Hacettepe Journal of Mathematics and Statistics
Volume 47 (2) (2018), 471 – 511

Estimation of population distribution function in the presence of non-response

Mazhar Yaqub*† and Javid Shabbir‡

Abstract

This article addresses the problem of estimating the population distribution function in the presence of non-response. We suggest a general class of estimators for estimating the cumulative distribution function using the auxiliary information. Expressions for bias and mean squared error of considered estimators are derived up to the first order of approximation. The performance of estimators are compared theoretically and numerically. A numerical study is carried out to evaluate the performances of estimators.

Keywords: Auxiliary variable; absolute relative bias; cumulative distribution function; mean squared error; relative efficiency.

2000 AMS Classification: Primary 62D05, Secondary 62P20.

Received : 30.10.2015 *Accepted* : 09.05.2016 *Doi* : 10.15672/HJMS.201613118919

1. Introduction

It is a well established phenomenon in the theory of sample survey that the non-response is an unavoidable fact, which is devastating and almost in every surveys of human respondents, suffer from some degree of non-response. Non-response mainly classified as: (i), unit non-response or total failure, in which entire unit is missing, for example, a person may totally refuse or unable to participate in the survey for some specified reasons and (ii), item non-response or partial failure, in which at least one item is missing from some measurements for the given observations. For example, a household may hesitate to give information about his income. The problem of non-response has already been tackled from different ways, is common and widespread in mail surveys than in personal interviewing. The usual approach to overcome non-response problem is to contact the non-respondent and obtain maximum information as much as possible.

*Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan., Email: mazharyaqubphd@gmail.com

†Corresponding Author.

‡Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan., Email: js@qau.edu.pk

Hansen and Hurwitz [11] were the first to suggest a non-response technique in mail surveys, combined the advantages of mailed questionnaires and personal interviews. They plan first use the economies involved in the use of questionnaires by mailing them to a sample of population under study. After this a follow-up is carried out by interviewing a subsample of the non-respondents.

Consider a finite population of size N and a random sample of size m is drawn from a population by using simple random sample without replacement (SRSWOR) sampling scheme. In survey of human populations, it is often the case that m_R units respond, but the remaining $m_M = (m - m_R)$ units do not. The initial survey may be conducted through the mail or by telephone, perhaps computer aided. Hansen and Hurwitz [11] suggested a two phase sampling scheme for estimating the population mean by using the following steps.

(a) a simple random sample of size m is selected and the questionnaire are mailed to the sampled units;

(b) a subsample of size $r = \frac{m_M}{k}$ for $(k > 1)$ is taken from m_M non-responding units.

The graphical illustration of non-response scheme is given in Figure 1. A widely debated

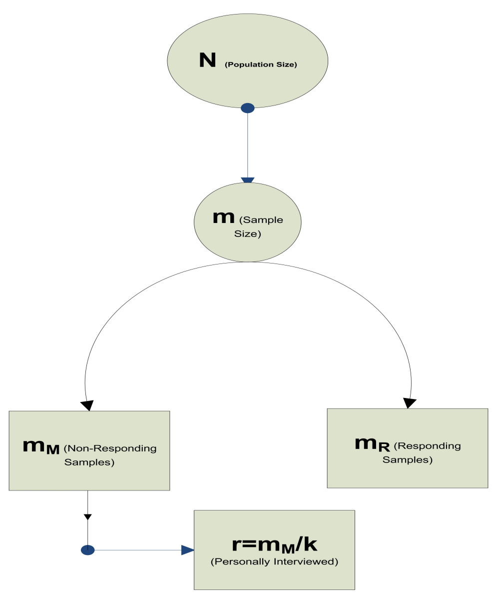

Figure 1. Illustration of Hansen and Hurwitz [11] non-response scheme

topic in sample survey is the estimation of population mean for the study variable by using the auxiliary variables in the presence of non-response. Several authors including Chambers and Dunstan [3], Rao [28], Rao et al. [27], Khare and Srivastava [17, 18, 19], Olkin [25] suggested different types of estimators for estimation of population mean using the auxiliary information under non-response. Okafor and Lee [24] presented ratio and regression estimation with sub-sampling the non-respondents in estimating the population mean $\bar{Y}$. Further, Khare and Sinha [14, 15, 16] proposed some classes of estimators for estimating population mean using multi-auxiliary characters in different way. For the estimating population mean under two-phase sampling scheme in presence of non-response, Singh and Kumar [34, 35, 36], Klein [20], Tabasum and Khan [40], and Shabbir and Nasir [33] have made significant contributions. Diana and Perri [5] suggested a class of estimators in two-phase sampling with sub-sampling of non respondents in estimating the finite population mean. For controlling the non-response bias and eliminating the need for call backs in survey sampling, John and Robert [13], citeBS, and Dunkelberg and Goerge [7], El-Badry [8], Diana and Perri [4], Hansen et al. [12], and Politz and Simmons [26], discussed some good techniques and plans.

An extensive literature is available on estimation of population mean under non-response, but lesser effort has been devoted in the development of efficient methods for population cumulative distribution function by using the auxiliary information.

We are often concerned with the proportion of y_i values in the population. Users of sample survey data commonly need to estimate the population distribution function, or, equivalently, the proportion of units in the population with values less than or equal to a specified value t_y. For example, we may be interested in the proportion of agricultural area for poisonous effect of pesticides less than zero, the proportion of filtration plants for the present of arsenic in portable water less than zero. Such a proportion is particular value of the cumulative distribution function (CDF) for the population.

$$F_Y(t_y) = \frac{1}{N}\sum_{i=1}^{N} I\left(y_i \leq t_y\right).$$

Above expression is just the average of the values of Bernoulli distribution $I\left(y_i \leq t_y\right)$ over all elements of the population, where $I\left(y_i \leq t_y\right) = 1$ for $y \leq t_y$ and $I\left(y_i \leq t_y\right) = 0$, for $y > t_y$. Often in survey sampling, we can only measure the study variable for those items in some sample, thus, the usual estimators of the distribution function depends exclusively on the selection of the sampling design and the sampled portion of the population. It is often seen the case, that some values of study variable are not available for non-sampled portion of the population, so we may use auxiliary information for improving the efficiency of population distribution function.

Chambers and Dunstan [3] and Chambers et al. [2] suggested the procedure and properties for estimating the finite population distribution function and the quantiles based on use of the auxiliary information. Rao et al. [27] used a general sampling design and proposed ratio and difference type estimators for population distribution function. Kuk [21], presented a classical as well as a prediction approach in estimating the distribution function from survey data. Some more work is due to Woodruff [42], Kuk and Mak [22, 23], Rueda et al. [30, 31], Rueda and Arcos [29], Dorfman [6], Ahmed and Abu-Dayyeh [1], and Singh and Joarder [39].

In presence of the auxiliary information, there exist several general estimation procedures. For more details see Wang and Alan [41], Kuk and Mak [23], Rao et al. [27], Rueda et al. [31], Garcia and Cebrian [9] and Singh et al. [37] to obtain more efficient estimates for the population mean or totals.

An extensive literature is available on estimation of population mean under non-response, but lesser effort has been devoted in the development of efficient methods for population cumulative distribution function (CDF) by using the auxiliary information. The present article focuses on the estimation of population distribution function of the study variable using the auxiliary information when data are not collected from all sampled units due to the problem of non-response.

We organize the rest of the article as follows: Section 2 introduces the notations and symbols. Section 3 gives detailed proof for estimating the population distribution function under non-response case. Section 4 contains the expressions for the bias and mean squared error (MSE). Section 5 gives a general class of estimators to first order of approximation. A numerical study is presented in Section 6 and cost of the survey is discussed in Section 7. Section 8 gives the conclusion.

2. Notations and symbols

Consider a finite population $\Omega = \{1, 2, ..., N\}$ having N distinct and identifiable units. Let $(y_1, y_2, ..., y_N)$ be the values of the study variable Y. For each index t_y, $(-\infty < t_y < +\infty)$, the cumulative distribution function (CDF) of Y is given by

$$(2.1) \qquad F_Y(t_y) = \frac{1}{N}\sum_{i=1}^{N} I\left(y_i \leq t_y\right),$$

where $I(.)$ is an indicator function.
Then the corresponding population β quantile $(0 < \beta < 1)$ is defined by

$$(2.2) \qquad Q_Y(\beta) = inf\left\{y|F_Y(y) \geq \beta\right\} = F_Y^{-1}(\beta),$$

where inf stands for infinimum. The problem is to estimate $F_Y(t_y)$ for any given t_y. We draw a random sample of size m from N by simple random sampling without replacement sampling scheme (SRSWOR). Then given t_y, the $F_Y(t_y)$ can be estimated by

$$(2.3) \qquad \hat{F}_Y(t_y) = \frac{1}{m}\sum_{i=1}^{m} I\left(y_i \leq t_y\right).$$

Following Garcia and Cebrian [9], it is easy to show that

$$(2.4) \qquad E\left(\hat{F}_Y(t_y)\right) = F_Y(t_y) \quad \text{and} \quad V\left(\hat{F}_Y(t_y)\right) = \frac{N-m}{m(N-1)}F_Y(t_y)\left(1 - F_Y(t_y)\right),$$

where $E(.)$ and $V(.)$ are the mathematical expectation and variance of (.), respectively. The layout of response stratum is given in Table 1.

Table 1. Layout of respondent stratum

	$X \leq F_X(t_x)$	$X > F_X(t_x)$	Total
$Y \leq F_Y(t_y)$	m_{11}/N_{11}	m_{12}/N_{12}	$N_{1.}$
$Y > F_Y(t_y)$	m_{21}/N_{21}	m_{22}/N_{22}	$N_{2.}$
Total	$N_{.1}$	$N_{.2}$	N

Here, N_{11}, N_{12}, N_{21}, and N_{22} be the number of units in the population in their respective cells for respondents. Similarly, m_{11}, m_{12}, m_{21}, and m_{22} be the number of units in the sample in their respective cells. Hence $(m_{11}, m_{12}, m_{21}, m_{22})$ is a trivariate Hyper Geometrically(THG) distributed random variable,
i.e., $(m_{11}, m_{12}, m_{21}, m_{22}) \sim THG\left(N, m, N_{11}, N_{12}, N_{21}\right)$.
Also $m\hat{F}_Y(t_y) = m_{11} + m_{12}$ and $m\hat{F}_X(t_x) = m_{11} + m_{21}$.
The non-response stratum layout is given in Table 2.

