in Figure 4.

In order to assess if the model is appropriate, the plots of the fitted MOAW, AW ant TAW cumulative distribution and the empirical cdf are displayed in Figure 5. We conclude that the MOAW and TAW distributions provides a good fit for these data. Note that the MOAW and TAW models are potentially competitors.

(a) (b)

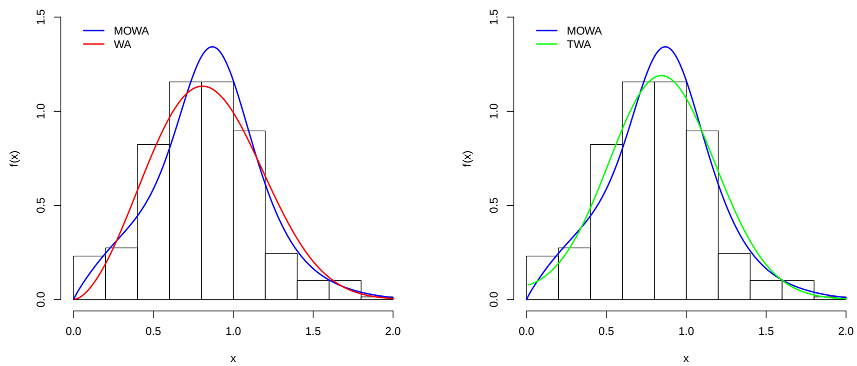

Figure 4. Fitted GL density for the data. (a) MOAW vs AW (b) MOAW vs TAW.

(a) (b) (c)

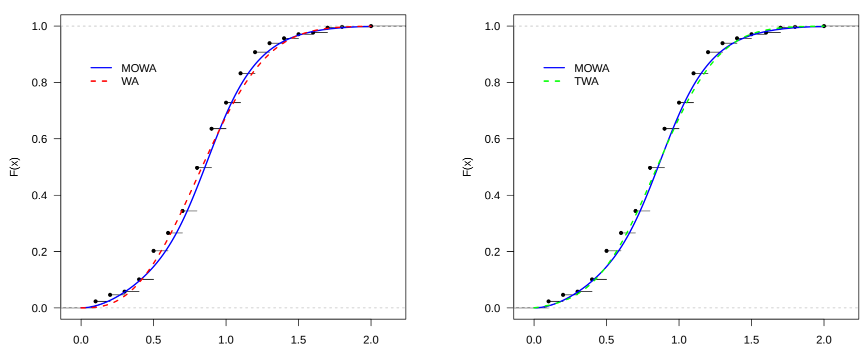

Figure 5. Estimated GL cumulative distributions for the data. (a) MOAW vs AW (b) MOAW vs TAW.

7. Conclusions

In this paper, we propose a five-parameter model, called the Marshall-Olkin additive Weibull (MOAW) distribution, which extends the additive Weibull (AW) distribution pioneered by Xie and Lai **[23]** and some other well-known distributions. An obvious reason for generalizing a standard distribution is the fact that the generated model can provide more flexibility to analyze real life data. We provide some of its mathematical and statistical properties. The MOAW density function can be expressed as a linear mixture of AW densities. We derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions and moments of the residual life and reversed residual life model. We also obtain the density function of the order statistics and their moments. We discuss the estimation of the model parameters by maximum likelihood. The proposed distribution is applied to a real data set. It provides a better fit than several other competitive nested and non-nested models. We hope that the proposed model will attract wider application in areas such as engineering, survival and lifetime data, meteorology, hydrology, economics and others.

Acknowledgement

The research of Abdus Saboor has been supported in part by the Higher Education Commission of Pakistan under NRPU project No. 3104.

References

[1] Abramowitz, M. and Stegun, I. E. Handbook of Mathematical Func- tions. Dover, 1964).

[2] Afify, A. Z., Nofal, Z. M. and Ebraheim, A. N. *Exponentiated transmuted generalized Rayleigh distribution: a new four parameter Rayleigh distribution*, Pakistan Journal of Statistics and Operation Research, 11, 115–134, 2015.

[3] Afify, A.Z., Nofal, Z.M., Yousof, H.M., Gebaly, Y.M. and Butt, N.S. *The transmuted Weibull Lomax distribution: properties and application*, Pakistan Journal of Statistics and Operation Research, 11, 135–152, 2015.

