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Elzaki transform combined with variational iteration method for partial differential equations of fractional order

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Abstract

The idea, which will be communicated through this paper is to make a change to the proposed method by Tarig M. Elzaki [21] and we extend it to solve nonlinear partial differential equations with time-fractional derivative. This document also includes illustrative examples show us how to apply this method, we also show the interest of combining these two methods is the speed of the calculates the terms, and not calculating the Lagrange multipliers.

1. Introduction

The nonlinear differential equations are a type of equations which are difficult to solve with respect to linear differential equations. Therefore, we find that a lot of researchers are working to discover new methods to enable us to solve this kind of equations. These efforts made, which is still ongoing, resulted in the promotion of this research in many methods, among them, we find the homotopy analysis method, Adomian decomposition method, variational iteration method [7, 8, 9] and homotopy perturbation method, which have become known in a large number of researchers in this area. A new option emerged recently, includes the composition of Laplace transform, Sumudu transform, natural transform or Elzaki transform with these methods. Among wich are the Laplace homotopy analysis method [15], homotopy analysis Sumudu transform method [18], modified fractional homotopy analysis transform method [11], Adomian decomposition method coupled with Laplace transform method [16], Sumudu decomposition method for nonlinear equations [5], Elzaki transform decomposition algorithm [13], natural decomposition method [14], variational iteration method coupled with Laplace transform method [4], variational iteration Sumudu transform method [3], Elzaki variational iteration method [21], homotopy perturbation transform method [17], homotopy perturbation Sumudu transform method [10], homotopy perturbation Elzaki transform method [12].
The motivation of this article is to make a change on the method proposed by Elzaki and suggested in [21], and then extend it to solve nonlinear partial differential equations with time-fractional derivative.
The present paper has been organized as follows: In Section 2 some basic definitions of ELzaki transform are montioned. In section 3 we will propose an analysis of the modified method. In section 4 it was presented three examples of application of this method (FEVIM). Finally, the conclusion follows

2. Basic definitions

2.1. Fractional calculus

In this section, we present some basic definitions and properties of fractional calculus [1, 6], and we focus specifically on the definitions of the following concepts: Riemann–Liouville fractional, Caputo fractional derivative, some important results, definition of Elzaki transform and Elzaki transform of fractional derivatives which are used further in this paper.

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Definition 2.1. *Let $\Omega = [\alpha, \beta]$ $(-\infty < \alpha < \beta < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integral $I^{\rho} h$ of order $\rho \in \mathbb{R}$ $(\rho > 0)$ is defined by*

$$\begin{aligned} (I^{\rho} h)(\tau) &= \frac{1}{\Gamma(\rho)} \int_0^{\tau} \frac{h(\zeta) d\zeta}{(\tau - \zeta)^{1-\rho}}, \quad \tau > 0,\ \rho > 0, \\ (I^{0} h)(\tau) &= h(\tau), \end{aligned} \tag{2.1}$$

where $\Gamma(x) = \int_0^{\infty} \tau^{x-1} e^{-\tau} d\tau$, $x > 0$, is called the gamma function of Euler.

Theorem 2.2. *Let $\rho \geqslant 0$ and let $m = [\rho] + 1$. If $h(\tau) \in AC^{m}[\alpha, \beta]$, then the Caputo fractional derivative $({}^{c}D^{\rho}_{0+} h)(\tau)$ exist almost evrywhere on $[\alpha, \beta]$.*
If $\rho \notin \mathbb{N}$, $({}^{c}D^{\rho}_{\tau} h)(\tau)$ is represented by

$$({}^{c}D^{\rho}_{\tau} h)(\tau) = \frac{1}{\Gamma(m-\rho)} \int_0^{\tau} \frac{h^{(m)}(\zeta) d\zeta}{(\tau - \zeta)^{\rho - m + 1}}, \tag{2.2}$$

where $D = \frac{d}{d\tau}$ and $m = [\rho] + 1$.