Table 2. Layout of non-response stratum

	$X_2 \leq F_X^{(2)}(t_{x_2})$	$X_2 > F_X^{(2)}(t_{x_2})$	Total
$Y_2 \leq F_Y^{(2)}(t_{y_2})$	$m_{11}^{(2)}/N_{11}^{(2)}$	$m_{12}^{(2)}/N_{12}^{(2)}$	$N_{1.}^{(2)}$
$Y_2 > F_Y^{(2)}(t_{y_2})$	$m_{21}^{(2)}/N_{21}^{(2)}$	$m_{22}^{(2)}/N_{22}^{(2)}$	$N_{2.}^{(2)}$
Total	$N_{.1}^{(2)}$	$N_{.2}^{(2)}$	N

Here, $N_{11}^{(2)}$, $N_{12}^{(2)}$, $N_{21}^{(2)}$, and $N_{22}^{(2)}$ be the number of units in the population in their respective cells for non-respondents. Similarly, $m_{11}^{(2)}$, $m_{12}^{(2)}$ $m_{21}^{(2)}$ and $m_{22}^{(2)}$ be the number of units in the sample in their respective cells.
Let $\{I(Y_i \leq t_y), I(X_i \leq t_x)\} = 1$ if ith unit possesses an attribute and $\{I(Y_i \leq t_y), I(X_i \leq t_x)\} = 0$ otherwise, which follows the uniform probability distribution. Let sample means $\left(\hat{F}^{*}_{Yi}(t_y), \hat{F}^{*}_{Xi}(t_x)\right)$ be the unbiased estimators of population means $(F_Y(t_y), F_X(t_x))$ based on m observations. Let $S^2_{F_Y}(t_y) = F_Y(t_y)(1 - F_Y(t_y))$, $S^2_{F_X}(t_x) = F_X(t_x)(1 - F_X(t_x))$,
$S^2_{F_{X2}}(t_{x_2}) = F_{X2}(t_{x_2})(1 - F_{X2}(t_{x_2}))$ be the population variances and
$S_{F_{YX}}(t_y, t_x) = F_{Y,X}(t_y, t_x) - F_Y(t_y)F_X(t_x)$ be the population covariance for Stratum 1 and Stratum 2 respectively.
Also $C_{F_Y}(t_y) = \frac{1-F_Y(t_y)}{F_Y(t_y)}$, $C_{F_X}(t_x) = \frac{1-F_Y(t_x)}{F_Y(t_x)}$, $C_{F_{X2}}(t_{x_2}) = \frac{1-F_{X_2}(t_{x_2})}{F_{X_2}(t_{x_2})}$ be the population coefficient of variations of X for Stratum 1 and Stratum 2 respectively.
Let $\beta_1(F_X(t_x)) = \frac{1-2F_X(t_x)}{\sqrt{F_X(t_x)(1-F_X(t_x))}}$ and $\beta_2(F_X(t_x)) = \frac{1-3F_X(t_x)+3F_X^2(t_x)}{F_X(t_x)(1-F_X(t_x))}$ be the population coefficients of skewness and kurtosis of X. Let $\left(\rho_{\left(F_Y(t_y), F_X(t_x)\right)}\right) = \frac{S_{F_{YX}}(t_y,t_x)}{S_{F_Y}(t_y)S_{F_X}(t_x)}$ be the phi-population correlation coefficient. To obtain Bias and MSE of estimators up to first order of approximation, we define the following relative error terms.
Let $e_0^* = \frac{\hat{F}_Y^*(t_y)-F_Y(t_y)}{F_Y(t_y)}$, $e_1^* = \frac{\hat{F}_X^*(t_x)-F_X(t_x)}{F_X(t_x)}$, $e_0 = \frac{\hat{F}_Y(t_y)-F_Y(t_y)}{F_Y(t_y)}$ and $e_1 = \frac{\hat{F}_X(t_x)-F_X(t_x)}{F_X(t_x)}$ such that $E(e_i^*) = E(e_i) = 0$, for $i = 0, 1$. To first order of approximation we have

$$E\left(e_0^{*2}\right) = \frac{1}{F_Y^2(t_y)}\left\{\lambda_1 F_Y(t_y)(1 - F_Y(t_y)) + \lambda_2 F_Y^{(2)}(t_{y_2})\left(1 - F_Y^{(2)}(t_{y_2})\right)\right\} \cong V_{20}^*,$$

$$E\left(e_1^{*2}\right) = \frac{1}{F_X^2(t_x)}\left\{\lambda_1 F_X(t_x)(1 - F_X(t_x)) + \lambda_2 F_X^{(2)}(t_{x_2})\left(1 - F_X^{(2)}(t_{x_2})\right)\right\} \cong V_{02}^*,$$

$$E(e_0^* e_1^*) = \frac{1}{F_Y(t_y)F_X(t_x)}\left\{\lambda_1\left(\frac{N_{11}N_{22}-N_{12}N_{21}}{N^2}\right) + \lambda_2\left(\frac{N_{11}^{(2)}N_{22}^{(2)}-N_{12}^{(2)}N_{21}^{(2)}}{(N_2^{(2)})^2}\right)\right\} \cong V_{11}^*,$$

$$E\left(e_0^2\right) = \frac{1}{F_Y^2(t_y)}\left\{\lambda_1 F_Y(t_y)(1 - F_Y(t_y))\right\} \cong V_{20},$$

$$E\left(e_1^2\right) = \frac{1}{F_X^2(t_x)}\left\{\lambda_1 F_X(t_x)(1 - F_X(t_x))\right\} \cong V_{02},$$

$$E(e_0^* e_1) = \frac{1}{F_Y(t_y)F_X(t_x)}\left\{\lambda_1\left(\frac{N_{11}N_{22}-N_{12}N_{21}}{N^2}\right)\right\} \cong V_{11}^{*'},$$

where

$$V_{rs}^* = \frac{E\left\{\left(\hat{F}_Y^*(t_y)-F_Y(t_y)\right)^r\left(\hat{F}_X^*(t_x)-F_X(t_x)\right)^s\right\}}{\left(F_{Yh}(t_y)\right)^r(F_{Xh}(t_x))^s},\ V_{rs}^{*'} = \frac{E\left\{\left(\hat{F}_Y^*(t_y)-F_Y(t_y)\right)^r\left(\hat{F}_X(t_x)-F_X(t_x)\right)^s\right\}}{\left(F_{Yh}(t_y)\right)^r(F_{Xh}(t_x))^s}.$$

$\lambda_1 = \left(\frac{1}{m} - \frac{1}{N}\right)$, and $\lambda_2 = \frac{W_M(k-1)}{m}$, with $W_M = \frac{N_M}{N}$, N_M be the number of units in the population corresponding to non-response group.

3. Estimation of population distribution function under non-response

In this section, we drive the expressions for mean, variance and covariance of the estimator, $\hat{F}_Y^{(*)}(t_y) = w_R\hat{F}_Y^{(1)}(t_y) + w_M\hat{F}_Y^{(2r)}(t_{y_2})$ under non-response for estimating the CDF.
Suppose that the underlying population is divided into two homogeneous strata: (i) response group and (ii) non-response group. Let N_R and N_M be the number of units in the population that correspond to the response group and the non-response group, respectively, where $N_R + N_M = N$. Given this information, following Gross [10], the finite population CDF, $F_Y(t_y)$, can be written as

$$F_Y(t_y) = W_R F_Y^{(1)}(t_y) + W_M F_Y^{(2)}(t_{y_2}), \tag{3.1}$$

where $W_i = N_i/N$ for $i = R, M$.
Out of m selected units, m_R units respond and m_M units do not respond, where $m_R + m_M = m$. In order to get response from m_M, the non-respondents are contacted once

again by personal interview. Then sub-sample of size $r = m_M/k$ for $(k > 1)$, is obtained from m_M non-responding units. It is assumed that all r units respond. Let $\hat{F}_Y^{(1)}(t_y)$ and $\hat{F}_Y^{(2r)}(t_{y2})$ be the CDF estimators based on m_R and r responding units. On the lines of Hansen and Hurwitz [11], the estimator of $F_Y(t_y)$ under non-response is given by

$$(3.2) \qquad \hat{F}_Y^{(*)}(t_y) = w_R \hat{F}_Y^{(1)}(t_y) + w_M \hat{F}_Y^{(2r)}(t_{y2}),$$

where $w_i = m_i/m$ for $i = R, M$.
Based on the estimator given in (3.2), we present the following theorem.

Theorem 1

(i) $\hat{F}_Y^{(*)}(t_y)$ is an unbiased estimator of $F_Y(t_y)$, i.e., $E\left(\hat{F}_Y^{(*)}(t_y)\right) = F_Y(t_y)$.

(ii) $Var\left(\hat{F}_Y^{(*)}(t_y)\right)$
$$= \left[\tfrac{N-m}{m(N-1)} F_Y(t_y)(1 - F_Y(t_y)) + \tfrac{W_M(k-1)}{m} \tfrac{N_M}{N_M - 1} F_Y^{(2)}(t_{y2}) \left(1 - F_Y^{(2)}(t_{y2})\right)\right].$$

(iii) $Cov\left(\hat{F}_Y^{(*)}(t_y), \hat{F}_X^{(*)}(t_x)\right)$
$$= \left[\tfrac{(1-f)}{m} \left(\tfrac{N_{11}N_{22} - N_{12}N_{21}}{(N)^2}\right) + \tfrac{W_M(k-1)}{m} \left(\tfrac{N_{11}^{(2)}N_{22}^{(2)} - N_{12}^{(2)}N_{21}^{(2)}}{(N_M^{(2)})^2}\right)\right].$$

Proof (i). Taking mathematical expectation on both sides of (3.2), we have

$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = E\left\{w_R \hat{F}_Y^{(1)}(t_y)\right\} + E\left\{w_M \hat{F}_Y^{(2r)}(t_{y2})\right\},$$
$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = E_1\left\{w_R E_2\left(\hat{F}_Y^{(1)}(t_y)|m_R\right)\right\} + E\left\{w_M E_2\left(\hat{F}_Y^{(2r)}(t_{y2})|m_R\right)\right\},$$
$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = E_1\left\{w_R \left(F_Y^{(1)}\right)(t_y)|m_R\right\} + E\left\{w_M E_2\left(E_3\left(\hat{F}_Y^{(2r)}(t_{y2})|r, m_R\right)\right)\right\},$$
$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = E_1\left\{w_R \left(F_Y^{(1)}\right)(t_y)|m_R\right\} + E\left\{w_M E_2\left(\hat{F}_Y^{(2)}(t_{y2})|m_R\right)\right\},$$
$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = E_1\left\{w_R \left(F_Y^{(1)}\right)(t_y)|m_R\right\} + E\left\{w_M \left(F_Y^{(2)}(t_{y2})|m_R\right)\right\},$$
$$E\left(\hat{F}_Y^{(*)}(t_y)\right) = W_R F_Y^{(1)}(t_y) + W_M F_Y^{(2)}(t_{y2}) = F_Y(t_y),$$

which completes the proof.
Proof (ii). From (3.2), we can write

$$(3.3) \qquad \hat{F}_Y^{(*)}(t) = \hat{F}_Y(t_y) + w_M\left(\hat{F}_Y^{(2r)}(t_{y2}) - \hat{F}_Y^{(2)}(t_{y2})\right),$$

where, $\hat{F}_Y(t_y) = w_R \hat{F}_Y^{(1)}(t_y) + w_M \hat{F}_Y^{(2)}(t_{y2})$.
It is easy to show that

$$(3.4) \qquad E\left(\hat{F}_Y(t_y)\right) = F_Y(t_y),$$

and

$$(3.5) \qquad Var\left(\hat{F}_Y(t_y)\right) = \left[\frac{N-m}{m(N-1)} F_Y(t_y)(1 - F_Y(t_y))\right].$$

