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A note on the multiple comparisons of exponential location parameters with several controls under heteroscedasticity

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Abstract

Several researchers have addressed the problem of constructing simultaneous confidence intervals (SCIs) for comparing exponential location parameters with a control or controls under heteroscedasticity when sample sizes are equal or unequal. They usually used simulation-based inference procedures or Lam's technique that leads to conservative SCIs. In this paper, we present a set of SCIs for comparing exponential location parameters with a control, controls and the best control under heteroscedasticity when sample sizes are possibly unequal. Our method is not a simulation-based inference procedure and our results show that the proposed SCIs have some advantages over others.

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1. Introduction

There are many applications of exponential distribution in reliability analysis, life testing, biological and epidemiological studies. For example, in dose-response analysis, the location and scale parameters are known as the guaranteed mean effective duration and the mean effective duration in addition to the guaranteed mean effective duration, respectively. In biological and epidemiological studies, these parameters are known as

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the latency period and the mean duration of a disease; for more applications of exponential distribution see [17] and [9]. We use notation $E(a,b)$ to denote the exponential distribution with the following density function

$$f(x;a,b)=\frac{1}{b}\exp\Big(\frac{x-a}{b}\Big)I_{[a,\infty)}(x),$$

where $I_A(\cdot)$ is the indictor function of event A and parameters $a\in(-\infty,\infty)$ and $b\in(0,\infty)$ are location and scale parameters, respectively. Suppose that there are I independent treatment groups and J independent control groups where the i-th treatment group follows $E(\theta_i,\sigma_i)$ and the j-th control group follows $E(\mu_j,\delta_j)$. We consider the problem of constructing simultaneous confidence intervals (SCIs) for $\theta_i-\mu_j$, $i=1,\ldots,I$; $j=1,\ldots,J$, under one-stage and two-stage sampling procedures and heteroscedasticity; i.e. when the scale parameters are unequal.

Under two-stage sampling scheme or the homogeneity assumption of scale parameters, the problem of comparing exponential location parameters with a control or controls has been addressed by several authors; e.g. [8] , [13] and [14]. Recently, Wu et al. [15] proposed conservative two-sided and one-sided SCIs for $\theta_i-\mu_1$, $i=1,\ldots,I$, under heteroscedasticity when sample sizes are equal. They used Lam's [6, 7] technique to show theoretically that the proposed SCIs are conservative. They also examined the coverage probability (CP) of SCIs via a simulation study. The results of simulation indicated that the proposed SCIs are too conservative. Maurya et al. [12] proposed one-sided and two-sided conservative SCIs for $\theta_i-\mu_j$, $i=1,\ldots,I;j=1,\ldots,J$, under heteroscedasticity when sample sizes are equal. They also used Lam's [6, 7] technique to show that their SCIs are conservative. Maurya et al. [11] proposed only two-sided conservative SCIs for $\theta_i-\mu_1$, $i=1,\ldots,I$, under heteroscedasticity when sample sizes are equal. They also used Lam's [6, 7] technique to obtain the required critical values. However, they did not examine the CP of their SCIs.

Kharrati-Kopaei et al. [5] proposed simultaneous fiducial generalized confidence intervals (SFGCIs) for the successive differences of the location parameters of several exponential distributions and indicated how their proposed procedure can be modified to obtain SFGCIs for comparisons of the location parameters with one or several controls. Malekzadeh et al. [10] proposed parametric bootstrap simultaneous confidence intervals (PBSCIs) for $\theta_i-\mu_j$, $i=1,\ldots,I;j=1,\ldots,J$, under heteroscedasticity when sample sizes are unequal. They showed theoretically that their SCIs have correct CP asymptotically; however, they did not investigate the CP of PBSCIs for small samples theoretically. Malekzadeh et al. [10] also compared the CP of PBSCIs with SFGCIs via simulation studies and concluded that PBSCIs outperform the SFGCIs for sample size as small as 10.

These proposed methods are based on intensive computer simulation or Lam's [6, 7] technique which leads to SCIs that are too conservative. In this paper, we present new one-stage and two-stage SCIs for $\theta_i-\mu_j$, $i=1,\ldots,I$; $j=1,\ldots,J$, under heteroscedasticity when sample sizes are possibly unequal. We also discuss constructing SCIs for comparing the treatment groups with the best control. Our proposed method, in contrast to PBSCIs and SFGCIs, is not a simulation-based inference procedure and it is very easy to implement. In addition, our method is not as conservative as the methods that are based on Lam's technique and consequently our SCIs have shorter lengths. The organization of the paper is as follows. In Section 2, we present some necessary backgrounds, notation and a useful lemma for constructing SCIs. In Sections 3 and 4, SCIs under two-stage and one-stage sampling procedures are presented. In Section 5, we discuss constructing SCIs for comparing the treatment groups with the best control. In Section 6, via a simulation study, we compare the proposed SCIs with the ones that

were proposed by [12], [5], and [10] in terms of CP and average volume (AV). Simulation results show that the proposed SCIs have higher CP and smaller AV in most cases. In Section 7, we illustrate our method with a real example that is about two test drugs and two control drugs used in the treatment of Leukemia, as measured by the duration of remission time. All proofs are given in the Appendix.