Proof. (see [1]). □

Remark 2.3. *The time-fractional derivative in the Caputo's sense, is given by*

$$\begin{aligned} ({}^{c}D^{\rho}_{\tau} w)(\varkappa, \tau) &= \frac{\partial^{\rho} w(\varkappa, \tau)}{\partial \tau^{\rho}} \\ &= \begin{cases} \frac{1}{\Gamma(k-\rho)} \int_0^{\tau} (\tau - \zeta)^{k-\rho-1} \frac{\partial^{\rho} w(\varkappa, \zeta)}{\partial \zeta^{\rho}}, & k-1 < \rho < k, \\ \frac{\partial^{k} w(\varkappa, \tau)}{\partial \tau^{k}}, & \rho = k, \end{cases} \end{aligned} \tag{2.3}$$

where $k \in \mathbb{N}^{*}$ and $\rho \in \mathbb{R}^{+}$.
(1) Let $\rho > 0$ and let $m = [\rho] + 1$ for $m \notin \mathbb{N}$, $m = \rho$ for $m \in \mathbb{N}$. If $h(\tau) \in AC^{m}[\alpha, \beta]$, then

$$(I^{\rho}_{0+}\, {}^{c}D^{\rho}_{0+} h)(\tau) = h(\tau) - \sum_{j=0}^{m-1} \frac{h^{(j)}(0)}{j!} \tau^{j}.$$

(2) $(I^{\rho}_{0+} x^{\lambda-1})(\tau) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\rho)} \tau^{\lambda+\rho-1}$, $\rho > 0$, $\lambda > 0$.
(3) $({}^{c}D^{\rho}_{0+} x^{\lambda-1})(\tau) = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\rho)} t^{\lambda-\beta-1}$, $\alpha > 0$, $\beta > m$.
(4) $({}^{c}D^{\rho}_{0+} C)(\tau) = 0$, where C is constant.

2.2. Definitions of Elzaki transform

A new integral transform called Elzaki transform [20] defined for functions of exponential order, is proclaimed. They consider functions in the set G defined by

$$G = \left\{ h(\tau) : \exists Q,\ p_1, p_2 > 0,\ |h(\tau)| < Q e^{\frac{|\tau|}{p_i}},\ if\ \tau \in (-1)^{i} \times [0, \infty) \right\}.$$

Definition 2.4. *If $h(\tau)$ is function defined for all $\tau \geqslant 0$, its Elzaki transform is defined by $E[h]$*

$$E[h(\tau)] = T(s) = s \int_0^{\infty} h(\tau) e^{-\frac{\tau}{s}} d\tau. \tag{2.4}$$

Theorem 2.5. *Elzaki transform amplifies the coefficients of the power series function*

$$h(\tau) = \sum_{n=0}^{\infty} a_n \tau^{n}, \tag{2.5}$$

on the new integral transform "Elzaki transform", given by

$$E[h(\tau)] = T(v) = \sum_{n=0}^{\infty} n! a_n v^{n+2}. \tag{2.6}$$

Theorem 2.6. *Let $h(\tau)$ be in G and Let $T_n(v)$ denote Elzaki transform of nth derivative $h^{(n)}(\tau)$ of $h(\tau)$, then for $n \geq 1$,*

$$T_n(v) = \frac{T(v)}{v^n} - \sum_{j=0}^{n-1} v^{2-n+j} h^{(j)}(0). \tag{2.7}$$

By using the integration by parts, Elzaki transform of partial derivative is given as

$$\begin{aligned} E\left(\frac{\partial h(\varkappa,\tau)}{\partial \tau}\right) &= \frac{1}{v}T(\varkappa,v) - vh(\varkappa,0), \\ E\left(\frac{\partial^2 h(\varkappa,\tau)}{\partial \tau^2}\right) &= \frac{1}{v^2}T(\varkappa,v) - h(\varkappa,0) - v\frac{\partial h(\varkappa,0)}{\partial \tau} \end{aligned} \tag{2.8}$$

2.3. Elzaki transform of fractional derivatives

To give the formula of Elzaki transform of Caputo fractional derivative, we use the Laplace transform formula for the Caputo fractional derivative [6]

$$L\{\, ({}^cD_\tau^\rho h)(\tau); u\} = u^\rho F(u) - \sum_{i=0}^{n-1} s^{\rho-i-1} f^{(i)}(0),$$

where $n-1 < \rho \leqslant n,\, n \in \mathbb{N}^*$.