If we consider $N - 1 \cong N$, then we can write (3.5) as

$$Var\left(\hat{F}_Y(t)\right) = \left[\frac{N-m}{mN} F_Y(t_y)(1 - F_Y(t_y))\right].$$

Applying variance on both sides of (3.3), we get

$$(3.6) \qquad Var\left(\hat{F}_Y^{(*)}(t_y)\right) = \begin{bmatrix} \dfrac{N-m}{mN} F_Y(t_y)\left(1 - F_Y(t_y)\right) \\ +Var\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2}) - \hat{F}_Y^{(2)}(t_{y2})\right)\right\} \\ +2Cov\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_{y2}) - \hat{F}_Y^{(2)}(t_{y2})\right)\right\} \end{bmatrix}.$$

From (3.6), we can write

$$(3.7)\quad Var\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\begin{bmatrix} V_1E_2\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\} \\ +E_1V_2\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}\end{bmatrix}.$$

Considering the terms on right hand side of (3.7), we have

$$\begin{aligned} V_1E_2\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\} &= V_1E_2E_3\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\} \\ (3.8)\qquad &= 0. \end{aligned}$$

$$(3.9)\qquad E_1V_2\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=E_1\left\{w_M^2V_2\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}.$$

Considering the term on right hand side of (3.9), we have
$V_2\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}$
$=V_2E_3\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}+E_2V_3\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\},$
or
$V_2\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=E_2V_3\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}.$
Finally, we get

$$(3.10)\qquad V_2\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\frac{m_M-r}{r\,(m_M-1)}E_2\left\{\hat{F}_Y^{(2)}(t_{y2})\left(1-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}.$$

Replace the values of $r=m_M/k$, $E_2\left(\hat{F}_Y^{(2)}(t_{y2})\right)=F_Y^{(2)}(t_{y2})$ and
$E_2\left(F_Y^{(2)}(t_{y2})\right)^2=V_2\left(\hat{F}Y^{(2)}(t_{y2})\right)+\left(F_Y^{(2)}(t_{y2})\right)^2$
$=\frac{N_M-m_M}{m_M(N_M-1)}F_Y^{(2)}(t_{y2})(-F_Y^{(2)}(t_{y2}))+\left(F_Y^{(2)}(t_{y2})\right)^2$ in (3.10), and after some simplifications, we get

$$(3.11)\qquad V_2\left\{\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\frac{N_M\,(k-1)}{m_M\,(N_M-1)}F_Y^{(2)}(t_{y2})\left(1-F_Y^{(2)}(t_{y2})\right).$$

Substituting (3.11) in (3.9), we have
$E_1\left\{w_M^2V_2\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}$
$=E_1\left\{\frac{m_M^2}{m^2}\frac{N_M(k-1)}{m_M(N_M-1)}F_Y^{(2)}(t_{y2})\left(1-F_Y^{(2)}(t_{y2})\right)\right\},$
or
$E_1\left\{w_M^2V_2\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\frac{W_M(k-1)}{m}\frac{N_M}{N_M-1}F_Y^{(2)}(t_{y2})\left(1-F_Y^{(2)}(t_{y2})\right).$
If we consider $N_M-1\cong N_M$, then above expression can be written as

$$(3.12)\qquad E_1\left\{w_M^2V_2\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\frac{W_M\,(k-1)}{m}F_Y^{(2)}(t_{y2})\left(1-F_Y^{(2)}(t_{y2})\right).$$

Substituting (3.8) and (3.12) in (3.7), we get

$$(3.13)\qquad Var\left\{w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}=\frac{W_M\,(k-1)}{m}F_Y^{(2)}(t_{y2})\left(1-F_Y^{(2)}(t_{y2})\right).$$

From (3.6), the covariance term can be written as
$Cov\left\{\hat{F}_Y(t_y),w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}$

$$(3.14)\qquad =\begin{bmatrix} E_1Cov_2\left\{\hat{F}_Y(t_y),w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\} \\ +Cov_1E_2\left\{\hat{F}_Y(t_y),w_M\left(\hat{F}_Y^{(2r)}(t_{y2})-\hat{F}_Y^{(2)}(t_{y2})\right)\right\}\end{bmatrix}.$$

Now considering the term $Cov_2\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_{y_2})-\hat{F}_Y^{(2)}(t_{y_2})\right)\right\}$ on right hand side of (3.14), we get

$$(3.15)\quad = \left[\begin{array}{l} E_2Cov_3\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_{y_2})-\hat{F}_Y^{(2)}(t_{y_2})\right)\right\} \\ +Cov_2E_3\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_y y_2)-\hat{F}_Y^{(2)}(t_{y_2})\right)\right\}\end{array}\right] = 0.$$

On similar steps, it can be shown that

$$(3.16)\quad Cov_1E_2\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_{y_2})-\hat{F}_Y^{(2)}(t_{y_2})\right)\right\} = 0.$$

Substituting (3.15) and (3.16) in (3.14), we get

$$(3.17)\quad Cov\left\{\hat{F}_Y(t_y), w_M\left(\hat{F}_Y^{(2r)}(t_{y_2})-\hat{F}_Y^{(2)}(t_{y_2})\right)\right\} = 0.$$

Again by using (3.13) and (3.17) in (3.6), we get

$$Var\left(\hat{F}_Y^{(*)}(t_y)\right) = \left[\begin{array}{l} \frac{N-m}{mN}F_Y(t_y)(1-F_Y(t_y)) \\ +\frac{W_M\,(k-1)}{m}F_Y^{(2)}(t_{y_2})\left(1-F_Y^{(2)}(t_{y_2})\right)\end{array}\right].$$

This completes the proof and on the same lines we have

$$Var\left(\hat{F}_X^{(*)}(t_x)\right) = \left[\begin{array}{l} \frac{N-m}{mN}F_X(t_x)(1-F_X(t_x)) \\ +\frac{W_M\,(k-1)}{m}F_X^{(2)}(t_{x_2})\left(1-F_X^{(2)}(t_{x_2})\right)\end{array}\right].$$

Proof (iii). See Appendix A, Page 53.

4. Suggested estimators of population distribution function

We suggest the following family of estimators for estimating the population distribution function.

4.1. General family of estimators. A general family of estimators for estimating CDF, is given by

$$(4.1)\quad \hat{F}_{MJ}(t_y) = \hat{F}_Y(t_y)\left[\frac{aF_X(t_x)+b}{\delta\left(a\hat{F}_X(t_x)+b\right)+(1-\delta)\left(aF_X(t_x)+b\right)}\right]^g,$$

where δ, g are suitably chosen constants and $a\,(\neq 0)$, b are either real numbers or function of known parameters of the auxiliary variable X, such as standard deviation $(S_{F_X}(t_x))$, co-efficient of variation $(C_{F_X}(t_x))$, co-efficient of skewness $(\beta_1(F_X(t_x)))$, co-efficient of kurtosis $(\beta_2(F_X(t_x)))$, and co-efficient of correlation $\left(\rho_{(F_Y(t_y),F_X(t_x))}\right)$.
Expressing (4.1) in terms of $e's$, we have

$$\hat{F}_{MJ}(t_y) = F_Y(t_y)\,(1+e_0)\,(1+\delta\alpha e_1)^{-g},$$

where $\alpha = \frac{aF_X(t_x)}{aF_X(t_x)+b}$.
Expanding the right hand side of the above expression and retaining the terms up to power 2 in $e's$, we have

$$(4.2)\quad \hat{F}_{MJ}(t_y) = F_Y(t_y)\left[1+e_0-\delta\alpha g e_1+\frac{g(g+1)}{2}\delta^2\alpha^2e_1^2-\delta\alpha g e_0e_1\right].$$

4.1.1. *Situation I - Non-response both on the study and the auxiliary variables:* $\hat{F}_Y^*(t_y)$, $\hat{F}_X^*(t_x)$. When non-response occurs on both the study and the auxiliary variables, and population mean $F_X(t_x)$ of the auxiliary variable X is known in advance. In agricultural survey, for instance, expenditures of fertilizer or pesticides on crop can be used as the auxiliary variable for the estimation, say, production of crop, there may be non-response on both the variables and for Situation I, Equation (4.1) can be written as,

$$\text{(4.3)} \qquad \hat{F}_{MJ}(t_y) = \hat{F}_Y^*(t_y)\left[\frac{aF_X(t_x)+b}{\delta\left(a\hat{F}_X^*(t_x)+b\right)+(1-\delta)\left(aF_X(t_x)+b\right)}\right]^g.$$

For this, (4.2) can be written as,

$$\text{(4.4)} \qquad \hat{F}^{(1)}_{MJ_i}(t_y) \cong F_Y(t_y)\left[1+e_0^*-\delta\alpha g e_1^*+\frac{g(g+1)}{2}\delta^2\alpha^2 e_1^{*2}-\delta\alpha g e_0^* e_1^*\right],$$

where $\hat{F}^{(1)}_{MJ_i}(t_y)$ for case of Situation I.
Subtracting $F_Y(t_y)$ from both sides of (4.4) and then taking expectation, we get the bias of $\hat{F}^{(1)}_{MJ_i}(t_y)$ up to first order of approximation, given as

$$\text{(4.5)} \qquad Bias\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) \cong F_Y(t_y)\left[\delta^2 g(g+1)\alpha_i^2\frac{1}{2}V_{02}^*-\delta g\alpha_i V_{11}^*\right].$$

Squaring both sides of (4.4) and then taking expectations, we get the MSE of $\hat{F}^{(1)}_{MJ_i}(t_y)$, up to first order of approximations, given by

$$\text{(4.6)} \qquad MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) \cong F_Y^2(t_y)\left[V_{20}^*+\delta^2 g^2\alpha_i^2 V_{02}^*-2\delta g\alpha_i V_{11}^*\right],$$

where $\alpha_i = \frac{aF_X(t_x)}{aF_X(t_x)+b}$ for $i = 0, 1, ..., 13$.
Different estimators can be generated from proposed class of estimators by substituting the suitable choices of (δ,a,b,g). The generated estimators are listed in Table 3. Many more estimators can also be generated from suggested family of estimators by substituting different values of (δ,a,b,g).
The biases of the suggested family of estimators $\hat{F}^{(1)}_{MJ_i}(t_y)$ up to the first order of approximations are given below.