2. Notation and necessary backgrounds

Let $X_{i1},\ldots,X_{im_i}$ and $Y_{j1},\ldots,Y_{jn_j}$ denote random samples of sizes m_i and n_j from the i-th treatment and j-th control, respectively. Let $X_i=\min\{X_{i1},\ldots,X_{im_i}\}$, $Y_j=\min\{Y_{j1},\ldots,Y_{jn_j}\}$, $S_{X_i}=\sum_{t=1}^{m_i}(X_{it}-X_i)/(m_i-1)$, $S_{Y_j}=\sum_{t=1}^{n_j}(Y_{jt}-Y_j)/(n_j-1)$, $C_{X_i}=S_{X_i}/m_i$, $C_{Y_j}=S_{Y_j}/n_j$, and

$$C=max\left\{\max_{i=1,\ldots,I}\{S_{X_i}/m_i\},\max_{j=1,\ldots,J}\{S_{Y_j}/n_j\}\right\}.$$

In addition, let $W_{X_i}=m_i(X_i-\theta_i)/S_{X_i}$ and $W_{Y_j}=n_j(Y_j-\mu_j)/S_{Y_j}$, $i=1,\ldots,I;j=1,\ldots,J$. It is well-known that W_{X_i}'s and W_{Y_j}'s are independently distributed as F distributions with 2 and $2m_i-2$ and 2 and $2n_j-2$ degrees of freedom, respectively; see [15] and [12]. Let $F^{-1}_{2,2n-2}(x)=(n-1)\left\{(1-x)^{-\frac{1}{n-1}}-1\right\}$, for $x\in(0,1)$, generally denote the quantile function of a random variable whose distribution is F distribution with 2 and $2n-2$ degrees of freedom.

The two-stage sampling scheme can be described as follows. Initial random samples of sizes m_i and n_j are taken from the i-th treatment and the j-th control group, respectively. Then random quantities

$$M_i=\max\left\{m_i,\left[\frac{S_{X_i}}{d}\right]+1\right\}\quad\text{and}\quad N_j=\max\left\{n_j,\left[\frac{S_{Y_j}}{d}\right]+1\right\},$$

$i=1,\ldots,I;j=1,\ldots,J$, are calculated where the constant $d>0$ is given and used to control the length of the confidence intervals for $\theta_i-\mu_j$ and $[x]$ denotes the largest integer smaller than or equal to x; see [8]. If $m_i<M_i$ ($n_j<N_j$), then M_i-m_i (N_j-n_j) additional observations $X_{im_i+1},\ldots,X_{iM_i}$ ($Y_{jn_j+1},\ldots,Y_{jN_j}$) are taken from the i-th (j-th) population. Let

$$\tilde{X}_i=\begin{cases}\min\{X_i,X_{im_i+1},\ldots,X_{iM_i}\} & \text{if}\quad m_i<M_i\\ X_i & \text{if}\quad m_i=M_i,\end{cases}$$

$$\tilde{Y}_j=\begin{cases}\min\{Y_j,Y_{jn_j+1},\ldots,Y_{N_j}\} & \text{if}\quad n_j<N_j\\ Y_j & \text{if}\quad n_j=N_j,\end{cases}$$

and $\tilde{C}_{X_i}=S_{X_i}/M_i$, $\tilde{C}_{Y_j}=S_{Y_j}/N_j$, $\tilde{W}_{X_i}=M_i(\tilde{X}_i-\theta_i)/S_{X_i}$ and $\tilde{W}_{Y_j}=N_j(\tilde{Y}_j-\mu_j)/S_{Y_j}$, $i=1,\ldots,I;j=1,\ldots,J$. Note that $\tilde{W}_{X_i}$'s are distributed as F distribution with 2 and $2m_i-2$ degrees of freedom and $\tilde{W}_{Y_j}$'s are also distributed as F distribution 2 and $2n_j-2$ degrees of freedom; see [12]. In addition, $\tilde{W}_{X_i}$'s and $\tilde{W}_{Y_j}$'s are independent.

The following lemma is useful for constructing SCIs for $\theta_i-\mu_j$; see [4]. We use this lemma for constructing SCIs under different sampling schemes.

2.1. Lemma. *Suppose that X and Y are two positive random variables and a, b, and d are three positive constants, then $[Y\le d]\subseteq[aX\ge bY-bd]$.*

3. SCIs under two-stage sampling procedure

In this section, we present a set of new two-stage SCIs under heteroscedasticity when sample sizes are possibly unequal. The following theorem shows how the additional information in the second stage of sampling can be utilized to construct SCIs for $\theta_i - \mu_j$.

3.1. Theorem. *Suppose that $\alpha \in (0,1)$ is given. Then*
(a) $(-\infty,\ \tilde{X}_i - \tilde{Y}_j + \tilde{C}_{Y_j} q_j^u)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous upper bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_j^u = F^{-1}_{2,2n_j-2}\left((1-\alpha)^{\frac{1}{J}}\right)$.
(b) $(\tilde{X}_i - \tilde{Y}_j - \tilde{C}_{X_i}\ q_i^l, \infty)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous lower bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_i^l = F^{-1}_{2,2m_i-2}\left((1-\alpha)^{\frac{1}{I}}\right)$.
(c) $(\tilde{X}_i - \tilde{Y}_j - \tilde{C}_{X_i} q_i^t, \tilde{X}_i - \tilde{Y}_j + \tilde{C}_{Y_j} d_j^t)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous two-sided bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_i^t = F^{-1}_{2,2m_i-2}\left((1-\alpha)^{\frac{1}{I+J}}\right)$ and $d_j^t = F^{-1}_{2,2n_j-2}\left((1-\alpha)^{\frac{1}{I+J}}\right)$.

