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Efficiency of D-optimal designs for quasi-likelihood estimation in Poisson regression model with random effects

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Abstract

Optimum experimental designs are most commonly used to obtain maximum likelihood estimators of parameters. However, obtaining an explicit form of these estimators is not feasible for generalized linear mixed models (GLMMs). Hence as an alternative method to handle this issue, the quasi-likelihood method is applied to Poisson regression models with random effects, a special case of GLMMs. In this paper, we consider this model and compare D-optimal designs for quasi-likelihood estimation and maximum likelihood estimation of fixed effects parameters. The empirical results in a simulated environment suggest that the optimal designs for quasi-likelihood estimation are efficient.

Keywords: Optimal Design, Poisson Regression, Quasi-Likelihood, relative efficiency, Laplace approximation.

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1. Introduction

Many experiment responses are not continuous and can not be described by a linear model with normally distributed errors. If responses are binary or count data, generalized linear models (GLMs), which are described in great detail by McCullagh and Nelder (1989), are established tools to model such data.
The maximum likelihood method can be applied to estimate the parameters in GLMs. As a result, the Fisher information matrix, which equals asymptotically the inverse of the variance-covariance matrix of the maximum likelihood estimator of fixed parameters, can be obtained.
The particular property of GLMs is to assume that all observations are independent of

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each other. Therefore, these models are not appropriate to analyze correlated data structures. In this case, generalized linear mixed models, which extend GLMs by including random effects in the predictors, are the general tools at hand to model the correlated data (McCulloch and Searle, 2001).
However, unlike GLMs, the likelihood function to estimate the fixed parameters of GLMMs can not be obtained explicitly, and hence there is no closed form for the Fisher information matrix.
Despite widespread theoretical and numerical work on GLMMs, there are few results on optimal designs for these models. Waterhouse (2005) has done extensive work on optimal designs for GLMMs. Recently, the quasi-likelihood approach has been applied to find optimal designs for some special cases of GLMMs, called Poisson regression models with random coefficients (Niaparast (2009), Niaparast and Schwabe (2013)).
In this paper we take these special cases of GLMMs into account. The quasi-likelihood approach is applied to this model. McCullagh (1983) has demonstrated that under suitable conditions, quasi-likelihood estimators are efficient.
In the present work, Also using simulation and numerical techniques, we compare the optimal designs for maximum likelihood estimators to these for quasi-likelihood estimators of the fixed effect parameters. To the best of the authors'knowledge, there is no published study on the relative efficiency of D-optimal designs for quasi-likelihood estimators.
In what follows, we will first review Poisson regression models with random effects, information matrices and quasi-information matrices, and discuss designed experiments. Section 3 provides criterion for the measurement relative efficiency of the quasi-likelihood method to obtain D-optimal designs for Poisson regression models with random effects and gives an approximation of the Fisher information matrix. Then in section 4 we obtain the relative efficiency of D-optimal designs for quasi-likelihood estimation for three cases of Poisson regression models with random effects. Finally we conclude with a short discussion of the results.

2. Preliminary

The results of this paper extend those of Niaparast (2009) and Niaparast and Schwabe (2013). We use their notation and results, hence we summarise them here.

2.1. Model. We consider a Poisson regression model with random effects, which can be written as,

$$(2.1) \qquad Y_{ij} \mid \mathbf{b}_i \overset{ind}{\sim} P(\mu_{ij}(\mathbf{b}_i)) \qquad i = 1, \ldots, s \qquad j = 1, \ldots, m_i \qquad \sum_{i=1}^{s} m_i = n,$$

where Y_{ij} stands for the jth observation for individual i at the experimental setting x_{ij} from the experimental region $\mathfrak{X}$.
The conditional mean of Y_{ij} given $\mathbf{b}_i$, $\mu_{ij}(\mathbf{b}_i)$, is specified as an exponential function of x_{ij} and $\mathbf{b}_i$; that is,