$$\text{(4.7)} \qquad Bias\left(\hat{F}^{(1)}_{MJ_1}(t_y)\right) \cong F_Y(t_y)\left(V_{02}^*-V_{11}^*\right),$$

$$\text{(4.8)} \qquad Bias\left(\hat{F}^{(1)}_{MJ_2}(t_y)\right) \cong F_Y(t_y)\left(V_{11}^*\right),$$

$$\text{(4.9)} \qquad Bias\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) \cong F_Y(t_y)\left(\alpha_i^2 V_{02}^*-\alpha_i V_{11}^*\right),$$

for $i = 3, 10, 12$, and

$$\text{(4.10)} \qquad Bias\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) \cong F_Y(t_y)\left(\alpha_i^2 V_{02}^*+\alpha_i V_{11}^*\right),$$

for $i = 4 - 9, 11, 13$.
The MSE of the suggested family of estimators $\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)$ up to first order of approximation are given below.

$$\text{(4.11)} \qquad MSE\left(\hat{F}^{(1)}_{MJ_0}(t_y)\right) \cong F_Y^2(t_y)V_{20}^*,$$

$$\text{(4.12)} \qquad MSE\left(\hat{F}^{(1)}_{MJ_1}(t_y)\right) \cong F_Y^2(t_y)\left(V_{20}^*+V_{02}^*-2V_{11}^*\right),$$

$$\text{(4.13)} \qquad MSE\left(\hat{F}^{(1)}_{MJ_2}(t_y)\right) \cong F_Y^2(t_y)\left(V_{20}^*+V_{02}^*+2V_{11}^*\right),$$

$$\text{(4.14)} \qquad MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) \cong F_Y^2(t_y)\left(V_{20}^*+\alpha_i^2 V_{02}^*-2\alpha_i V_{11}^*\right),$$

Table 3. Some members of the suggested classes of estimators $\hat{F}^{(1)}_{MJ_i}(t_y)$

δ	a	b	g	Estimator
0	0	0	0	$\hat{F}^{(1)}_{MJ_0}(t_y)=\hat{F}^*_Y(t_y)$
1	1	0	1	$\hat{F}^{(1)}_{MJ_1}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)}{\hat{F}^*_X(t_x)}\right)$
1	1	0	-1	$\hat{F}^{(1)}_{MJ_2}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\hat{F}^*_X(t_x)}{F_X(t_x)}\right)$
1	1	$C_{F_X}(t_x)$	1	$\hat{F}^{(1)}_{MJ_3}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+C_{F_X}(t_x)}{\hat{F}^*_X(t_x)+C_{F_X}(t_x)}\right)$
1	1	$C_{F_X}(t_x)$	-1	$\hat{F}^{(1)}_{MJ_4}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\hat{F}^*_X(t_x)+C_{F_X}(t_x)}{F_X(t_x)+C_{F_X}(t_x)}\right)$
1	$\beta_2(F_X(t_x))$	$C_{F_X}(t_x)$	-1	$\hat{F}^{(1)}_{MJ_5}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\beta_2(F_X(t_x))\hat{F}^*_X(t_x)+C_{F_X}(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+C_{F_X}(t_x)}\right)$
1	$C_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(1)}_{MJ_6}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{C_{F_X}(t_x)\hat{F}^*_X(t_x)+\beta_2(F_X(t_x))}{C_{F_X}(t_x)F_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	1	$S_{F_X}(t_x)$	-1	$\hat{F}^{(1)}_{MJ_7}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\hat{F}^*_X(t_x)+S_{F_X}(t_x)}{F_X(t_x)+S_{F_X}(t_x)}\right)$
1	$S_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(1)}_{MJ_8}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{S_{F_X}(t_x)\hat{F}^*_X(t_x)+\beta_2(F_X(t_x))}{S_{F_X}(t_x)F_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	$\beta_2(F_X(t_x))$	$S_{F_X}(t_x)$	-1	$\hat{F}^{(1)}_{MJ_9}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\beta_2(F_X(t_x))\hat{F}^*_X(t_x)+S_{F_X}(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+S_{F_X}(t_x)}\right)$
1	1	$\rho_{(F_Y(t_y),F_X(t_x))}$	1	$\hat{F}^{(1)}_{MJ_{10}}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}{\hat{F}^*_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
1	1	$\rho_{(F_Y(t_y),F_X(t_x))}$	-1	$\hat{F}^{(1)}_{MJ_{11}}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\hat{F}^*_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
1	1	$\beta_2(F_X(t_x))$	1	$\hat{F}^{(1)}_{MJ_{12}}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+\beta_2(F_X(t_x))}{\hat{F}^*_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	1	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(1)}_{MJ_{13}}(t_y)=\hat{F}^*_Y(t_y)\left(\frac{\hat{F}^*_X(t_x)+\beta_2(F_x(t_x))}{F_X(t_x)+\beta_2(F_X(t_x))}\right)$

for $i=3,10,12.$, and

$$MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)\cong F^2_Y(t_y)\left(V^*_{20}+\alpha_i^2V^*_{02}+2\alpha_iV^*_{11}\right), \tag{4.15}$$

for $i=4$–$9,11,13$.

Also here,

$\alpha_1=0$, $\alpha_2=\alpha_3=1$, $\alpha_3=\alpha_4=\frac{F_X(t_x)}{F_X(t_x)+C_{F_X}}$, $\alpha_5=\frac{\beta_2(F_X(t_x))F_X(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+C_{F_X}}$,
$\alpha_6=\frac{C_{F_X}F_X(t_x)}{C_{F_X}F_X(t_x)+C_{F_X}}$, $\alpha_7,=\frac{F_X(t_x)}{F_X(t_x)+S_{F_X}}$, $\alpha_8=\frac{S_{F_X}F_X(t_x)}{S_{F_X}F_X(t_x)+\beta_2(F_X(t_x))}$,
$\alpha_9=\frac{\beta_2(F_X(t_x))F_X(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+S_{F_X}}$, $\alpha_{(10,11)}=\frac{F_X(t_x)}{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}$,
$\alpha_{(12,13)}=\frac{F_X(t_x)}{F_X(t_x)+\beta_2(F_X(t_x))}$.

4.1.2. *Situation II - Non-response only on the study variable:* $\hat{F}^*_Y(t_y)$. When non-response occurs only on the study variable, information on the auxiliary variable X is obtained from all sampled units and population mean $F_X(t_x)$ of the auxiliary variable X is known. In household survey, for example, by using the household size as the auxiliary variable for the estimation of family expenditures. Information can be obtained completely on family size, while there may be non-response on household expenditure. For Situation II, (4.1) can be written as

$$\hat{F}^{(2)}_{MJ_i}(t_y)=\hat{F}^*_Y(t_y)\left[\frac{aF_X(t_x)+b}{\delta\left(a\hat{F}_X(t_x)+b\right)+(1-\delta)\left(aF_X(t_x)+b\right)}\right]^g. \tag{4.16}$$

Expression (4.16) in terms of $e's$, we have

$$(4.17)\quad \hat{F}^{(2)}_{MJ_i}(t_y) \cong F_Y(t_y)\left[1 + e_0^* - \delta\alpha g e_1 + \frac{g(g+1)}{2}\delta^2\alpha^2 e_1^2 - \delta\alpha g e_0^* e_1\right],$$

where $\hat{F}^{(2)}_{MJ_i}(t_y)$, for the case of Situation II.
Different estimators can be generated from suggested family of estimators by substituting the suitable choices of (δ, a, b, g) and are listed in Table 4.

The biases of the suggested family of estimators $\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right)$ up to the first order are

Table 4. Some members of the suggested class of estimators $\hat{F}^{(2)}_{MJ_i}(t_y)$

δ	a	b	g	Estimator
0	0	0	0	$\hat{F}^{(2)}_{MJ_0}(t_y) = \hat{F}^*_Y(t_y)$
1	1	0	1	$\hat{F}^{(2)}_{MJ_1}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)}{\hat{F}_X(t_x)}\right)$
1	1	0	-1	$\hat{F}^{(2)}_{MJ_2}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\hat{F}_X(t_x)}{F_X(t_x)}\right)$
1	1	$C_{F_X}(t_x)$	1	$\hat{F}^{(2)}_{MJ_3}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+C_{F_X}(t_x)}{\hat{F}_X(t_x)+C_{F_X}(t_x)}\right)$
1	1	$C_{F_X}(t_x)$	-1	$\hat{F}^{(2)}_{MJ_4}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\hat{F}_X(t_x)+C_{F_X}(t_x)}{F_X(t_x)+C_{F_X}(t_x)}\right)$
1	$\beta_2(F_X(t_x))$	$C_{F_X}(t_x)$	-1	$\hat{F}^{(2)}_{MJ_5}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\beta_2(F_X(t_x))\hat{F}_X(t_x)+C_{F_X}(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+C_{F_X}(t_x)}\right)$
1	$C_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(2)}_{MJ_6}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{C_{F_X}(t_x)\hat{F}_X(t_x)+\beta_2(F_X(t_x))}{C_{F_X}(t_x)F_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	1	$S_{F_X}(t_x)$	-1	$\hat{F}^{(2)}_{MJ_7}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\hat{F}_X(t_x)+S_{F_X}(t_x)}{F_X(t_x)+S_{F_X}(t_x)}\right)$
1	$S_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(2)}_{MJ_8}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{S_{F_X}(t_x)\hat{F}_X(t_x)+\beta_2(F_X(t_x))}{S_{F_X}(t_x)F_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	$\beta_2(F_X(t_x))$	$S_{F_X}(t_x)$	-1	$\hat{F}^{(2)}_{MJ_9}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\beta_2(F_X(t_x))\hat{F}_X(t_x)+S_{F_X}(t_x)}{\beta_2(F_X(t_x))F_X(t_x)+S_{F_X}(t_x)}\right)$
1	1	$\rho_{(F_Y(t_y),F_X(t_x))}$	1	$\hat{F}^{(2)}_{MJ_{10}}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}{\hat{F}_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
1	1	$\rho_{(F_Y(t_y),F_X(t_x))}$	-1	$\hat{F}^{(2)}_{MJ_{11}}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\hat{F}_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
1	1	$\beta_2(F_X(t_x))$	1	$\hat{F}^{(2)}_{MJ_{12}}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{F_X(t_x)+\beta_2(F_X(t_x))}{\hat{F}_X(t_x)+\beta_2(F_X(t_x))}\right)$
1	1	$\beta_2(F_X(t_x))$	-1	$\hat{F}^{(2)}_{MJ_{13}}(t_y) = \hat{F}^*_Y(t_y)\left(\frac{\hat{F}_X(t_x)+\beta_2(F_x(t_x))}{F_X(t_x)+\beta_2(F_X(t_x))}\right)$

given below.