3.2. Remark. When $m_i = n_j = n$ for all i and j, [15] and [12] also proposed two-stage SCIs for $\theta_i - \mu_j$. However, they used d in their SCIs instead of $\tilde{C}_{X_i}$ and $\tilde{C}_{Y_j}$. It is clear that the proposed two-stage SCIs performs better than the SCIs proposed by [15] and [12, 11] since $d \geq \max_{i=1,\ldots,I} \tilde{C}_{X_i}$ and $d \geq \max_{j=1,\ldots,J} \tilde{C}_{Y_j}$.

Note that the additional sample for the second stage may not be available due to an experimental budget shortage; see [12]. In this view, we do not discuss any more the SCIs that are based on a two-stage sampling procedure. In the next section, we discuss constructing SCIs under one-stage sampling scheme when sample sizes and scale parameters are possibly unequal.

4. SCIs under one-stage sampling procedure

Suppose that the additional sample for the second stage is not available and random samples of sizes m_i and n_j are taken from the i-th treatment and j-th control, respectively. The following theorem presents a set of one-stage SCIs for $\theta_i - \mu_j$.

4.1. Theorem. *Suppose that $\alpha \in (0,1)$ is given. Then*
(a) $(-\infty,\ X_i - Y_j + C_{Y_j} q_j^u)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous upper bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_j^u = F^{-1}_{2,2n_j-2}\left((1-\alpha)^{\frac{1}{J}}\right)$.
(b) $(X_i - Y_j - C_{X_i}\ q_i^l, \infty)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous lower bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_i^l = F^{-1}_{2,2m_i-2}\left((1-\alpha)^{\frac{1}{I}}\right)$.
(c) $(X_i - Y_j - C_{X_i} q_i^t, X_i - Y_j + C_{Y_j} d_j^t)$, $i = 1,\ldots,I; j = 1,\ldots,J$, are conservative simultaneous two-sided bounds for $\theta_i - \mu_j$, $i = 1,\ldots,I; j = 1,\ldots,J$, with confidence coefficient $1-\alpha$ where $q_i^t = F^{-1}_{2,2m_i-2}\left((1-\alpha)^{\frac{1}{I+J}}\right)$ and $d_j^t = F^{-1}_{2,2n_j-2}\left((1-\alpha)^{\frac{1}{I+J}}\right)$.

4.2. Remark. Suppose that $m_i = n_j = n$ for all i and j. In this case, the proposed SCIs in Theorem 4.1 are similar to ones proposed by [15] and [12]. The difference is that they used C in their SCIs instead of C_{X_i} and C_{Y_j}. It is clear that the proposed SCIs outperform the SCIs proposed by [15] and [12] since C_{X_i} and C_{Y_j} are smaller than C for all i and j.

4.3. Remark. By using Lemma 2.1, the two-sided SCIs proposed by [11] can be improved similarly. However, we do not discuss it any more since we are interested in constructing both one-sided and two-sided SCIs.

5. SCIs for comparing the treatment groups with the best control

Suppose that the best control is defined as a control group that has the largest location parameter. We denote the location parameter of the best control by $\mu_{[J]}$. In practice, it may be of interest to compare the treatment groups with the best control. In the following theorem, we present one-stage SCIs for $\theta_i - \mu_{[J]}$, $i = 1, \ldots, I$, under heteroscedasticity when sample sizes are possibly unequal.

5.1. Theorem. *Suppose that $0 < \alpha < 1$ is given. Then*
(a) $\left(-\infty, X_i - \min_{j=1,\ldots,J}\{Y_j\} + \min_{j=1,\ldots,J}\left\{C_{Y_j}\, q_j^u\right\}\right)$, $i = 1, \ldots, I$, *are conservative simultaneous upper bounds for* $\theta_i - \mu_{[J]}$, $i = 1, \ldots, I$, *with confidence coefficient* $1 - \alpha$.
(b) $\left(X_i - \max_{j=1,\ldots,J}\{Y_j\} - C_{X_i} q_i^l, \infty\right)$, $i = 1, \ldots, I$, *are conservative simultaneous lower bounds for* $\theta_i - \mu_{[J]}$, $i = 1, \ldots, I$, *with confidence coefficient* $1 - \alpha$.
(c) $\left(X_i - \max_{j=1,\ldots,J}\{Y_j\} - C_{X_i} q_i^t, X_i - \min_{j=1,\ldots,J}\{Y_j\} + \min_{j=1,\ldots,J}\left\{C_{Y_j} d_j^t\right\}\right)$, $i = 1, \ldots, I$, *are conservative simultaneous two-sided bounds for* $\theta_i - \mu_{[J]}$, $i = 1, \ldots, I$, *with confidence coefficient* $1 - \alpha$.

5.2. Remark. We note that Maurya et al. [12] proposed SCIs for comparing the treatment groups with the best control when sample sizes are equal. However, similar to the Remarks 3.2 and 4.2, one can see that our proposed SCIs in Theorem 5.1 perform better than the SCIs proposed by [12].

6. Simulation studies

In this section, we present the results of simulation studies in which we compare our two-sided one-stage SCIs in terms of CP and AV with the ones that were proposed by [12], [5], and [10]. We use abbreviations M11, SFGCI, and PBSCI for these three methods. (A similar procedure can be used for examining and comparing one-sided SCIs.) Note that in M11, it was assumed that sample sizes are equal; however, our SCIs, SFGCI and PBSCI methods can be used even when sample sizes are unequal. In this regard, we compare the methods under two scenarios that sample sizes are equal or unequal. We first review the methods briefly.