Theorem 2.7. *[19] Let G defined as above. With Laplace transform $F(u)$, then the Elzaki transform $T(v)$ of $h(\tau)$ is given by*

$$T(v) = vF(\frac{1}{v}).$$

Theorem 2.8. *Suppose $T(v)$ is the Elzaki transform of the function $h(\tau)$ then*

$$E\{\, ({}^cD_t^\rho h)(\tau), v\} = \frac{T(v)}{v^\rho} - \sum_{i=0}^{n-1} v^{i-\rho+2} h^{(i)}(0). \tag{2.9}$$

Proof. (see [2]). □

3. Fractional Elzaki Variational Iteration Method (FEVIM)

The work that we will do in this paragraph, is to make a change to the method proposed in [21], and we extend to solve nonlinear partial differential equations of order ρ, $(\, n-1 < \rho \leqslant n,\, n = 1,2,\ldots)$. For this cause, we consider a general nonlinear partial differential equation with time-fractional derivative

$${}^cD_\tau^\rho Z(\varkappa,\tau) + RZ(\varkappa,\tau) + NZ(\varkappa,\tau) = f(\varkappa,\tau), \tag{3.1}$$

subject to the initial conditions

$$\left[\frac{\partial^{n-1} Z(\varkappa,\tau)}{\partial \tau^{n-1}}\right]_{\tau=0} = g_{n-1}(\varkappa), \tag{3.2}$$

where ${}^cD_\tau^\rho = \frac{\partial^\rho}{\partial \tau^\rho}$ is the Caputo fractional derevative, R is the linear differential operator, N represents the general nonlinear differential operator, and $g(\varkappa,\tau)$ is the source term.
Applying Elzaki transform on both sides of (3.1), we obtain

$$E\left[{}^cD_\tau^\rho Z(\varkappa,\tau)\right] + E\left[RZ(\varkappa,\tau)\right] + E\left[NZ(\varkappa,\tau)\right] = E\left[f(\varkappa,\tau)\right]. \tag{3.3}$$

Depending on the properties of Elzaki transform, the equation (3.3) becomes

$$E\left[Z(\varkappa,\tau)\right] = \sum_{i=0}^{n-1} v^{i+2} g_i(\varkappa) + v^\rho E\left[f(\varkappa,\tau)\right] - v^\rho E\left[RZ(\varkappa,\tau) + NZ(\varkappa,\tau)\right]. \tag{3.4}$$

Operating with the inverse Elzaki transform on both sides of (3.4), we obtain

$$Z(\varkappa,\tau) = K(\varkappa,\tau) - E^{-1}\left(v^\rho E\left[RZ(\varkappa,\tau) + NZ(\varkappa,\tau)\right]\right). \tag{3.5}$$

where $K(\varkappa,\tau) = \sum_{i=0}^{n-1} v^{i+2} g_i(\varkappa) + v^\rho E\left[f(\varkappa,\tau)\right]$.
Applying $\frac{\partial}{\partial t}$ on both sides of (3.5), we have

$$\frac{\partial Z(\varkappa,\tau)}{\partial \tau}+\frac{\partial}{\partial \tau}E^{-1}\left(v^{\rho}E\left[RZ(\varkappa,\tau)+NZ(\varkappa,\tau)\right]\right)-\frac{\partial K(\varkappa,\tau)}{\partial \tau}=0. \tag{3.6}$$

According to the variational iteration method [8], we can construct a correct functional as follows

$$Z_{m+1}(\varkappa,\tau)= Z_m(\varkappa,\tau)-\int_0^{\tau}\left[\frac{\partial Z_m}{\partial \zeta}+\frac{\partial}{\partial \zeta}E^{-1}\left(v^{\rho}E\left[RZ_m+NZ_m\right]\right)-\frac{\partial K}{\partial \zeta}\right]d\zeta. \tag{3.7}$$

Or alternately

$$Z_{m+1}(\varkappa,\tau)=K(\varkappa,\tau)-E^{-1}\left(v^{\rho}E\left[RZ_m(\varkappa,\tau)+NZ_m(\varkappa,\tau)\right]\right). \tag{3.8}$$

Recall that $Z(\varkappa,\tau)=\lim_{m\to\infty}Z_m(\varkappa,\tau)$.
According to the preceding limit, we can obtain the exact solution if it exists, or we obtain an approximate solution for the considered equation.