$$(4.18)\quad Bias\left(\hat{F}^{(2)}_{MJ_1}(t_y)\right) \cong F_Y(t_y)\left(V_{02} - V_{11}^{*'}\right),$$

$$(4.19)\quad Bias\left(\hat{F}^{(2)}_{MJ_2}(t_y)\right) \cong F_Y(t_y)\left(V_{11}^{*'}\right),$$

$$(4.20)\quad Bias\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) \cong F_Y(t_y)\left[\alpha_i\left(\alpha_i V_{02} - V_{11}^{*'}\right)\right],$$

for $i = 3, 10, 12$, and

$$(4.21)\quad Bias\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) \cong F_Y(t_y)\left[\alpha_i\left(\alpha_i V_{02} + V_{11}^{*'}\right)\right],$$

for $i = 4\text{–}9, 11, 13$.
The *MSE* of the suggested family of estimators $\hat{F}^{(2)}_{MJ_i}(t_y)$ up to the first order are given below,

$$(4.22)\quad MSE\left(\hat{F}^{(2)}_{MJ_0}(t_y)\right) \cong F_Y^2(t_y)V_{20}^*,$$

$$(4.23)\quad MSE\left(\hat{F}^{(2)}_{MJ_1}(t_y)\right) \cong F^2_Y(t_y)\left(V^*_{20}+V_{02}-2V^{*'}_{11}\right),$$

$$(4.24)\quad MSE\left(\hat{F}^{(2)}_{MJ_2}(t_y)\right) \cong F^2_Y(t_y)\left(V^*_{20}+V_{02}+2V^{*'}_{11}\right),$$

$$(4.25)\quad MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) \cong F^2_Y(t_y)\left[V^*_{20}+\alpha_i\left\{\alpha_i V_{02}-2V^{*'}_{11}\right\}\right],$$

for $i = 3, 10, 12$, and

$$(4.26)\quad MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) \cong F^2_Y(t_y)\left[V^*_{20}+\alpha_i\left\{\alpha_i V_{02}+2V^{*'}_{11}\right\}\right],$$

for $i = 4–9, 11, 13$.

4.2. Exponential family of estimators. A general family of exponential estimators for estimating population distribution function is given by

$$(4.27)\quad \hat{F}_{Re}(t_y) = \hat{F}_Y(t_y)exp\left[\frac{(cF_X(t_x)+d)-\left(c\hat{F}_X(t_x)+d\right)}{(cF_X(t_x)+d)+\left(c\hat{F}_X(t_x)+d\right)}\right],$$

or

$$(4.28)\quad \hat{F}_{Re}(t_y) = \hat{F}_Y(t_y)exp\left[\frac{c\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{c\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2d}\right],$$

Expressing (4.28) in terms of $e's$, we have

$$(4.29)\quad \hat{F}_{Re}(t_y) \cong F_Y(t_y)\,(1+e_0)\left(1-\frac{1}{2}\psi e_1+\frac{3}{8}\psi^2 e_1^2\right),$$

with $\psi = \frac{cF_X(t_x)}{cF_X(t_x)+d}$.
Expanding the right hand side of (4.29) and retaining the terms up to the second order of $e's$, we have

$$(4.30)\quad \hat{F}_{Re}(t_y) \cong F_Y(t_y)\left(1+e_0-\frac{1}{2}\psi e_1+\frac{3}{8}\psi^2 e_1^2-\frac{1}{2}\psi e_0 e_1\right).$$

4.2.1. *Situation I - Non-response on both the study and the auxiliary variables:* $\hat{F}^*_Y(t_y)$, $\hat{F}^*_X(t_x)$. For Situation I, (4.28) can be written as

$$(4.31)\quad \hat{F}_{Re}(t_y) = \hat{F}^*_Y(t_y)exp\left[\frac{c\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{c\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2d}\right],$$

and from (4.30), we have

$$(4.32)\quad \hat{F}^{(1)}_{Re}(t_y) \cong F_Y(t_y)\left(1+e^*_0-\frac{1}{2}\psi e^*_1+\frac{3}{8}\psi^2 e^{*2}_1-\frac{1}{2}\psi e^*_0 e^*_1\right).$$

where $\hat{F}^{(1)}_{Re}(t_y)$ for case of Situation I.
The biases and MSE of the suggested family of estimators $\hat{F}^{(1)}_{Re_j}(t_y)$ up to the first order are given below.

$$(4.33)\quad Bias\left(\hat{F}^{(1)}_{Re_j}(t_y)\right) \cong F_Y(t_y)\left(\frac{3}{8}\psi_j^2 V^*_{02}-\frac{1}{2}\psi_j V^*_{11}\right),$$

$$(4.34)\quad MSE\left(\hat{F}^{(1)}_{Re_j}(t_y)\right) \cong F^2_Y(t_y)\left(V^*_{20}+\frac{1}{4}\psi_j^2 V^*_{02}-\psi_j V^*_{11}\right),$$

for $(j = 1, 2, ..., 10)$, with
$\psi_1 = 0$, $\psi_2 = \frac{F_X(t_x)}{F_X(t_x)+C_{F_X}}$, $\psi_3 = \frac{F_X(t_x)}{F_X(t_x)+\beta_2(F_X(t_x))}$, $\psi_4 = \frac{\beta_2(F_X(t_x))F_X(t_x)}{\beta_2(F_X(t_x))+C_{F_x}}$,

$\psi_5 = \frac{F_X(t_x)C_{F_X}}{F_X(t_x)\beta_2(F_X(t_x))+C_{F_X}}$, $\psi_6 = \frac{F_X(t_x)}{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}$, $\psi_7 = \frac{F_X(t_x)}{F_X(t_x)+\rho_{(F_Y(t_y),F_X(t_x))}}$,
$\psi_8 = \frac{F_X(t_x)\rho_{(F_Y(t_y),F_X(t_x))}}{F_X(t_x)\rho_{(F_Y(t_y),F_X(t_x))}+C_{F_X}}$, $\psi_9 = \frac{\beta_2(F_X(t_x))F_X(t_x)}{\beta_2(t_x)+C_{F_x}}$,
$\psi_{10} = \frac{F_X(t_x)\rho_{(F_Y(t_y),F_X(t_x))}}{F_X(t_x)\rho_{(F_Y(t_y),F_X(t_x))}+\beta_2(F_X(t_x))}$.
Different estimators can be generated from suggested family by substituting the suitable choices of (c, d). The generated estimators are in Table 5.

Table 5. Some members of the suggested class of estimators $\hat{F}^{(1)}_{Re_j}(t_y)$.

c	d	Estimator
1	0	$\hat{F}^{(1)}_{Re_1}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)}\right)$
1	$C_{F_X}(t_x)$	$\hat{F}^{(1)}_{Re_2}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\left(F_X(t)-\hat{F}^*_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
1	$\beta_2(F_X(t_x))$	$\hat{F}^{(1)}_{Re_3}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\left(F_X(t)-\hat{F}^*_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$
$\beta_2(F_X(t_x))$	$C_{F_X}(t_x)$	$\hat{F}^{(1)}_{Re_4}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\beta_2(F_X(t_x))\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{\beta_2(F_X(t_x))\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
$C_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	$\hat{F}^{(1)}_{Re_5}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{C_{F_X}(t_x)\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{C_{F_X}(t_x)\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$
1	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(1)}_{Re_6}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$C_{F_X}(t_x)$	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(1)}_{Re_7}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{C_{F_X}(t_x)\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{C_{F_x}\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$\rho_{(F_Y(t_y),F_X(t_x))}$	$C_{F_X}(t_x)$	$\hat{F}^{(1)}_{Re_8}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)-\hat{F}^*_X(t_x)\right)}{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
$\beta_2(F_X(t_x))$	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(1)}_{Re_9}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\beta_2(F_X)\left(F_X(t)-\hat{F}^*_X(t_x)\right)}{\beta_2(F_X(t_x))\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$\rho_{(F_Y(t_y),F_X(t_x))}$	$\beta_2(F_X(t)$	$\hat{F}^{(1)}_{Re_{10}}(t_y) = \hat{F}^*_Y(t_y)\text{exp}\left(\frac{\rho_{(F_Y(t_y),F_X(t_x))}\left(F^*_X(t_x)-\hat{F}_X(t_x)\right)}{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)+\hat{F}^*_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$

4.2.2. *Situation II - Non-response only on the study variable:* $\hat{F}^*_Y(t_y)$. For Situation II, (4.28) can be written as

$$\hat{F}_{Re}(t_y) = \hat{F}^*_Y(t_y)exp\left[\frac{c\left(F_X(t_x) - \hat{F}_X(t_x)\right)}{c\left(F_X(t_x) + \hat{F}_X(t_x)\right) + 2d}\right], \quad (4.35)$$

and from (4.30) we have,

$$\hat{F}^{(2)}_{Re}(t_y) \cong F_Y(t_y)\left\{\left(1 + e^*_0 - \frac{1}{2}\psi e_1 + \frac{3}{8}\psi^2 e_1^2 - \frac{1}{2}\psi e^*_0 e_1\right)\right\}. \quad (4.36)$$

where $\hat{F}^{(2)}_{Re}(t_y)$ for case of Situation II.
The biases of the suggested family of estimators $\left(\hat{F}^{(2)}_{Re_j}(t_y)\right)$ up to first order are given below.

$$Bias\left(\hat{F}^{(2)}_{Re_j}(t)\right) \cong F_Y(t_y)\left[\psi_j\left(\frac{3}{8}\psi_j V_{02} - \frac{1}{2}V^{*'}_{11}\right)\right]. \quad (4.37)$$

The MSE of the suggested family of estimators $\left(\hat{F}^{(2)}_{Re_j}(t_y)\right)$ up to first order are given below.

$$MSE\left(\hat{F}^{(2)}_{Re_j}(t)\right) \cong F^2_Y(t_y)\left[V^*_{20} + \left(\frac{1}{4}\psi_j^2 V_{02} - \psi_j V^{*'}_{11}\right)\right]. \quad (4.38)$$

Different estimators can be generated from proposed class of estimators by substituting the suitable choices of c, and d are listed in Table 6.