M11. When sample sizes are equal; i.e. $m_i = n_j = n$, $i = 1, \ldots, I; j = 1, \ldots, J$, Maurya et al. [12] used Lam's [6, 7] technique and showed that

$$(X_i - Y_j - Cq_t, X_i - Y_j + Cq_t), i = 1, \ldots, I; j = 1, \ldots, J,$$

are conservative simultaneous two-sided bounds for $\theta_i - \mu_j$, $i = 1, \ldots, I; j = 1, \ldots, J$, with confidence coefficient $1-\alpha$ where $q_t = F_{2,2n-2}^{-1}\left((1-\alpha)^{\frac{1}{I+J}}\right)$. Note that when there is only one control group (i.e. $J = 1$), the above SCIs reduce to the SCIs proposed by [15].

SFGCI. This procedure is based on the concept of fiducial generalized pivotal quantities (FGPQ); see [2, 3] and [1]. Let X_i^*, Y_j^*, $S_{X_i}^*$, and $S_{Y_j}^*$ denote independent copies of X_i, Y_j, S_{X_i}, and S_{Y_j}, and let

$$R_{ij} = X_i - \frac{S_{X_i}}{S_{X_i}^*}\left(X_i^* - \theta_i\right) - Y_j + \frac{S_{Y_j}}{S_{Y_j}^*}\left(Y_j^* - \mu_j\right).$$

Now, let

$$T^F = \max_{1 \le i \le I; 1 \le j \le J} \left|(X_i - Y_j - R_{ij})/\sqrt{V_{ij}}\right|,$$

where V_{ij}'s are

$$V_{ij} = \frac{(m_i - 1)S_{X_i}^2}{m_i^3} + \frac{(n_j - 1)\, S_{Y_j}^2}{n_j^3}. \tag{6.1}$$

Then $100(1-\alpha)\%$ two-sided SFGCIs for $\theta_i - \mu_j$ are

$$X_i - Y_j \pm q_\alpha^F \sqrt{V_{ij}}, \quad i = 1, \ldots, I; j = 1, \ldots, J,$$

where q_α^F is the $(1-\alpha)$-th quantile of the conditional distribution of T^F given the observed values. The value of q_α^F is obtained by a Monte Carlo procedure. For more details, see [5].

PBSCI. Let s_{x_i} and s_{y_j} denote the observed values of S_{X_i} and S_{Y_j} and let

$$T^B = \max_{1 \le i \le I, 1 \le j \le J} \left| (X_i^B - Y_j^B)/\sqrt{V_{ij}^B} \right|,$$

where

$$V_{ij}^B = \frac{(m_i - 1)\left(S_{X_i}^B\right)^2}{m_i^3} + \frac{(n_j - 1)\left(S_{Y_j}^B\right)^2}{n_j^3},$$

in which X_i^B and Y_j^B are taken from $(s_{x_i}/m_i)\, E(0,1)$ and $(s_{y_j}/n_j)E(0,1)$, respectively, and

$$S_{X_i}^B \sim \frac{s_{x_i}}{(2m_i - 2)}\chi_{2m_i-2}^2 \quad \text{and} \quad S_{Y_j}^B \sim \frac{s_{y_j}}{(2n_j - 2)}\chi_{2n_j-2}^2,$$

where generally χ_n^2 denotes a chi-square distribution with n degrees of freedom. Then $100(1-\alpha)\%$ two-sided PBSCI for $\theta_i - \mu_j$ are

$$X_i - Y_j \pm q_\alpha^B \sqrt{V_{ij}}, \quad i = 1, \ldots, I;\ j = 1, \ldots, J,$$

where q_α^B denotes the $(1-\alpha)$-th quantile of the distribution of T^B and V_{ij} is given in (6.1). For more details see [10].

6.1. When sample sizes are equal. We used the following procedure to estimate the CPs and AVs of the methods when $m_i = n_j = n$, $i = 1, \ldots, I; j = 1, \ldots, J$. Let $(\boldsymbol{\mu}, \boldsymbol{\theta})$ and $(\boldsymbol{\sigma}, \boldsymbol{\delta})$ denote the vector of location and scale parameters, respectively. To avoid any bias and for a realistic comparison, we decided to generate the values of $(\boldsymbol{\sigma}, \boldsymbol{\delta})$ at random from a known distribution. We refer to a "case "for a generated value of $(\boldsymbol{\sigma}, \boldsymbol{\delta})$. For given $(\boldsymbol{\sigma}, \boldsymbol{\delta})$ and n, random samples of sizes n are generated from $E(\boldsymbol{\mu}, \boldsymbol{\sigma})$ and $E(\boldsymbol{\theta}, \boldsymbol{\delta})$. For each method, the two-sided SCIs are obtained and the results whether or not all the values of $\theta_i - \mu_j$ fall in their corresponding confidence intervals are recorded. The $(I \times J)$-th root of volumes is calculated. This process is repeated 5000 times and the fraction of times that all $\theta_i - \mu_j$ are in their corresponding SCIs is calculated as an estimate of the true CP. In addition, the average of the $(I \times J)$-th root of volumes is calculated as AV. Finally, the whole process is repeated for M cases.

The results are shown in Figure 1 for $I = 3$ and $J = 2$, $n = 3, 5, 10$, and 20, $1 - \alpha = 0.95$, $M = 100$ and when the values of scale parameters are generated from chi-square distributions with two degrees of freedom. In addition, the descriptive statistics of the AV of methods have been shown in Table 1. In this simulation study, we assumed that $\boldsymbol{\mu} = \boldsymbol{\theta} = \mathbf{0}$ without loss of generality and for SFGCI and PBSCI methods, we used 50,000 samples to obtained the required quantiles q_α^B and q_α^F.