4. Applications

In the following examples, we'll apply the method proposed in the previous paragraph to solve nonlinear time-fractional partial differential equations.

Example 4.1. *First, we consider the nonlinear time-fractional partial differential equation*

$${}^{c}D_{\tau}^{\rho}Z+ZZ_{\varkappa}-Z_{\varkappa}=0,\ \ 0<\rho\leqslant 1, \tag{4.1}$$

subject to the initial condition

$$Z(\varkappa,0)=\varkappa+1. \tag{4.2}$$

If $\rho=1$, *we obtain*

$$Z_{\tau}+ZZ_{\varkappa}-Z_{\varkappa}=0. \tag{4.3}$$

The exact solution of (4.3), is

$$Z(\varkappa,\tau)=1+\frac{\varkappa}{1+\tau}. \tag{4.4}$$

According to (3.8), we can construct the folowing formula

$$Z_{m+1}(\varkappa,\tau)=\varkappa+1-E^{-1}\left(v^{\rho}E\left[(Z_m-1)(Z_m)_{\varkappa}\right]\right). \tag{4.5}$$

Using the iteration formula (4.5), we get

$$\begin{gathered}Z_0(\varkappa,\tau)=\varkappa+1,\\ Z_1(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)},\\ Z_2(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-\varkappa\frac{\Gamma(2\rho+1)}{\Gamma^2(\rho+1)}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)},\\ Z_3(\varkappa,\tau)=1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-a_1\varkappa\tau^{3\rho}+a_2\varkappa\tau^{4\rho}-a_3\varkappa\tau^{5\rho}\\ +a_4\varkappa\tau^{6\rho}-a_5\varkappa\tau^{7\rho},\\ \vdots\end{gathered} \tag{4.6}$$

where
$a_1=\left[\frac{4}{\Gamma(2\rho+1)}+\frac{1}{\Gamma^2(\rho+1)}\right]\frac{\Gamma(2\rho+1)}{\Gamma(3\rho+1)}$, $a_2=\left[\frac{4}{\Gamma(\rho+1)\Gamma(2\rho+1)}+\frac{2\Gamma(2\rho+1)}{\Gamma^2(\rho+1)\Gamma(3\rho+1)}\right]\frac{\Gamma(3\rho+1)}{\Gamma(4\rho+1)}$,
$a_3=\left[\frac{2\Gamma(2\rho+1)}{\Gamma^3(\rho+1)\Gamma(3\rho+1)}+\frac{4}{\Gamma^2(2\rho+1)}\right]\frac{\Gamma(4\rho+1)}{\Gamma(5\rho+1)}$, $a_4=\frac{4}{\Gamma^2(\rho+1)\Gamma(3\rho+1)}\times\frac{\Gamma(5\rho+1)}{\Gamma(6\rho+1)}$,
$a_5=\frac{\Gamma^2(2\rho+1)}{\Gamma^4(\rho+1)\Gamma^2(3\rho+1)}\times\frac{\Gamma(6\rho+1)}{\Gamma(7\rho+1)}$.
Recall that, the exact solution of Eq.(4.1) is calculated by

$$Z(\varkappa,\tau)=\lim_{m\to\infty}Z_m(\varkappa,\tau).$$

From (4.6), the approximate solution of (4.1), is

$$\begin{aligned} Z(\varkappa,\tau) &= 1+\varkappa-\varkappa\frac{\tau^{\rho}}{\Gamma(\rho+1)}+2\varkappa\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}-a_1\varkappa\tau^{3\rho}+a_2\varkappa\tau^{4\rho}-a_3\varkappa\tau^{5\rho} \\ &\quad +a_4\varkappa\tau^{6\rho}-a_5\varkappa\tau^{7\rho}\cdots, \end{aligned}$$