Table 6. Some members of the suggested class of estimators $\hat{F}^{(2)}_{Re_j}(t_y)$.

c	d	Estimator
1	0	$\hat{F}^{(2)}_{Re_1}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}_X(t_x)\right)}\right)$
1	$C_{F_X}(t_x)$	$\hat{F}^{(2)}_{Re_2}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\left(F_X(t)-\hat{F}_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
1	$\beta_2(F_X(t_x))$	$\hat{F}^{(2)}_{Re_3}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\left(F_X(t)-\hat{F}_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$
$\beta_2(F_X(t_x))$	$C_{F_X}(t_x)$	$\hat{F}^{(2)}_{Re_4}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\beta_2(F_X(t))\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{\beta_2(F_X(t_x))\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
$C_{F_X}(t_x)$	$\beta_2(F_X(t_x))$	$\hat{F}^{(2)}_{Re_5}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{C_{F_X}(t_x)\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{C_{F_X}(t_x)\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$
1	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(2)}_{Re_6}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$C_{F_X}(t_x)$	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(2)}_{Re_7}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{C_{F_X}(t_x)\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{C_{F_x}\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$\rho_{(F_Y(t_y),F_X(t_x))}$	$C_{F_X}(t_x)$	$\hat{F}^{(2)}_{Re_8}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2C_{F_X}(t_x)}\right)$
$\beta_2(F_X(t_x))$	$\rho_{(F_Y(t_y),F_X(t_x))}$	$\hat{F}^{(2)}_{Re_9}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\beta_2(F_X)\left(F_X(t)-\hat{F}_X(t_x)\right)}{\beta_2(F_X(t_x))\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\rho_{(F_Y(t_y),F_X(t_x))}}\right)$
$\rho_{(F_Y(t_y),F_X(t_x))}$	$\beta_2(F_X(t_x)$	$\hat{F}^{(2)}_{Re_{10}}(t_y) = \hat{F}^{*}_{Y}(t_y)\text{exp}\left(\frac{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)-\hat{F}_X(t_x)\right)}{\rho_{(F_Y(t_y),F_X(t_x))}\left(F_X(t_x)+\hat{F}_X(t_x)\right)+2\beta_2(F_X(t_x))}\right)$

5. Proposed generalized class of exponential ratio type estimators

We propose a generalized class of exponential ratio type estimators, given by

$$\hat{F}_{MJP}(t_y) = K_1\left(\hat{F}_{MJ_i}(t_y)\right) + (1-K_1)\left(\hat{F}_{Re_j}(t_y)\right), \tag{5.1}$$

for $i = 1, 2, 3, ..., 13$, $j = 1, 2, 3, ..., 10$ and K_1 is suitably chosen constant.

5.1. Situation I - Non-response on both the study and the auxiliary variables: $\hat{F}^{*}_{Y}(t_y)$, $\hat{F}^{*}_{X}(t_x)$.

For Situation I, (5.1) can be written as

$$\hat{F}^{(1)}_{MJP}(t_y) = K_1\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) + (1-K_1)\left(\hat{F}^{(1)}_{Re_j}(t_y)\right), \tag{5.2}$$

where $\hat{F}^{(1)}_{MJP}(t_y), \hat{F}^{(1)}_{MJ_i}(t_y)$ and $\hat{F}^{(1)}_{MJ_j}(t_y)$ for the case of Situation I respectively. Expressing (5.2) in terms of $e's$, we have

$$\hat{F}^{(1)}_{MJP}(t_y) \cong F_Y(t_y)\begin{bmatrix} e_0^* + \left(\frac{1}{2}K_1 - \frac{1}{2} - \alpha g K_1\right)\psi e_1^* + \frac{1}{2}\left(g+\psi^2\right)K_1 g\alpha^2 e_1^{*2} \\ +\frac{3}{8}\left(1+K_1\right)\psi^2 e_1^{*2} + \left(\frac{1}{2}+\frac{1}{2}K_1 - K_1\alpha g\right)\psi e_0^* e_1^* \end{bmatrix}. \tag{5.3}$$

The bias and MSE of $\hat{F}^{(1)}_{MJP}(t_y)$, up to first order of approximation, is given by

$$(5.4)\qquad Bias\left(\hat{F}^{(1)}_{MJP}(t_y)\right) \cong F_Y(t_y)\left[\begin{array}{r}\left\{\begin{array}{r}\left(\frac{1}{2}\psi-\alpha g\psi\right)V^*_{11}\\ +\left(\frac{1}{2}g\alpha^2+\frac{1}{2}g^2\alpha^2-\frac{3}{8}\right)V^*_{02}\end{array}\right\}K_1\\ +\left(\frac{3}{8}\psi^2V^*_{02}-\psi V^*_{11}\right)\end{array}\right].$$

and

$$(5.5)\qquad MSE\left(\hat{F}^{(1)}_{MJP}(t_y)\right) \cong \frac{F^2_Y(t_y)}{4}\left[\begin{array}{r}4V^*_{20}+\left\{\begin{array}{r}4\left(K_1-K_1^2+\alpha gK_1^2\right)\alpha g\psi^2\\ +\left(1-\psi K_1\right)^2\end{array}\right\}V^*_{02}\\ +4\left(K_1-1-8\alpha g\right)\psi V^*_{11}\end{array}\right].$$

By differentiating (5.5) with respect to K_1, we get the optimum value as $K_1^{(opt)}=\frac{2V^{*2}_{11}-\psi V^{*2}_{02}}{(2\alpha g-1)\psi V^{*2}_{02}}$.
The minimum MSE of $\hat{F}^{(1)}_{MJP}(t_y)$ at optimum value of K_1 , up to first order of approximation is given by

$$(5.6)\qquad MSE_{min}\left(\hat{F}^{(1)}_{MJP}(t_y)\right) \cong F^2_Y(t_y)V^*_{20}\left(1-\rho^2_{(F_Y(t_y),F_X(t_x))}\right),$$

where $\rho^2_{(F_Y(t_y),F_X(t_x))}=\frac{(V^*_{11})^2}{V^*_{20}V^*_{02}}$.

5.2. Situation II - Non-response only on the study variable: $\hat{F}^*_Y(t_y)$. For Situation II, (5.1) can be written as,

$$(5.7)\qquad \hat{F}^{(2)}_{MJP}(t_y)=K_1\left(\hat{F}^{(2)}_{MJ_i}(t)\right)+(1-K_1)\left(\hat{F}^{(2)}_{Re_j}(t)\right),$$

where $\hat{F}^{(2)}_{MJP}(t_y), \hat{F}^{(2)}_{MJ_i}(t_y)$ and $\hat{F}^{(2)}_{MJ_j}(t_y)$ for the case of Situation II.
The bias and minimum MSE up to first order of approximation at optimum value $K_1^{(opt)}=\frac{2(V^{*'}_{11})^2-V^2_{02}}{V^2_{02}}$ is given by

$$(5.8)\qquad Bias\left(\hat{F}^{(2)}_{MJP}(t_y)\right) \cong F_Y(t_y)\left[(\frac{5}{8}V_{02}-\frac{1}{2}V^{*'}_{11})K_1^{(opt)}+\frac{3}{8}V_{02}-V^{*'}_{11}\right],$$

$$(5.9)\qquad MSE_{min}\left(\hat{F}^{(2)}_{MJP}(t_y)\right) \cong F^2_Y(t_y)V^*_{20}\left(1-\rho^2_{(2)(F_Y(t_y),F_X(t_x))}\right),$$

where $\rho^2_{(2)(F_Y(t_y),F_X(t_x))}=\frac{(V^{*'}_{11})^2}{V^*_{20}V_{02}}$ for Situation II .

5.2.1. *Efficiency comparisons for general family of estimators.* In this section, suggested estimators under Situation I are compared in terms of MSEs.
Condition i: By Equation (4.11) and Equation (5.6)

$$MSE\left(\hat{F}^{(1)}_{MJ_0}(t_y)\right)-MSE_{min}\left(\hat{F}^{(1)}_{MJP}(t_y)\right)>0,\qquad \text{if}$$
$$V^*_{20}\rho^2_{(F_Y(t_y),F_X(t_x))}>0.$$

Condition ii: By Equations (4.12), (4.14) and Equation (5.6)

$$\begin{aligned}MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)-MSE_{min}\left(\hat{F}^{(1)}_{MJP}(t_y)\right) &> 0,\qquad \text{for } i\text{=1,3, 10, 12, if}\\ V^*_{20}\rho^2_{(F_Y(t_y),F_X(t_x))}+\alpha_i\left(\alpha_iV^*_{02}-2V^*_{11}\right) &> 0.\end{aligned}$$

Condition iii: By Equations (4.13), (4.15) and Equation (5.6)

$$\begin{aligned} MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right) - MSE_{min}\left(\hat{F}^{(1)}_{MJP}(t_y)\right) &> 0, \quad \text{for } i\text{=2, 4-9, 11, 13, if} \\ V^*_{20}\rho^2_{(F_Y(t_y),F_X(t_x))} + \alpha_i\left(\alpha_i V^*_{02} + 2V^*_{11}\right) &> 0. \end{aligned}$$

Note that the proposed estimator $\left(\hat{F}^{(1)}_{MJP}(t_y)\right)$ is more efficient than the other suggested estimators
$\left(\hat{F}^{(1)}_{MJ_1}(t_y)\right), ..., \left(\hat{F}^{(1)}_{MJ_{13}}(t_y)\right)$, when above conditions are satisfied.
The comparisons of estimators for Situation II are given below.
Condition i: By Equation (4.22) and Equation (5.9)

$$\begin{aligned} MSE\left(\hat{F}^{(2)}_{MJ_0}(t_y)\right) - MSE_{min}\left(\hat{F}^{(2)}_{MJP}(t_y)\right) &> 0, \quad \text{if} \\ V^*_{20}\rho^2_{(2)(F_Y(t_y),F_X(t_x))} &> 0. \end{aligned}$$

Condition ii: By Equations (4.23), (4.25) and Equation (5.9)

$$MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) - MSE_{min}\left(\hat{F}^{(2)}_{MJP}(t_y)\right) > 0, \quad \text{for } i\text{=1,3, 10, 12, if}$$

$$V^*_{20}\rho^2_{(2)(F_Y(t_y),F_X(t_x))} + \alpha_i\left(\alpha_i V^*_{02} - 2V^{*'}_{11}\right) > 0.$$

Condition iii: By Equations (4.24), (4.26) and Equation (5.9)

$$MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right) - MSE_{min}\left(\hat{F}^{(2)}_{MJP}(t_y)\right) > 0, \quad \text{for } i\text{=2, 4-9, 11, 13, if}$$

$$V^*_{20}\rho^2_{(2)(F_Y(t_y),F_X(t_x))} + \alpha_i\left(\alpha_i V^*_{02} + 2V^{*'}_{11}\right) > 0.$$

Note that the proposed estimator $\left(\hat{F}^{(2)}_{MJP}(t_y)\right)$ is more efficient than the other suggested estimators
$\left(\hat{F}^{(2)}_{MJ_1}(t_y)\right), ..., \left(\hat{F}^{(2)}_{MJ_{13}}(t_y)\right)$, when above conditions are satisfied.