Note that an estimated CP is a normal random variable approximately with mean $1-\alpha$ and variance $\alpha(1-\alpha)/5000$. Thus a 95% lower bound for the true CP $(1 - \alpha = 0.95)$ can be obtained as 0.9449. In Figure 1, this lower bound for the true CP has been also drawn. If the estimated CP of a method is less than this lower bound, we can conclude that the CP of that method is less than the true CP significantly.

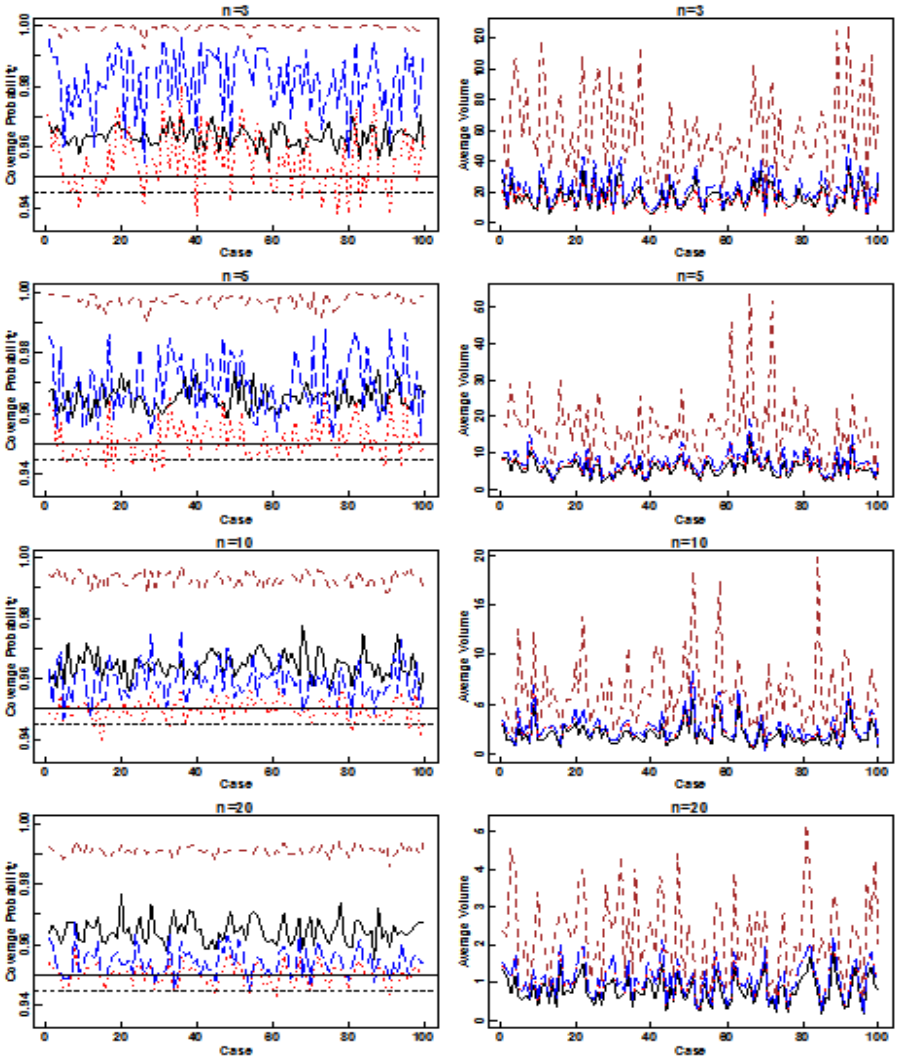
The following conclusions can be drawn from Figure 1 and Table 1:

(a) M11 has the highest CP and AV for different cases. This method is too conservative and has the worst performance in terms of AV.

(b) SFGCI performs better than M11 in terms of AV. The AV of SFGCI is larger than our method and PBSCI. The AV of SFGCI is close to that of PBSCI for large sample size ($n = 20$). When sample size is small ($n = 3$ and 5), the SFGCI method is conservative. However, it is not as conservative as M11 method. The CP of SFGCI is higher than our method when $n = 3$ and they are competitive when $n = 5$. For moderate and large sample sizes ($n = 10$ and 20), our method has higher CP than SFGCI. The CP of SFGCI tends to 0.95 when sample size increases; see [5].

(c) The AV of PBSCI is much less than M11. Although there are cases that the AV of PBSCI is less than the AV of our method (see Table 1), there are several cases that the CP of PBSCI is significantly less than 0.95 (especially when sample sizes are small i.e. $n = 3$ and 5).There are a few cases that the CP of PBSCI is significantly less than 0.95 for large sample size (i.e. n =20). When sample size increases, our method has higher CP than PBSCI and the number of cases that the AV of our method is less than PBSCI increases. It seems that PBSCI method is somewhat liberal for some cases (especially for small sample sizes).

Figure 1. The CPs and AVs of our (solid line), M11 (dashed line), SFGCI (long dashed line), and PBSCI (dotted line) methods when the scale parameters are generated from chi-square distributions with two degrees of freedom.



(d) The AV of our proposed method is much less than M11 and it is smaller than the AV of SFGCI method (see Table 1). Therefore, our method performs better than M11 and SFGCI in terms of AV. The CP of our method is higher than 0.95 in contrast to PBSCI when $n = 3$ and $n = 5$ and it has smaller AV than PBSCI for most cases when $n = 5, 10$ and 20. The CP of our method varies between 0.96 and 0.97 approximately for all cases and sample sizes that we considered here.