$$\mu_{ij}(\mathbf{b}_i) = \exp(\mathbf{f}^\top(x_{ij})\boldsymbol{\beta} + \mathbf{g}^\top(x_{ij})\mathbf{b}_i)$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^\top$ is the p-dimensional vector of fixed effects and $\mathbf{b}_i$ is the q-dimensional vector of random effects for individual i. These are assumed to be independent and identically normally distributed with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Also we suppose that $\mathbf{f} = (f_1, \ldots, f_p)^\top$ and $\mathbf{g} = (g_1, \ldots, g_q)^\top$ are the vectors of known regression functions corresponding to the fixed effect parameters and random effect parameters, respectively. We assume that $\mathbf{f}$ and $\mathbf{g}$ are the same for all individuals. Moreover, we suppose that the random effects are uncorrelated for different individuals.

According to Niaparast and Schwabe (2013), the mean and the variance-covariance structure for the responses of the ith individual are as follows:

$$\begin{aligned} \mathrm{E}(Y_{ij}) &= \exp(\mathbf{f}(x_{ij})^\top \boldsymbol{\beta} + \frac{1}{2}\,\sigma(x_{ij}, x_{ij})) = \mu(x_{ij})\,, \\ \mathrm{Var}\,(Y_{ij}) &= \mu(x_{ij}) + \mu(x_{ij})^2\, c(x_{ij}, x_{ij})\,, \\ \mathrm{Cov}\,(Y_{ij},\, Y_{ij'}) &= \mu(x_{ij})\mu(x_{ij'})\, c(x_{ij},\, x_{ij'}), \end{aligned}$$

where $\sigma(x, x') = \mathbf{g}(x)^\top\, \boldsymbol{\Sigma}\, \mathbf{g}(x')$ and $c(x, x') = \exp\left(\sigma(x, x')\right) - 1$.
Let $\mathbf{Y}_i = (Y_{i1}, \ldots, Y_{im_i})^\top$ be the vector of m_i observations for individual i with mean vector $\mathrm{E}(\mathbf{Y}_{ij}) = \mu(x_{ij})$. Therefore,

$$Var(\mathbf{Y}_i) = \mathbf{V}_i = \mathbf{A}_i + \mathbf{A}_i\mathbf{C}_i\mathbf{A}_i,$$

with $\mathbf{A}_i = diag\{\mu(x_{i1}),\, \ldots\,, \mu(x_{im_i})\}$ and $\mathbf{C}_i = (c(x_{ij}, x_{ij'}))_{j,j'=1,\ldots,m_i}$.
Since that the covariance between observations from different individuals are zero, the variance-covariance of the vector of all observations is a block diagonal matrix

$$\mathbf{V} = Var(\mathbf{Y}) = diag\{\mathbf{V}_1, \cdots, \mathbf{V}_s\}$$

and depends on $\boldsymbol{\beta}$ through the mean vector $\mathrm{E}(\mathbf{Y})$.

2.2. Quasi-information matrix. The log-likelihood for model (2.1) can be obtained as

$$\ell(\boldsymbol{\beta}; \mathbf{y}) = \sum_{i=1}^{s} \log\left(\int \prod_{j=1}^{m_i} \frac{\mu_{ij}(\mathbf{b}_i)^{y_{ij}}}{y_{ij}!}\, \exp(-\mu_{ij}(\mathbf{b}_i))\, \mathrm{P}_{\mathrm{N}(0,\,\boldsymbol{\Sigma})}\,(\mathrm{d}\,\mathbf{b}_i)\right) \tag{2.2}$$

It involves an integration over $\mathbf{b}_i$ with respect to the Normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Unfortunately this integral cannot be simplified further or evaluated in closed form and hence the Fisher information cannot be expressed in closed form either. There are several attempts to find some numerical methods, but there is no guaranteed method to establish stable solutions (see e.g. Davidian and Giltinan (1995) and Pinheiro and Bates (2010)). For a general discussion of appropriateness of various approximations of the Fisher information matrix see Mielke (2012). As an alternative method the quasi-likelihood method can be considered.
Let $\mathbf{Y}$ be a vector of observations with the mean $E(\mathbf{Y}) = \boldsymbol{\mu}(\boldsymbol{\beta})$ and variance-covariance matrix $Var(\mathbf{Y}) = V(\boldsymbol{\mu}(\boldsymbol{\beta}))$ which is related to $\boldsymbol{\mu}(\boldsymbol{\beta})$ through the known variance function $V(.)$.
The quasi-score function to estimate the regression parameters $\boldsymbol{\beta}$ is defined as