5.2.2. *Efficiency comparisons for exponential family of estimators.* In this section, suggested estimators are compared under Situation I in terms of $MSEs$.
Condition (i–x): By Equation (4.11) and Equation (4.34)

$$MSE\left(\hat{F}^{(1)}_{MJ_0}(t_y)\right) - MSE\left(\hat{F}^{(1)}_{Re_j}(t_y)\right) > 0, \quad \text{for } j\text{=1, 2,..,10, if}$$

$$\left(\frac{1}{4}\psi_j^2 V^*_{02} - \psi_j V^*_{11}\right) > 0.$$

Condition xi: By Equations (4.34) and Equation (5.6)

$$MSE\left(\hat{F}^{(1)}_{Re_j}(t_y)\right) - MSE_{min}\left(\hat{F}^{(1)}_{MJP}(t_y)\right) > 0.$$

$$V^*_{20}\rho^2_{(F_Y(t_y),F_X(t_x))} + \psi_i\left(\frac{1}{2}\psi_i V^*_{02} - V^*_{11}\right) > 0.$$

Note that the proposed estimator $\left(\hat{F}^{(1)}_{MJP}(t_y)\right)$ is more efficient than the other suggested estimators
$\left(\hat{F}^{(1)}_{Re_1}(t_y)\right), ..., \left(\hat{F}^{(1)}_{Re_{10}}(t_y)\right)$, when above conditions are satisfied.

Proposed and existing estimators Under Situation II are compared in terms of $MSEs$.
Condition (i–x): By Equation (4.11) and Equation (4.38)

$$MSE\left(\hat{F}^{(2)}_{MJ_0}(t_y)\right) - MSE\left(\hat{F}^{(2)}_{Re_j}(t_y)\right) > 0, \quad \text{for } j\text{=1, 2,..,10, if}$$
$$\left(\frac{1}{4}\psi_j^2 V^*_{02} - \psi_j V^*_{11}\right) > 0.$$

Condition xi: By Equations (4.38) and Equation (5.9)

$$MSE\left(\hat{F}^{(2)}_{Re_j}(t_y)\right) - MSE_{min}\left(\hat{F}^{(2)}_{MJP}(t_y)\right) > 0.$$
$$V^*_{20}\rho^2_{(2)(F_Y(t_y),F_X(t_x))} + \psi_i\left(\frac{1}{2}\psi_i V^*_{02} - V^*_{11}\right) > 0.$$

Note that the proposed estimator $\left(\hat{F}^{(2)}_{MJP}(t_y)\right)$ is more efficient than the other suggested estimators
$\left(\hat{F}^{(2)}_{Re_1}(t_y)\right)$, ..., $\left(\hat{F}^{(2)}_{Re_{10}}(t_y)\right)$, when above conditions are satisfied.

6. Numerical study

In this section, we consider the following data set for numerical comparisons of suggested estimators considered here.
Population I: Source: Sarndal et al. [32], (P-662)
The CO 120 data is based on 120 countries across five continents.
Let $Y =$ P–83, 1983 population (in million) and $X =$ P–80, 1980 population (in million).
$N = 120$, $m = 50$, $W_M = 0.25$, $k = 2,3,4$, $f = 0.41667$, $\lambda_1 = 0.0117$, $\lambda_2 = 0.0050$, $F_Y(t_y) = 0.816667$, $F_X(t_x) = 0.808333$, $N_{11} = 97$, $N_{12} = 01$, $N_{21} = 00$, $N_{22} = 22$, $\rho_{(F_Y(t_y),F_X(t_x))} = 0.9730$, $C_{F_Y(t_y)} = 0.47578$, $C_{F_X(t_x)} = 0.48889$, $\beta_1(F_X(t_x)) = -1.5667$, $\beta_2(F_X(t_x)) = 3.454419$. Let $I(Y_i \le t_y) = 1$ for $Y \le 0.816667$, $I(Y_i \le t_y) = 0$ for all $Y > 0.816667$ and $I(X_i \le t_x) = 1$ for $X \le 0.808333$, $I(X_i \le t_x) = 0$ for all $X > 0.808333$
The non-response rate in the given population is considered to be 25 percent, taken as last 30 units of the population.
$N_M = 30$, $F^{(2)}_Y(t_{y_2}) = 0.66667$, $F^{(2)}_X(t_{x_2}) = 0.66667$, $N^{(2)}_{11} = 20$, $N^{(2)}_{12} = 00$, $N^{(2)}_{21} = 00$, $N^{(2)}_{22} = 10$.
Let $I\left(Y^{(2)}_i \le t_{y_2}\right) = 1$ for $Y_2 \le 0.66667$, $I\left(Y^{(2)}_i \le t_{y_2}\right) = 0$ for all $Y_2 > 0.66667$ and $I\left(X^{(2)}_i \le t_{x_2}\right) = 1$ for $X_2 \le 0.66667$, $I\left(X^{(2)}_i \le t_{x_2}\right) = 0$ for all $X_2 > 0.66667$.
Population II: **Source: Sarjinder Singh [38], P.1113**
Let $Y =$ Duration of sleep (in minutes) and $X =$ Age of old persons(≥ 50) years.
$N = 30$, $m = 12$, $W_M = 0.25$, $k = 2$, $f = 0.400$, $\lambda_1 = 0.020$, $\lambda_2 = 0.0833$, $F_Y(t) = 0.5000$, $F_X(t) = 0.5333$, $N_{11} = 02$, $N_{12} = 13$, $N_{21} = 14$, $N_{22} = 01$, $\rho_{(F_X,F_Y)} = -0.80178$, $C_{F_Y(t)} = 1.00$, $C_{F_X(t)} = 0.9355$, $\beta_1(F_X(t)) = -0.1335$, $\beta_2(F_X(t)) = 1.0177$. Let $I(Y_i \le t_y) = 1$ for $Y \le 0.5000$, $I(Y_i \le t_y) = 0$ for all $Y > 0.5000$ and $I(X_i \le t_x) = 1$ for $X \le 0.5333$, $I(X_i \le t_x) = 0$ for all $X > 0.5333$
The non-response rate in the given population is considered to be 25 percent, taken as last 08 units of the population.
$N_M = 08$, $F^{(2)}_Y(t) = 0.25$, $F^{(2)}_X(t) = 0.875$, $N^{(2)}_{11} = 01$, $N^{(2)}_{12} = 01$, $N^{(2)}_{21} = 06$, $N^{(2)}_{22} = 00$.

We use the following expressions for Percentage Relative Efficiency (PRE).

$$\mathbf{i}: PRE\left(\hat{F}^{(\cdot)}_{MJ_0}(t_y), \hat{F}^{(\cdot)}_{MJ_i}(t_y)\right) = \frac{\hat{F}^{(\cdot)}_{MJ_0}(t_y)}{\hat{F}^{(\cdot)}_{MJ_i}(t_y)} \times 100,$$

$$\mathbf{ii}: PRE\left(\hat{F}^{(.)}_{MJ_0}(t_y), \hat{F}^{(.)}_{Re_j}(t_y)\right) = \frac{\hat{F}^{(.)}_{MJ_0}(t_y)}{\hat{F}^{(.)}_{Re_j}(t_y)} \times 100,$$

where (.) can be replaced by (1) and (2) under Situation I and Situation II respectively. We have computed the Absolute Relative Bias (ARB) for different suggested estimators by using the following expression, given by

$$ARB^{(.)} = \frac{|Bias\left(\hat{F}^{(.)}_{MJ_i}(t_y)\right)| \quad \text{or} \quad |Bias\left(\hat{F}^{(.)}_{Re_j}(t_y)\right)|}{|F_Y(t_y)|},$$

for $i = 1, 2, ..., 13$ and for $j = 1, 2, .., 10$.

MSE , PRE and ARB values based on given data set under both Situations I and II are given in Tables 7–14.

Table 7. MSE, PRE and ARB values of $\hat{F}^{(1)}_{MJ_i}(t_y)$ for different values of k of Population I

	k=2			k =3			k=4		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(1)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00397	100.00	-	0.00508	100.00	-
$\hat{F}^{(1)}_{MJ_1}(t_y)$	0.00009	2906.35	0.00002	0.00009	4031.93	0.00003	0.00009	5155.06	0.00005
$\hat{F}^{(1)}_{MJ_2}(t_y)$	0.01157	24.68	0.00430	0.01607	24.70	0.00598	0.02056	24.71	0.00767
$\hat{F}^{(1)}_{MJ_3}(t_y)$	0.00044	651.29	0.00102	0.00059	671.22	0.00142	0.00074	682.97	0.00182
$\hat{F}^{(1)}_{MJ_4}(t_y)$	0.00434	37.65	0.00434	0.01054	37.65	0.00604	0.01349	37.65	0.00773
$\hat{F}^{(1)}_{MJ_5}(t_y)$	0.00990	28.87	0.00676	0.01374	28.88	0.00940	0.01758	28.89	0.01204
$\hat{F}^{(1)}_{MJ_6}(t_y)$	0.00348	82.16	0.00049	0.00483	82.14	0.00068	0.00618	82.13	0.00087
$\hat{F}^{(1)}_{MJ_7}(t_y)$	0.00806	35.48	0.00482	0.01112	35.48	0.00670	0.01432	35.48	0.00858
$\hat{F}^{(1)}_{MJ_8}(t_y)$	0.00336	84.93	0.00039	0.00467	84.91	0.00055	0.00598	84.90	0.00070
$\hat{F}^{(1)}_{MJ_9}(t_y)$	0.01017	28.10	0.00705	0.01412	28.11	0.00981	0.01807	28.12	0.01256
$\hat{F}^{(1)}_{MJ_{10}}(t_y)$	0.00087	329.53	0.00107	0.00119	332.66	0.00149	0.00153	334.45	0.00191
$\hat{F}^{(1)}_{MJ_{11}}(t_y)$	0.00607	47.04	0.00284	0.00844	47.03	0.00394	0.01080	47.03	0.00505
$\hat{F}^{(1)}_{MJ_{12}}(t_y)$	0.00188	152.27	0.00066	0.00260	152.48	0.00092	0.00333	152.60	0.00118
$\hat{F}^{(1)}_{MJ_{13}}(t_y)$	0.00405	70.51	0.00097	0.00563	70.49	0.00135	0.00721	70.47	0.00173
$\hat{F}^{(1)}_{MJP}(t_y)$	0.00009	**3031.09**	0.00203	0.00009	**4184.14**	0.00286	0.00009	**5337.46**	0.00369