We also compared the methods when the values of $\boldsymbol{\sigma}$ and $\boldsymbol{\delta}$ are generated from chi-square distributions with two and five, five and two degrees of freedom, respectively, and

Table 1. The descriptive statistics of the AV of methods.

n	Methods	Min	Max	Mean	S.D.	Number of cases that the AV of our method is less than
3	**M11**	14.80	127.70	58.81	26.22	100
	SFGCI	6.41	50.40	22.07	9.85	100
	PBSCI	4.60	33.93	15.08	6.29	26
	New	5.07	37.14	16.86	7.99	-
5	**M11**	5.24	54.25	17.04	8.36	100
	SFGCI	2.56	18.98	7.50	3.00	100
	PBSCI	2.35	15.36	6.48	2.58	76
	New	1.68	14.41	5.69	2.35	-
10	**M11**	0.80	19.78	16.19	3.50	100
	SFGCI	0.48	8.43	2.71	1.41	100
	PBSCI	0.43	8.10	2.56	1.35	96
	New	0.39	5.70	2.03	1.09	-
20	**M11**	0.36	5.22	2.21	1.04	100
	SFGCI	0.24	2.26	1.08	0.48	100
	PBSCI	0.24	2.22	1.06	0.47	99
	New	0.17	1.85	0.81	0.38	-

from a half-normal distribution with mean two (the results are not shown here). The results were similar to Figure 1.

6.2. When sample sizes are unequal. Simulation results in the previous section showed that the AV of our and PBSCI methods are less than those of other methods. Therefore, we decided to compare our method with PBSCI when sample sizes are unequal. We used the same procedure as in Section 6.1 to estimate the CPs and AVs of methods. The difference is that the values of sample sizes are generated from a truncated Poisson distribution at zero and one points with parameter λ in each case. The results for $\lambda = 3$, 10, and 20 are shown in Figure 2 and Table 2. It is seen that although the AV of PBSCI is less than our method in some cases when λ is small (sample sizes are small), the CP of PBSCI is less than 0.95 significantly in most cases (54% of cases). In addition, there are cases that the CP of PBSCI is less than 0.95 significantly when λ is moderate and large (28% and 6% of cases) while the CP of our method is higher than 0.95 in all cases. Therefore, it seems that PBSCI is not a reliable method when sample sizes are small which is commonly occurred in practice. It is worth mentioning that our method not only have higher CP than PBSCI but also its AV is smaller than PBSCI for $\lambda = 10$ and 20.

Figure 2. The CPs and AVs of our (solid line) and PBSCI (dashed line) methods when the scale parameters are generated from chi-square distributions with two degrees of freedom and sample sizes are generated from a truncated Poisson distribution at zero and one points with $\lambda = 3, 10$, and 20.

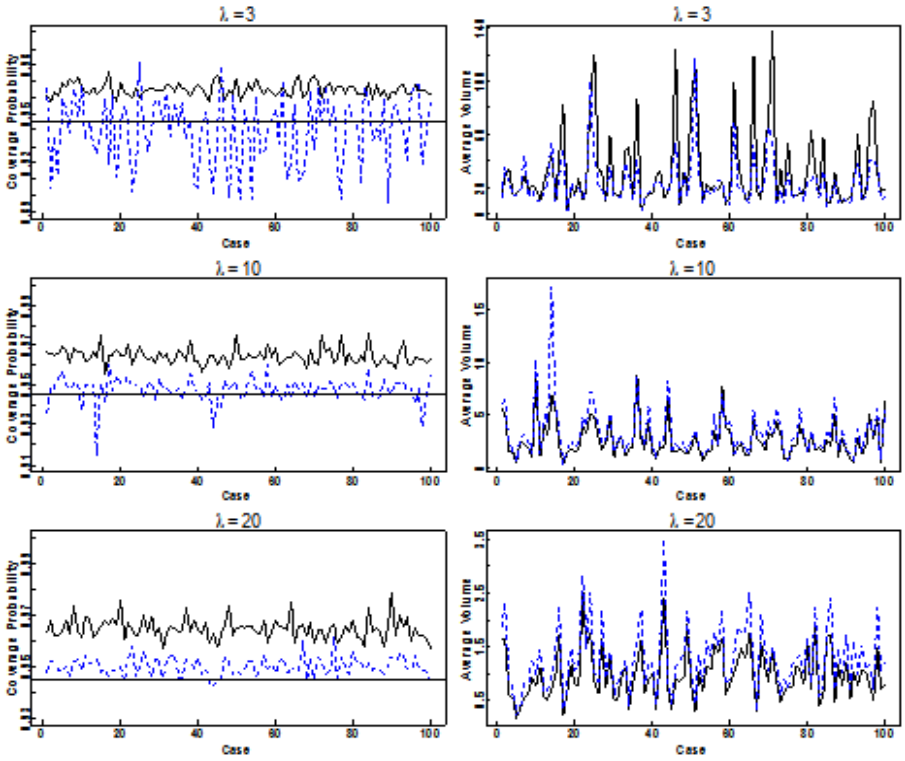


Table 2. The descriptive statistics of the AV of our and PBSCI methods for λ=3, 10, and 20.