$$U(\boldsymbol{\beta}, \mathbf{y}) = \phi^2 D^\top (V(\boldsymbol{\mu}(\boldsymbol{\beta})))^{-1}(\mathbf{y} - \boldsymbol{\mu}(\boldsymbol{\beta}))$$

where D is the partial derivative of the components of $\boldsymbol{\mu}(\boldsymbol{\beta})$ with respect to the entries in $\boldsymbol{\beta}$ and hence it is a function of $\boldsymbol{\beta}$ (McCullagh and Nelder,1989).
$\hat{\boldsymbol{\beta}}_Q$ is called the quasi-likelihood estimator of $\boldsymbol{\beta}$, if $U(\hat{\boldsymbol{\beta}}_Q, \mathbf{y}) = 0$.
Under mild conditions (McCullagh, 1983), we have

$$\hat{\boldsymbol{\beta}}_Q \sim AN(\boldsymbol{\beta}, \mathfrak{M}_{\boldsymbol{\beta}}^{-1})\,,$$

where $\mathfrak{M}_{\boldsymbol{\beta}} = -\mathrm{E}(\frac{\partial}{\partial\boldsymbol{\beta}}U(\boldsymbol{\beta}, \mathbf{y})) = D^\top V^{-1}(\boldsymbol{\mu}(\boldsymbol{\beta}))D$ is the quasi-information matrix. This matrix plays the same role as Fisher information for ordinary likelihood function.
In the model (2.1), the quasi-information can be written as

$$\mathfrak{M}_{\boldsymbol{\beta}} = \mathbf{F}^\top\, (\mathbf{A}^{-1} + \mathbf{C})^{-1}\mathbf{F} = \sum_{i=1}^{s} \mathbf{F}_i^\top\, (\mathbf{A}_i^{-1} + \mathbf{C}_i)^{-1}\mathbf{F}_i\,, \tag{2.3}$$

where $\mathbf{C} = \mathrm{diag}(\mathbf{C}_\mathrm{i})_{\mathrm{i}=1,\ldots,\mathrm{s}}$, $\mathbf{F} = (\mathbf{F}_1^\top, \ldots, \mathbf{F}_s^\top)^\top$ and $\mathbf{A} = \mathrm{diag}(\mu_\mathrm{i}(\boldsymbol{\beta}))_{\mathrm{i}=1,\ldots,\mathrm{s}}$ (Niaparast and Schwabe, 2013).

2.3. Designs. In carrying out an experiment, subjects such as expend and the extend of reliance of the result of experiment, led researchers to design the experiment for getting the best result before doing that.

2.1. Definition. An approximate design for individual i, ξ_i, is a probability measure with finitely many support points $x_{i1}, \dots, x_{it_i}$ and weights $p_{i1}, \dots, p_{it_i}$ which sum up to 1. In other words x_{ij}s are experimental settings and p_{ij} $(j = 1, \dots, t_i)$ are the proportion of individual i which is taken at x_{ij}.
The approximate design for individual i can be represented as

$$\xi_i = \left\{ \begin{array}{ccc} x_{i1} & \dots & x_{it_i} \\ p_{i1} & \dots & p_{it_i} \end{array} \right\}.$$

Based on this definition and to emphasis the design, we can represent equation (2.3) as

$$\mathfrak{M}_{\boldsymbol{\beta}}(\xi_i) = \mathbf{F}_{\xi_i}^{\top} \, (\mathbf{A}_{\xi_i}^{-1} + \mathbf{C}_{\xi_i})^{-1} \mathbf{F}_{\xi_i}.$$

The population design is defined as

$$\zeta = \left\{ \begin{array}{ccc} \xi_1 & \dots & \xi_s \\ w_1 & \dots & w_s \end{array} \right\},$$

where $w_i (i = 1, \dots, s)$ is the proportion of the individuals that have been observed under the individual setting ξ_i so the population quasi-likelihood information matrix will be

$$\mathfrak{M}_{\boldsymbol{\beta}}(\zeta) = \sum w_i \mathfrak{M}_{\boldsymbol{\beta}}(\xi_i).$$

If all individuals are observed under the same individual design ξ, then the quasi-information for population design equals the quasi-information for an individual.
Here, we suppose that all individuals are treated under the same design and hence we can ignore the index i in the experimental settings.