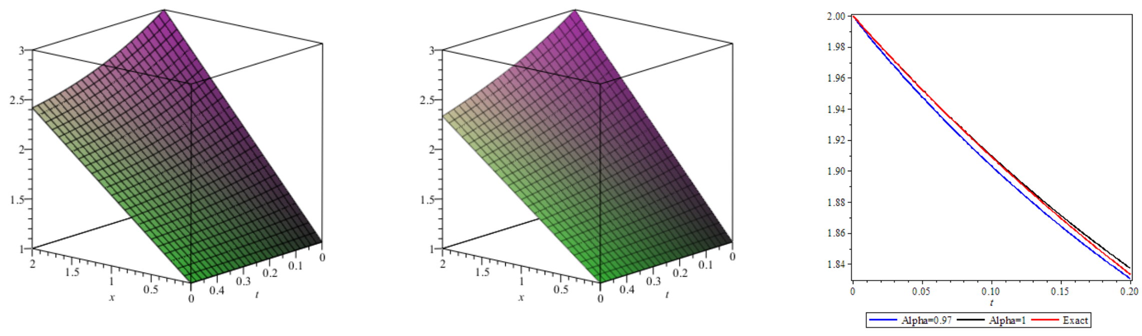
and in the special case $\rho = 1$, is

$$Z(\varkappa,\tau) = 1+\varkappa\left(1-\tau+\tau^2-\tau^3+\frac{2}{3}\tau^4-\frac{1}{3}\tau^5+\frac{1}{9}\tau^6-\frac{1}{63}\tau^7+\cdots\right).$$

When $m \longrightarrow +\infty$, we get the following exact solution

$$Z(\varkappa,\tau) = 1+\frac{\varkappa}{1+\tau}, \;\; |\tau|<1.$$

which is an exact solution to the nonlinear partial differential equation.



(a) (b) (c)

Figure 4.1: (a) Exact solution, (b) The approximate solution in the case $\rho = 1$, (c) The exact solution and approximate solutions to (4.1) for different values of ρ when $\varkappa = 1$. From (c) noted that the graphics have changed his position based on ρ values, if ρ took values closer to 1, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

Example 4.2. *Next, we consider the nonlinear time-fractional partial differential equation*

$$^{c}D^{\alpha}_{\tau}Z+\frac{2}{\tau}ZZ_{\varkappa} = 0, \;\tau>0, \;\varkappa\neq 0, \;\; 1<\rho\leqslant 2, \tag{4.7}$$

with

$$Z(\varkappa,0) = 0, \;\; Z_{\tau}(\varkappa,0) = \frac{1}{\varkappa}. \tag{4.8}$$

If $\rho = 2$, we obtain

$$Z_{\tau}+\frac{2}{\tau}ZZ_{\varkappa} = 0, \;\; \tau>0, \;\; \varkappa\neq 0 \tag{4.9}$$

The exact solution of (4.9), is

$$Z(\varkappa,\tau) = \tan\left(\frac{\tau}{\varkappa}\right). \tag{4.10}$$

According to (3.8), we can construct the iteration formula as follows

$$Z_{m+1} = \frac{\tau}{\varkappa}-E^{-1}\left[v^{\rho}E\left[\frac{2}{\tau}Z_m(Z_m)_{\varkappa}\right]\right]. \tag{4.11}$$