λ	Methods	Min	Max	Mean	S.D.	Number of cases that the AV of our method is less than
3	**PBSCI**	1.91	117.50	23.78	18.42	33
	New	1.96	136.90	32.62	29.73	-
10	**PBSCI**	0.26	17.28	3.28	2.39	96
	New	0.23	8.73	2.60	1.77	-
20	**PBSCI**	0.21	3.52	1.26	0.61	98
	New	0.17	2.52	0.98	0.47	-

It is clear that in practice a desirable set of SCIs must have the *highest* CP and the *lowest* lengths. In this view, it seems that our method has some advantages over the other methods. The CP of our method is higher than the nominal CP for all sample sizes and scale parameters (see Theorem 4.1) in contrast to PBSCI method (especially when sample size is small). Our simulation results show that our method is not as conservative

as M11 method and consequently has smaller AV; see remark 4.2. Although the CP of SFGCI is higher than our method for small sample sizes, our method has smaller AV. For moderate and large sample size, our method has higher CP and smaller AV than SFGCI. The CP of our method is higher than PBSCI and the AV of our method is smaller than the AV of PBSCI in most cases. In addition, our method is not a simulation-based method in contrast to SFGCI and PBSCI methods and it is very easy to implement. Totally, it seems that our method can be recommended for multiple comparisons of exponential location parameters with several controls since our method performs better than the other methods in terms of either CP or AV.

7. Example

Consider the data in Table 3 that is taken from [12] and [16]. They are representing the effectiveness of two test drugs and two control drugs used in the treatment of Leukemia, as measured by the duration of remission time. Maurya et al. [12] showed that each of the four groups of data is exponentially distributed.

Table 3. Remission duration by four drugs.

Test Drug 1	Test Drug 2	Control Drug 1	Control Drug 2
1.034	5.115	2.214	4.158
2.344	4.498	4.976	4.025
1.266	4.617	8.154	5.170
1.563	4.651	2.686	11.909
1.169	4.533	2.271	4.912
4.118	4.513	3.139	4.629
1.013	7.641	2.214	3.955
1.509	5.971	4.480	6.735
1.109	12.130	8.847	3.140
1.965	4.699	2.239	12.446
5.136	4.914	3.473	8.777
1.533	17.169	2.761	6.321
1.716	5.497	2.833	3.256
2.778	11.332	2.381	8.250
2.546	18.922	3.548	3.759
2.626	13.712	2.414	5.205
3.413	6.309	2.832	3.071
1.929	10.086	5.551	3.147
2.061	9.293	3.376	9.773
2.951	11.787	2.968	10.218

Maurya et al. [12] and Malekzadeh et al. [10] previously analyzed this data set. Our two-sided SCIs for comparing the test drugs with the control drugs are shown in Table 4. For comparisons, the results for PBSCI and SFGCI methods are also shown in this table. It is seen that our conclusion with confidence coefficients 0.95 and 0.975 is the same as M11, PBSCI and SFGCI methods. With respect to 99% SCIs, our conclusion is the same as 99% SFGCIs: test drug 1 is worse than "both"the control drugs and only test drug 2 is better than control drug 1. It is seen that the lengths of our two-sided SCIs are typically shorter than the other methods as reflected in the values of volume.

Table 4. The results of two-sided SCIs for our, M11, PBSCI, and SFGCI methods for comparing the test drugs with the control drugs.

	Our		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\alpha = 0.025$	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_1$	(-1.505,-0.826)	(-1.560,-0.757)	(-1.637,-0.663)
$\theta_2 - \mu_1$	(1.285,2.659)	(1.101,2.728)	(0.850,2.823)
$\theta_1 - \mu_2$	(-2.362,-1.265)	(-2.417,-1.120)	(-2.494,-0.920)
$\theta_2 - \mu_2$	(0.428,2.220)	(0.244,2.365)	(-0.007,2.565)
Volume	1.833	3.598	7.776
	PBSCI		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\boldsymbol{\alpha}$**=0.025**	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_1$	(-1.552,-0.851)	(-1.623,-0.779)	(-1.718,-0.684)
$\theta_2 - \mu_1$	(1.509,3.059)	(1.352,3.216)	(1.141,3.427)
$\theta_1 - \mu_2$	(-2.675,-1.441)	(-2.799,-1.317)	(-2.967,-1.149)
$\theta_2 - \mu_2$	(0.501,2.354)	(0.313,2.541)	(0.061,2.793)
Volume	2.48	5.20	11.74
	SFGCI		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\boldsymbol{\alpha}$**=0.025**	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_1$	(-1.571,-0.831)	(-1.648,-0.754)	(-1.754,-0.648)
$\theta_2 - \mu_1$	(1.465,3.103)	(1.296,3.272)	(1.062,3.506)
$\theta_1 - \mu_2$	(-2.709,-1.407)	(-2.844,-1.272)	(-3.030, -1.086)
$\theta_2 - \mu_2$	(0.448,2.406)	(0.246,2.608)	(-0.034,2.888)
Volume	3.09	6.56	15.35
	M11		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\boldsymbol{\alpha}$**=0.025**	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_1$	(-2.200,-0.202)	(-2.384,-0.018)	(-2.635,0.233)
$\theta_2 - \mu_1$	(1.285,3.283)	(1.101,3.467)	(0.850,3.718)
$\theta_1 - \mu_2$	(-3.057,-1.059)	(-3.241,-0.875)	(-3.492,-0.624)
$\theta_2 - \mu_2$	(0.428,2.426)	(0.244,2.610)	(-0.007,2.861)
Volume	15.94	31.34	67.66

Our two-sided SCIs for comparing the test drugs with the best control, $\theta_i - \mu_{[2]}$ for $i = 1, 2$, are shown in Table 5. For comparison, the SCIs obtained by Maurya et al. [12] are also shown in this table. It is seen that our conclusion is different from Maurya et al. [12] at confidence coefficient 0.99: our SCIs show that the test drug 1 is worse than the best control in contrast to Maurya's et al. [12] method. It is also seen that our SCIs have much smaller volume than the SCIs that proposed by Maurya et al. [12].