2.2. Definition. ξ^* is D-optimal design if

$$\xi^* = arg \min_{\xi \in \Xi} - \log \det(\mathfrak{M}_{\boldsymbol{\beta}}(\xi)).$$

In other word, ξ^* is a D-optimal design if it achieves the maximum determinant of the quasi-information matrix.

3. Results

Niaparast (2009) and Niaparast and Schwabe (2013) have obtained D-optimal designs for quasi-likelihood estimator of the fixed effect parameters of Poisson regression model with random effects. In this section we are going to measure the relative efficiency of D-optimal designs for quasi-likelihood estimators.

3.1. Lemma. *Suppose that $\hat{\beta}_Q$ is the quasi-likelihood estimator of β. Then*

$$\mathfrak{M}_{\boldsymbol{\beta}}(\xi) \leq \mathbf{I}_{\boldsymbol{\beta}}(\xi) \qquad \forall \xi \in \Xi.$$

Proof. According to the Cramer-Rao inequality, the inverse of the Fisher information matrix for $\boldsymbol{\beta}$ is lower bound for any unbiased estimator of $\boldsymbol{\beta}$. Since $\hat{\boldsymbol{\beta}}_Q$ is an unbiased estimator of $\boldsymbol{\beta}$ asymptotically, then

$$Var_{\hat{\boldsymbol{\beta}}_Q}(\xi) > \mathbf{I}_{\boldsymbol{\beta}}^{-1}(\xi) \qquad \forall \xi \in \Xi.$$

This inequality means $Var_{\hat{\boldsymbol{\beta}}_Q}(\xi)$ is greater than inverse of Fisher information matrix, $\mathbf{I}_{\boldsymbol{\beta}}(\xi)$, in the sense of Loewner ordering (Pukelsheim (1993)).

On the other hand $Var_{\hat{\boldsymbol\beta}_Q}(\xi)$ equals the inverse quasi-information matrix asymptotically (McCullagh, 1983).
Regarding to Pukelsheim (1993),

$$Var_{\hat{\boldsymbol\beta}_Q}(\xi) > \mathbf{I}_{\boldsymbol\beta}^{-1}(\xi) \quad \Rightarrow \quad \mathfrak{M}_{\boldsymbol\beta}(\xi) < \mathbf{I}_{\boldsymbol\beta}(\xi) \quad \forall \xi \in \Xi$$

□

3.2. Corollary. *For any design $\xi \in \Xi$ with $\mathfrak{M}_{\boldsymbol\beta}(\xi) < \mathbf{I}_{\boldsymbol\beta}(\xi)$, we have*

- $\det(\mathfrak{M}_{\boldsymbol\beta}(\xi)) < \det(\mathbf{I}_{\boldsymbol\beta}(\xi))$.
- $tr(\mathfrak{M}_{\boldsymbol\beta}(\xi)) < tr(\mathbf{I}_{\boldsymbol\beta}(\xi))$.

To compare the Fisher information matrix and the quasi-information matrix, we define the $Q - I_{\text{re}}$ criterion by

$$Q - I_{\text{re}} = \left(\frac{\det(\mathfrak{M}_{\boldsymbol\beta}(\xi^*))}{\det(\mathbf{I}_{\boldsymbol\beta}(\xi^*))} \right)^{\frac{1}{p}}$$