Using the iteration formula (4.11), we obtain

$$\begin{gathered}
Z_0(\varkappa,\tau)=\tfrac{\tau}{\varkappa},\\
Z_1(\varkappa,\tau)=\tfrac{\tau}{\varkappa}+\tfrac{2}{\varkappa^3}\tfrac{\tau^{\rho+1}}{\Gamma(\rho+2)},\\
Z_2(\varkappa,\tau)=\tfrac{\tau}{\varkappa}+\tfrac{2}{\Gamma(\rho+2)}\tfrac{\tau^{\rho+1}}{\varkappa^3}+\tfrac{16}{\Gamma(2\rho+2)}\tfrac{\tau^{2\rho+1}}{\varkappa^5}+\tfrac{24\Gamma(2\rho+2)}{\Gamma^2(\rho+2)\Gamma(3\rho+2)}\tfrac{\tau^{3\rho+1}}{\varkappa^7},\\
\vdots
\end{gathered}$$

The approximate solution is giveb by

$$\begin{aligned}
Z(\varkappa,\tau) &= \frac{\tau}{\varkappa}+\frac{2}{\Gamma(\rho+2)}\frac{\tau^{\rho+1}}{\varkappa^3}+\frac{16}{\Gamma(2\rho+2)}\frac{\tau^{2\rho+1}}{\varkappa^5}\\
&\quad+\frac{24\Gamma(2\rho+2)}{\Gamma^2(\rho+2)\Gamma(3\rho+2)}\frac{\tau^{3\rho+1}}{\varkappa^7}+\cdots
\end{aligned} \tag{4.12}$$

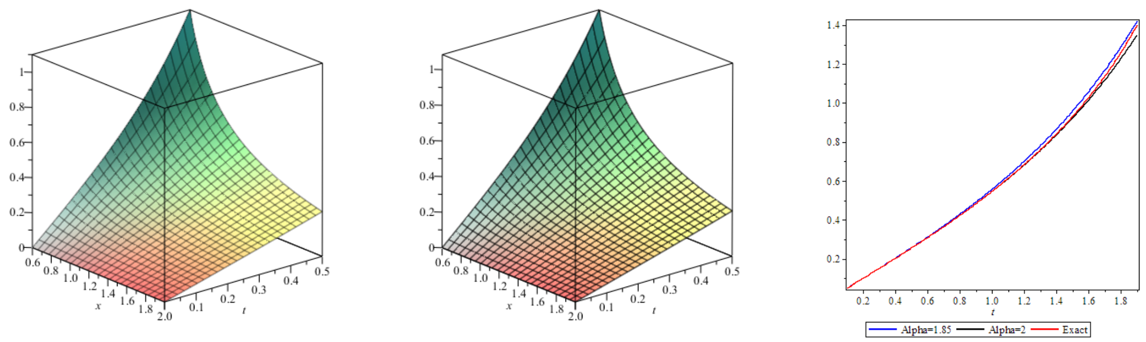
As $\rho\longrightarrow 2$, *we obtain*

$$Z(\varkappa,\tau)=\frac{\tau}{\varkappa}+\frac{1}{3}\left(\frac{\tau}{\varkappa}\right)^3+\frac{2}{15}\left(\frac{\tau}{\varkappa}\right)^5+\frac{1}{63}\left(\frac{\tau}{\varkappa}\right)^7.$$

And in closed form, is given by

$$Z(\varkappa,\tau)=\tan\left(\frac{\tau}{\varkappa}\right),$$

we get the exact solution of (4.7) when $\rho=2$.



(a') (b') (c')

Figure 4.2: (a') Exact solution, (b') the approximate solution in the case $\rho=2$, (c') The exact solution and approximate solutions to (4.7) for different values of ρ when $\varkappa=2$. From (c') noted that the graphics have changed his position based on ρ values, if ρ took values closer to 2, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

Example 4.3. *finally, we consider the nonlinear time-fractional partial differential equation*

$${}^{c}D_{\tau}^{\rho}Z-\frac{3}{8}\left[(Z_{\varkappa\varkappa})^2\right]_{\varkappa}=\frac{3}{2}\tau,\ \ 2<\rho\leqslant 3, \tag{4.13}$$

with

$$Z(\varkappa,0)=\frac{1}{2}\varkappa^2,\ \ Z_{\tau}(\varkappa,0)=\frac{1}{3}\varkappa^3,\ \ Z_{\tau\tau}(\varkappa,0)=0. \tag{4.14}$$