[4] Ahmad, I., Kayid, M. and Pellerey, F. *Further results involving the MIT order and the IMIT class* , Probability in the Engineering and Informational Science, 19, 377–395, 2005.

[5] Almalki, S.J. and Yuan, J. *A new modified Weibull distribution*, Reliability Engineering and System Safety, 111, 164–170, 2011.

[6] Alshangiti, A.M., Kayid, M. and Alarfaj, B. *A new family of Marshall–Olkin extended distributions*, Journal of Computational and Applied Mathematics, 271, 369–379, 2014.

[7] Bebbington, M., Lai, C.D. and Zitikis, R. *A flexible Weibull extension*, Reliability Engineering and System Safety, 92, 719–726, 2007.

[8] Elbatal, I. and Aryal, G. *On the transmuted additive Weibull distribution*, Austrian Journal of Statistics, 42, 117–132, 2013.

[9] Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. *Higher Transcendental Functions*, volume II. McGraw Hill, 1953.

[10] Gupta, R.C. *On characterization of distributions by conditional expectations*, Communication in Statistics: Simulation and Computation, 4, 99–103, 1975.

[11] Guess, F. and Proschan, F. *Mean residual life, theory and applications. In: Krishnaiah, P.R., Rao, C. R. (Eds.), Handbook of Statistics*, Reliability and Quality Control, 215–224, 1988.

[12] Hosking, J.R.M.. *L-moments: analysis and estimation of distributions using linear combinations of order statistics.* Journal of the Royal Statistical Society, Series B, 52: 105—124, 1990.

[13] Kotz, S. and Shanbhag, D. N. *Some new approaches to probability distributions*, Advances in Applied Probability, 12, 903–921, 1980.

[14] Kayid, M. and Ahmad, I. *On the mean inactivity time ordering with reliability applications*, Probability in the Engineering and Informational Science, 18, 395–409, 2004.

[15] Lai, C.D. and Xie, M. *Stochastic ageing and dependence for reliability*, Springer, New York, 2006.

[16] Marshall, A. W. and Olkin, I. *A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*, Biometrika, 84, 641–652, 1997.

[17] Merovci, F. and Elbatal, I. *A new generalization of linear exponential distribution: theory and application*, Journal of Statistics and Applied Probability Letters, 2, 1–14, 2015.

[18] Nadarajah S., Cordeiro, G.M. and Ortega Edwin, M.M. *General results for the beta–modified Weibull distribution*, Journal of Statistical Computation and Simulation, 81, 121–132, 2011.

[19] Navarro, J. Franco, M. and Ruiz, J.M. *Characterization through moments of the residual life and conditional spacing*, The Indian Journal of Statistics, Vol. 60, Series A, Pt. 1, 36–48, 1998.

[20] Nofal, Z. M., Afify, A. Z., Yousof, H. M. and Cordeiro, G.M. *The generalized transmuted-G family of distributions*, Communications in Statistics-Theory and Methods, To Appear, 2015.

[21] Pescim, R.R., Cordeiro, G.M., Nararajah, S., Demetrio, C.G.B. and Ortega, E.M.M. *The Kummer beta Birnbaum-Saunders: An alternative fatigue life distribution*, Hacettepe Journal of Mathematics and Statistics, 43, 473–510, 2014.

[22] Pundir, S., Arora, S., and Jain, K. (2005). Bonferroni curve and the related statistical inference. Statistics and Probability Letters, 75:140–150.

[23] Rayleigh, J.W.S. *On the resultant of a large number of vibration of the same pitch and arbitrary phase*, Philosophical magazine, 5th Series, 10, 73–78, 1880.

[24] Saboor, A., Kamal, M. and Ahmad, M. *The transmuted exponential–Weibull distribution with applications*, Pakistan Journal of Statistics, 31, 229–250, 2015.

[25] Sarhan A.M. and Zaindin, M. *Modified Weibull distribution*, Applied Sciences, 11, 123–36, 2009.

[26] Weibull, W. *A statistical distribution function of wide applicability*, Journal of Applied Mechanics and Transections, ASME 18, 293–297, 1951.

[27] Xie, M., and Lai, C.D. *Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function*, Reliability Engineering and System Safety, 52, 87–93, 1995.

[28] Zoroa, P., Ruiz, J.M. and Marin, J. *A characterization based on conditional expectations*, Communications in Statistics: Theory and Methods, 19, 3127–3135, 1990.