where $\mathbf{I}_{\boldsymbol\beta}(\xi^*)$ and $\mathfrak{M}_{\boldsymbol\beta}(\xi^*)$ are the Fisher information matrix and the quasi-information matrix for the same experimental setting of D-optimal designs, respectively.
In fact, $\mathbf{I}_{\boldsymbol\beta}^{-1}(\xi^*)$ and $\mathfrak{M}_{\boldsymbol\beta}^{-1}(\xi^*)$ are the asymptotic variances of $\hat{\boldsymbol\beta}$ based on the quasi-likelihood method and the likelihood method respectively, if ξ^* is the D-optimal design based on both methods. Therefore this criterion measures the difference between two asymptotic variances of two unbiased estimators of $\hat{\boldsymbol\beta}$. The Fisher information matrix for design ξ is given by the $p \times p$ symmetric matrix whose (k, l)-th element is given by the covariance between the first partial derivatives of the log-likelihood with respect to the parameters, i.e. $Cov(\frac{\partial \ell(\boldsymbol\beta;\mathbf{y})}{\partial \beta_k}, \frac{\partial \ell(\boldsymbol\beta;\mathbf{y})}{\partial \beta_l})$ where

$$\text{(3.1)} \quad \frac{\partial}{\partial \beta_k} \ell(\boldsymbol\beta; \mathbf{y}) \quad = \quad \sum_{j=1}^{t} n_j f_k(x_j) y_j - \sum_{j=1}^{t} n_j f_k(x_j) \exp(\mathbf{f}^\top(x_j)\boldsymbol\beta) \frac{\int e^{P_j(\mathbf{b})} \mathrm{d}\,\mathbf{b}}{\int e^{P_0(\mathbf{b})} \mathrm{d}\,\mathbf{b}}$$

Here $n_j = m \cdot p_j$ stand for the number of observations which are taken at x_j,
$P_0(\mathbf{b}) = -\sum_{j=1}^{t} n_j \exp(\mathbf{f}^\top(x_j)\boldsymbol\beta + \mathbf{g}^\top(x_j)\mathbf{b}) + \sum_{j=1}^{t} \mathbf{g}^\top(x_j)\mathbf{b} y_j - \frac{\mathbf{b}^\top \Sigma^{-1} \mathbf{b}}{2}$ and $P_j(\mathbf{b}) = \mathbf{g}^\top(x_j)\mathbf{b} - \sum_{j=1}^{t} n_j \exp(\mathbf{f}^\top(x_j)\boldsymbol\beta + \mathbf{g}^\top(x_j)\mathbf{b}) + \sum_{j=1}^{t} \mathbf{g}^\top(x_j)\mathbf{b} y_j - \frac{\mathbf{b}^\top \Sigma^{-1} \mathbf{b}}{2}$.
Since the relation (3.1) involves two integrals which cannot be expressed explicitly, the Laplace approximation is considered. Therefore relation (3.1) can be represented as follows:

$$\begin{aligned} \frac{\partial}{\partial \beta_k} \ell(\boldsymbol\beta; \mathbf{y}) \quad &= \quad \sum_{j=1}^{t} n_j f_k(x_j) y_j - \\ & \sum_{j=1}^{t} n_j f_k(x_j) \exp(\mathbf{f}^\top(x_j)\boldsymbol\beta) \left(\frac{-\det(H(\tilde{\mathbf{b}}_0))}{-\det(H(\tilde{\mathbf{b}}_j))} \right)^{\frac{1}{2}} e^{P(\tilde{\mathbf{b}}_j) - P(\tilde{\mathbf{b}}_0)}, \end{aligned}$$

where $\tilde{\mathbf{b}}_0$ and $\tilde{\mathbf{b}}_j$ $(j = 1, \ldots, t)$ are local extrema of the functions $P_0(\mathbf{b})$ and $P_j(\mathbf{b})$, respectively. Also, we have

$$\mathbf{H}(\tilde{\mathbf{b}}) = \sum_{j=1}^{t} (-n_j \mathbf{g}(x_j) \mu_j(\tilde{\mathbf{b}}) \mathbf{g}^\top(x_j)) - \Sigma^{-1}.$$

4. Simulation

In the following we will measure the relative efficiency of the quasi-likelihood approach for three special cases of Poisson regression model with random effects. Note that in these cases D-optimal designs for quasi-estimators of parameters have been obtained by Niaparast (2009) and Niaparast and Schwabe (2013).