If $\rho=3$, *we obtain*

$$Z_{\tau\tau\tau}-\frac{3}{8}\left[(Z_{\varkappa\varkappa})^2\right]_{\varkappa}=\frac{3}{2}\tau. \tag{4.15}$$

According to (3.8), we can construct the following iteration formula

$$Z_{m+1}=-\frac{1}{2}\varkappa^2+\frac{1}{3}\varkappa^3\tau+\frac{3}{2}\frac{\tau^{\rho+1}}{\Gamma(\rho+2)}-E^{-1}\left(v^{\rho}E\left[-\frac{3}{8}\left[(Z_{m\varkappa\varkappa})^2\right]_{\varkappa}\right]\right). \tag{4.16}$$

Use the (4.16) to get

$$\begin{aligned} Z_0(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2+\tfrac{1}{3}\varkappa^3\tau, \\ Z_1(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2+\tfrac{1}{3}\varkappa^3\tau+6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ Z_2(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2+\tfrac{1}{3}\varkappa^3\tau+6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ Z_3(\varkappa,\tau) &= -\tfrac{1}{2}\varkappa^2+\tfrac{1}{3}\varkappa^3\tau+6\varkappa\tfrac{\tau^{\rho+2}}{\Gamma(\rho+3)}, \\ &\vdots \end{aligned} \tag{4.17}$$

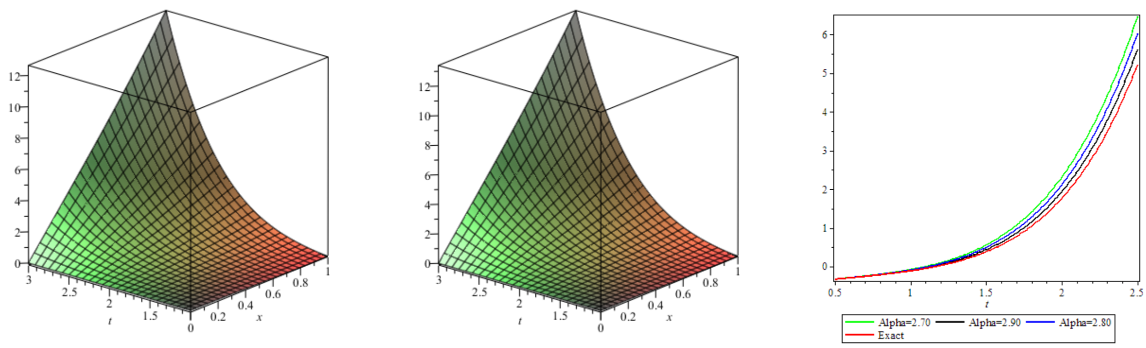
The approximate solution in a series form, is given by

$$Z(\varkappa,\tau) = -\frac{1}{2}\varkappa^2+\frac{1}{3}\varkappa^3\tau+6\varkappa\frac{\tau^{\rho+2}}{\Gamma(\rho+3)}. \tag{4.18}$$

As $\rho \longrightarrow 3$, *we obtain the following exact solution*

$$Z(\varkappa,\tau) = \frac{1}{20}\varkappa\tau^5+\frac{1}{3}\varkappa^3\tau-\frac{1}{2}\varkappa^2.$$

That gives the exact solution of (4.13) when $\rho = 3$.



(a”) (b”) (c”)

Figure 4.3: (a”) Exact solution, (b”)the approximate solution in the case $\rho = 2.90$, (c”) The exact solution and approximate solutions to (4.13) for different values of ρ when $\varkappa = 1$. From (c”) noted that the graphics have changed his position based on ρ values, if ρ took values closer to 3, we see that the graph corresponding to this value is barely graphical representation of the exact solution.

5. Conclusion

Coupling of Variational Iteration Method and Elzaki Transform, to be an effective method for solving nonlinear partial differential equations with time-fractional derivative. The proposed algorithm is suitable for such problems and is very efficient. From the results, it is clear that the FVIETM yields very accurate approximate solutions using only a few iterates. It provides a solution as a more realistic series, which converges rapidly to the exact solution.

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