Table 5. The two-sided SCIs for comparing the test drugs with the best control.

	Our		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\boldsymbol{\alpha}$**=0.025**	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_{[2]}$	(-2.362,-1.576)	(-2.417,-1.645)	(-2.494,-1.740)
$\theta_2 - \mu_{[2]}$	(0.428,1.909)	(0.244,1.840)	(-0.007,1.746)
Volume	1.16	1.23	1.32
	[12]		
Parameters	$\boldsymbol{\alpha}$**=0.050**	$\boldsymbol{\alpha}$**=0.025**	$\boldsymbol{\alpha}$**=0.010**
$\theta_1 - \mu_{[2]}$	(-3.058,-0.201)	(-3.242,-0.017)	(-3.494,0.235)
$\theta_2 - \mu_{[2]}$	(0.427,3.284)	(0.243,3.648)	(-0.009,3.720)
Volume	8.16	10.40	13.91

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Appendix. The proofs of theorems.

Proof of Theorem 3.1. We prove only part (a). Part (b) is proved similarly and part (c) is proved by combining parts (a) and (b). For (a), we have

$$\begin{aligned}
P &= \Pr\left(\theta_i - \mu_j \le \tilde{X}_i - \tilde{Y}_j + \tilde{C}_{Y_j} q_j^u, i = 1, \ldots, I; j = 1, \ldots, J\right) \\
&= \Pr\left(\frac{S_{X_i}}{M_i}\tilde{W}_{X_i} \ge \frac{S_{Y_j}}{N_j}\tilde{W}_{Y_j} - \tilde{C}_{Y_j} q_j^u, i = 1, \ldots, I; j = 1, \ldots, J\right) \\
&= \Pr\left(\frac{S_{X_i}}{M_i}\tilde{W}_{X_i} \ge \frac{S_{Y_j}}{N_j}\tilde{W}_{Y_j} - \frac{S_{Y_j}}{N_j} q_j^u,\ i = 1, \ldots, I; j = 1, \ldots, J\right) \\
&= E_{S_{X_1}, \ldots, S_{X_I}, S_{Y_1}, \ldots, S_{Y_J}}\left(\Pr\left\{\frac{S_{X_i}}{M_i}\tilde{W}_{X_i} \ge \frac{S_{Y_j}}{N_j}\tilde{W}_{Y_j} - \frac{S_{Y_j}}{N_j} q_j^u, i = 1, \ldots, I; j = 1, \ldots, J\right\}\right),
\end{aligned}$$

where $E_{X_1,\ldots,X_k}(\cdot)$ generally denotes the expectation over $X_1, \ldots, X_k$. Note that the probability inside the expectation operator is actually conditional probability (given $S_{X_1}, \ldots, S_{X_I}, S_{Y_1}, \ldots, S_{Y_J}$). Since $\tilde{W}_{X_i}$'s and $\tilde{W}_{Y_j}$'s are independently distributed as F distributions with 2 and $2m_i - 2$ and 2 and $2n_j - 2$ degrees of freedom, respectively, by using Lemma 2.1, we have

$$\mathrm{P} \ge \Pr\left(\tilde{W}_{Y_j} \le q_j^u, j = 1, \ldots, J\right) = 1 - \alpha.$$

It completes the proof. □

Proof of Theorem 4.1. The proof is similar to the proof of Theorem 3.1. □

Proof of Theorem 5.1. We prove only part (a). Part (b) is proved similarly and part (c) is proved by combining parts (a) and (b). Note that

$$\theta_i - \mu_{[J]} \le X_i - \min_{j=1,\ldots,J}\{Y_j\} + \min_{j=1,\ldots,J}\left\{C_{Y_j} q_j^u\right\}, i = 1, \ldots, I,$$

if and only if

$$\theta_i - \mu_{[J]} \le X_i - \min_{j=1,\ldots,J}\{Y_j\} + C_{Y_j} q_j^u, i = 1, \ldots, I; j = 1, \ldots, J.$$

For $i = 1, \ldots, I; j = 1, \ldots, J$, let

$$\begin{aligned}
A_{ij} &= \left\{\left(\theta_i, \mu_{[J]}\right) | \theta_i - \mu_{[J]} \in (-\infty, X_i - \min_{j=1,\ldots,J}\{Y_j\} + C_{Y_j} q_j^u)\right\}, \\
C_{ij} &= \left\{(\theta_i, \mu_j) | \theta_i - \mu_j \in (-\infty, X_i - \min_{j=1,\ldots,J}\{Y_j\} + C_{Y_j} q_j^u)\right\}, \\
B_{ij} &= \left\{(\theta_i, \mu_j) | \theta_i - \mu_j \in (-\infty, X_i - Y_j + C_{Y_j} q_j^u)\right\}.
\end{aligned}$$

Since $\mu_{[J]} \ge \mu_j$ and $Y_j \ge \min_{j=1,\ldots,J}\{Y_j\}$ for all j, we have $B_{ij} \subseteq C_{ij} \subseteq A_{ij}$ for all i and j. Now, by using part (a) of Theorem 4.1, the proof is completed. □