In practice, we obtain ξ^* for the maximum quasi-likelihood estimation of parameters, and then we simulate the Fisher information matrix under the condition that ξ^* be also the D-optimal design for maximum likelihood estimation of $\boldsymbol{\beta}$. Then we calculate $Q-I_{\text{re}}$

(1) Simple Poisson regression model with random intercept (SMI):
This model is obtained by assuming $\boldsymbol{\beta}^\top = (\beta_0, \beta_1)$, $\mathbf{f}^\top(x_j) = (1, x_j)$, $g(x_j) = 1$ and $var(b) = \sigma^2$. As in Niaparast (2009) and Stufken and Yang (2012) we consider designs with two support points. Table 1 contains $Q-I_{\text{re}}$, $\det(\mathbf{I}_{\boldsymbol{\beta}}(\xi^*))$ and $\det(\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*))$ for some representative values of σ.
Generally, the variance of observations decreases as σ decreases. Therefore the determinant of the Fisher information matrix and the determinant of the quasi information matrix properly decreased. Also, with increasing σ the distance between $\det(\mathbf{I}_{\boldsymbol{\beta}}(\xi^*))$ and $\det(\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*))$ increased, but values of the $Q-I_{\text{re}}$ demonstrated that D-optimal designs for quasi-likelihood estimation of parameters are efficient for different σ.

Table 1. $Q-I_{\text{re}}$ for SMI

	$\beta_0=3,\beta_1=-2,m{=}100$			$\beta_0=3,\beta_1=-5,m{=}100$		
σ	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q-I_{\text{re}}$	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q-I_{\text{re}}$
0.1	13471.22	13415.03	0.998	2149.35	2146.41	0.999
0.2	3805.49	3736.66	0.991	608.53	597.86	0.991
0.3	1772.64	1699.24	0.979	284.61	271.87	0.977
0.4	1042.83	962.89	0.961	167.03	154.06	0.960
0.5	701.52	617.95	0.938	111.66	98.87	0.941
0.6	514.15	428.83	0.913	82.23	68.61	0.913
0.7	405.29	313.99	0.880	64.76	50.23	0.881
0.8	332.84	238.90	0.847	53.61	38.22	0.844
0.9	288.47	186.94	0.805	46.16	29.91	0.805
1	256.24	149.34	0.763	41.36	23.89	0.760

(2) Quadratic Poisson regression model with random intercept (QMI):
It might happen that the effect of the explanatory variables are stronger than that which in SMI describes as the relation between the explanatory variable and response variable. Thus the quadratic model which is a special case of model (2.1) where $\boldsymbol{\beta}^\top = (\beta_0, \beta_1, \beta_2)$, $\mathbf{f}^\top(x_j) = (1, x_j, x_j^2)$, $g(x_j) = 1$ and $var(b) = \sigma^2$. As we mentioned in the simple poisson regression model with random intercept (SMI), we have also considered the saturated designs with three support points. The $Q-I_{\text{re}}$, $\det(\mathbf{I}_{\boldsymbol{\beta}}(\xi^*))$ and $\det(\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*))$ for this model according to different values of σ are listed in table (2).
The results for here are similar to those obtained in the SMI example.

(3) Simple Poisson regression model with random slope (SMS):
An assumption that might sometimes be in contention is whether the effect of the explanatory variable is constant across the different subjects. Contrary to a random intercept model, a random slope model allows the explanatory variable to have a different effect for each individual. This model also arises as a special

Table 2. $Q - I_{\text{re}}$ for QMI

	$\beta_0 = 3,\beta_1 = -5,\beta_2 = -2,m{=}100$			$\beta_0 = 3,\beta_1 = -2,\beta_2 = -5,m{=}100$		
σ	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q - I_{\text{re}}$	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q - I_{\text{re}}$
0.1	1909.96	1903.23	0.999	671.68	671.11	0.999
0.2	537.75	529.39	0.995	193.58	191.06	0.996
0.3	257.35	245.58	0.984	93.42	89.25	0.985
0.4	155.71	143.87	0.974	56.32	52.38	0.976
0.5	109.40	96.45	0.959	39.92	35.14	0.958
0.6	85.18	70.68	0.940	30.80	25.76	0.942
0.7	71.35	55.14	0.918	26.03	20.13	0.918
0.8	63.30	45.27	0.894	23.00	16.51	0.895
0.9	59.73	38.56	0.864	21.56	14.06	0.867
1	58.64	33.87	0.833	20.95	12.35	0.838

case of model (2.1) where $\mu_j(b) = \exp(\beta_0 + \beta_1 x_j + b x_j)$ and $b \sim N(0, \sigma^2)$. For SMS, D-optimal designs are searched among two-point designs (Niaparast and Schwabe, 2013). Table 3 contains three values $Q - I_{\text{re}}$, $\det(\mathbf{I}_{\boldsymbol{\beta}}(\xi^*))$ and $\det(\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*))$.

Table 3. $Q - I_{\text{re}}$ for SMS

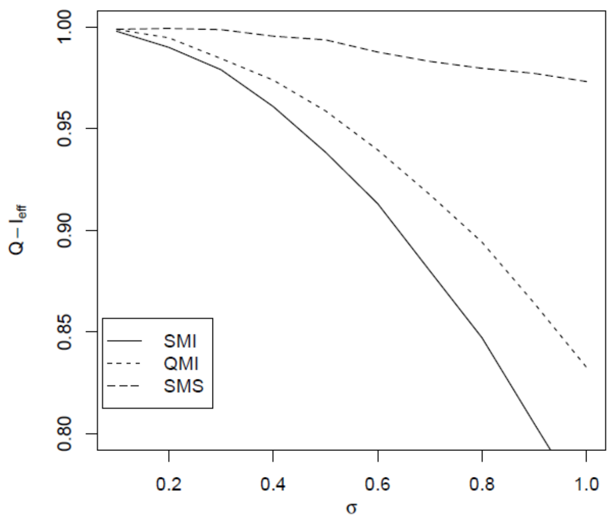
	$\beta_0 = 2,\beta_1 = -3,m{=}100$			$\beta_0 = -2,\beta_1 = -3,m{=}100$		
σ	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q - I_{\text{re}}$	$\lvert\mathbf{I}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$\lvert\mathfrak{M}_{\boldsymbol{\beta}}(\xi^*)\rvert$	$Q - I_{\text{re}}$
0.1	6797.00	6783.35	0.999	2.7489	2.7481	0.999
0.2	4632.30	4625.97	0.999	2.7428	2.7334	0.9982
0.3	3136.28	3128.35	0.999	2.716	2.708	0.998
0.4	2213.48	2193.96	0.995	2.673	2.672	0.999
0.5	1619.13	1599.07	0.994	2.631	2.627	0.999
0.6	1235.64	1205.62	0.988	2.575	2.570	0.999
0.7	968.33	935.92	0.983	2.510	2.500	0.998
0.8	775.98	744.94	0.980	2.428	2.417	0.998
0.9	633.89	605.32	0.977	2.343	2.316	0.994
1	527.87	500.76	0.974	2.233	2.199	0.992

5. Discussion

In this paper we evaluate the relative efficiency of the quasi-likelihood method in obtaining D-optimal designs for Poisson regression models with random effects. For this purpose, the quasi-information matrix was compared with approximations of the Fisher information matrix. To gain the Fisher information matrix of usual methods, the covariance between first partial derivatives of the log-likelihood with respect to parameters must be computed. The likelihood function includes the integral over the product of the probability density functions within any individual. Since these functions vary between zero and one, then their product tends to zero. To solve this problem, the Laplace approximation and weak law of large numbers were used to approximate the Fisher information.

The obtained results demonstrate that the D-optimal designs for quasi-likelihood estimator of parameters are efficient. Figure 1 shows D-optimal designs for quasi-likelihood

Figure 1. $Q - I_{\text{re}}$ for three models, SMI, QMI and RMS



estimators in simple model with random slope are more efficient.
Since the theoretical results can be obtained for quasi-likelihood approach in GLMMs, an interesting subject for further study is to extend the results to other optimality criteria.

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