Table 8. MSE, PRE and ARB values of $\hat{F}^{(2)}_{MJ_i}(t_y)$ for different values of k Population I

	k=2			k =3			k=4		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(2)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00397	100.00	-	0.00508	100.00	-
$\hat{F}^{(2)}_{MJ_1}(t_y)$	0.00121	236.31	0.00166	0.00232	171.04	0.00333	0.00343	148.04	0.00500
$\hat{F}^{(2)}_{MJ_2}(t_y)$	0.00820	34.87	0.00262	0.00931	42.64	0.00261	0.01042	48.76	0.00262
$\hat{F}^{(2)}_{MJ_3}(t_y)$	0.00140	206.51	0.00003	0.00251	158.22	0.00068	0.00362	140.35	0.00132
$\hat{F}^{(2)}_{MJ_4}(t_y)$	0.00575	49.69	0.00330	0.00686	57.84	0.00394	0.00797	63.72	0.00459
$\hat{F}^{(2)}_{MJ_5}(t_y)$	0.00717	39.79	0.00533	0.00828	47.95	0.00653	0.00939	54.11	0.00774
$\hat{F}^{(2)}_{MJ_6}(t_y)$	0.00324	88.31	0.00031	0.00435	91.30	0.00033	0.00546	93.07	0.00035
$\hat{F}^{(2)}_{MJ_7}(t_y)$	0.00604	47.34	0.00369	0.00715	55.53	0.00444	0.00826	61.51	0.00516
$\hat{F}^{(2)}_{MJ_8}(t_y)$	0.00317	90.24	0.00025	0.00428	92.78	0.00026	0.00539	94.27	0.00028
$\hat{F}^{(2)}_{MJ_9}(t_y)$	0.00733	38.97	0.00558	0.00844	47.00	0.00686	0.00956	53.11	0.00814
$\hat{F}^{(2)}_{MJ_{10}}(t_y)$	0.00165	172.94	0.00031	0.00276	143.62	0.00004	0.00387	131.11	0.00038
$\hat{F}^{(2)}_{MJ_{11}}(t_y)$	0.00482	59.25	0.00207	0.00593	66.88	0.00241	0.00704	72.11	0.00276
$\hat{F}^{(2)}_{MJ_{12}}(t_y)$	0.00226	126.36	0.00034	0.00337	117.67	0.00028	0.00448	113.29	0.00022
$\hat{F}^{(2)}_{MJ_{13}}(t_y)$	0.00359	79.68	0.00065	0.00470	84.49	0.00071	0.00581	87.45	0.00077
$\hat{F}^{(2)}_{MJP}(t_y)$	0.00120	**237.34**	0.00121	0.000231	**171.43**	0.00121	0.00343	**148.26**	0.00121

Table 9. MSE, PRE and ARB values of $\hat{F}^{(1)}_{Re_j}(t_y)$ for different values of k Population I

	k=2			k=3			k=4		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(1)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00397	100.00	-	0.00508	100.00	-
$\hat{F}^{(1)}_{Re_1}(t_y)$	0.00073	389.71	0.00048	0.00100	397.76	0.00068	0.00128	397.67	0.00088
$\hat{F}^{(1)}_{Re_2}(t_y)$	0.00136	210.26	0.00203	0.00188	211.01	0.00283	0.00240	211.44	0.00363
$\hat{F}^{(1)}_{Re_3}(t_y)$	0.00234	122.10	0.00076	0.00325	122.16	0.00105	0.00416	122.19	0.00135
$\hat{F}^{(1)}_{Re_4}(t_y)$	0.00096	299.10	0.00245	0.00132	301.45	0.00342	0.00167	302.79	0.00439
$\hat{F}^{(1)}_{Re_5}(t_y)$	0.00252	113.27	0.00049	0.00350	113.30	0.00069	0.00448	113.32	0.00088
$\hat{F}^{(1)}_{Re_6}(t_y)$	0.00171	167.20	0.00161	0.00237	167.52	0.00224	0.00303	167.69	0.00287
$\hat{F}^{(1)}_{Re_7}(t_y)$	0.00209	136.66	0.00110	0.00290	136.78	0.00154	0.00371	136.85	0.00197
$\hat{F}^{(1)}_{Re_8}(t_y)$	0.00137	208.34	0.00202	0.00190	209.06	0.00281	0.00242	209.48	0.00361
$\hat{F}^{(1)}_{Re_9}(t_y)$	0.00114	250.78	0.00227	0.00157	252.13	0.00317	0.00201	252.90	0.00406
$\hat{F}^{(1)}_{Re_{10}}(t_y)$	0.00235	121.54	0.00074	0.00326	121.60	0.00103	0.00418	121.63	0.00132
$\hat{F}^{(1)}_{MJP}(t_y)$	0.00009	**3031.09**	0.00203	0.00009	**4184.14**	0.00286	0.00009	**5337.46**	0.00369

Table 10. MSE, PRE and ARB values of $\hat{F}^{(2)}_{Re_j}(t_y)$ for different values of k Population I

	k=2			k=3			k=4		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(2)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00397	100.00	-	0.00508	100.00	-
$\hat{F}^{(2)}_{Re_1}(t_y)$	0.00157	181.76	0.00027	0.00268	147.90	0.00027	0.00379	133.88	0.00027
$\hat{F}^{(2)}_{Re_2}(t_y)$	0.00195	146.66	0. 00283	0.00306	129.72	0. 00283	0.00417	121.80	0. 00283
$\hat{F}^{(2)}_{Re_3}(t_y)$	0.00254	112.37	0.00046	0.00365	108.61	0.00046	0.00476	106.60	0.00046
$\hat{F}^{(2)}_{Re_4}(t_y)$	0.00170	167.57	0.00148	0.00282	140.92	0.00148	0.00393	129.34	0.00148
$\hat{F}^{(2)}_{Re_5}(t_y)$	0.00265	107.68	0.00030	0.00376	105.41	0.00030	0.00488	104.18	0.00030
$\hat{F}^{(2)}_{Re_6}(t_y)$	0.00216	132.30	0.00097	0.00327	121.33	0.00097	0.00438	115.92	0.00097
$\hat{F}^{(2)}_{Re_7}(t_y)$	0.00239	119.49	0.00067	0.00350	113.31	0.00067	0.0046	110.10	0.00067
$\hat{F}^{(2)}_{Re_8}(t_y)$	0.00196	146.10	0.00122	0.00307	129.40	0.00122	0.00418	121.58	0.00122
$\hat{F}^{(2)}_{Re_9}(t_y)$	0.00182	157.36	0.00137	0.00293	135.59	0.00137	0.00404	125.80	0.00137
$\hat{F}^{(2)}_{Re_{10}}(t_y)$	0.00255	112.08	0.00045	0.00366	108.42	0.00045	0.00478	106.46	0.00045
$\hat{F}^{(2)}_{MJP}(t_y)$	0.00120	**237.34**	0.00121	0.00231	**171.43**	0.00121	0.00343	**148.26**	0.00121

Table 11. Values of MSE, PRE and ARB, of $\hat{F}^{(1)}_{MJ_i}(t)$ for different values of k of Population II

	k=2			k =3			k=4			k =5		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator												
$\hat{F}^{(2)}_{MJ_0}(t)$	0.01641	100.00	-	0.02031	100.00	-	0.02422	100.00	-	0.02812	100.00	-
$\hat{F}^{(2)}_{MJ_1}(t)$	0.05176	31.69	0.11045	0.06133	33.12	0.13340	0.07090	34.15	0.15635	0.08048	34.95	0.17930
$\hat{F}^{(2)}_{MJ_2}(t)$	0.00693	236.58	0.04483	0.00918	221.23	0.05215	0.01143	211.92	0.05948	0.01367	205.66	0.06680
$\hat{F}^{(2)}_{MJ_3}(t)$	0.02625	62.50	0.02493	0.03175	63.97	0.02964	0.03725	65.01	0.03437	0.04275	65.79	0.03908
$\hat{F}^{(2)}_{MJ_4}(t)$	0.00997	164.48	0.00762	0.01281	158.51	0.00822	0.01565	154.70	0.00882	0.01850	152.06	0.00942
$\hat{F}^{(2)}_{MJ_5}(t)$	0.00992	165.36	0.00761	0.01275	159.27	0.00820	0.01558	155.40	0.00878	0.01842	152.72	0.00936
$\hat{F}^{(2)}_{MJ_6}(t)$	0.01043	157.24	0.00764	0.01335	152.13	0.00836	0.01627	148.85	0.00908	0.01919	146.57	0.00980
$\hat{F}^{(2)}_{MJ_7}(t)$	0.00828	198.12	0.00564	0.01083	187.57	0.00525	0.01338	181.03	0.00487	0.01593	176.59	0.00448
$\hat{F}^{(2)}_{MJ_8}(t)$	0.02049	80.05	0.00928	0.02507	81.03	0.01093	0.02964	81.71	0.01258	0.03421	82.21	0.01424
$\hat{F}^{(2)}_{MJ_9}(t)$	0.00824	199.07	0.00554	0.01078	188.37	0.00511	0.01332	181.75	0.00469	0.01587	177.26	0.00426
$\hat{F}^{(2)}_{MJ_{10}}(t)$	0.02295	71.49	0.00654	0.02748	73.91	0.006769	0.03202	75.64	0.00732	0.03655	76.94	0.00881
$\hat{F}^{(2)}_{MJ_{11}}(t)$	0.02503	65.55	0.007208	0.03049	66.610	0.00749	0.0.599	67.29	0.00821	0.04080	68.92	0.00881
$\hat{F}^{(2)}_{MJ_{12}}(t)$	0.02564	63.98	0.02317	0.03104	65.43	0.02754	0.03645	66.45	0.03190	0.04185	67.20	0.03626
$\hat{F}^{(2)}_{MJ_{13}}(t)$	0.01023	160.38	0.00765	0.01311	154.90	0.00832	0.01600	151.39	0.00900	0.01888	148.96	0.00967
$\hat{F}^{(2)}_{MJP}(t)$	0.00670	**244.80**	0.08538	0.00893	**227.25**	0.09956	0.01117	**216.75**	0.11374	0.01341	**209.76**	0.12792

Table 12. Values of MSE, PRE and ARB, of $\hat{F}^{(2)}_{MJ_i}(t)$ for different values of k of Population II

	k=2			k=3			k=4			k=5		
Estimator	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
$\hat{F}^{(1)}_{MJ_0}(t)$	0.01641	100.00	-	0.02031	100.00	-	0.02422	100.00	-	0.02812	100.00	-
$\hat{F}^{(1)}_{MJ_1}(t)$	0.04610	35.59	0. 10313	0.05000	40.62	0.11875	0.05391	44.92	0.13438	0.05781	48.65	0.15000
$\hat{F}^{(1)}_{MJ_2}(t)$	0.00859	190.90	0.03750	0.01250	162.50	0.03750	0.01641	147.62	0.03750	0.02031	138.46	0.03750
$\hat{F}^{(1)}_{MJ_3}(t)$	0.02466	66.54	0.02227	0.02856	71.11	0.02433	0.03247	74.59	0.02639	0.03637	77.32	0.02845
$\hat{F}^{(1)}_{MJ_4}(t)$	0.01104	148.61	0.00496	0.01495	135.90	0.00290	0.01885	128.46	0.00084	0.02276	123.58	0.00121
$\hat{F}^{(1)}_{MJ_5}(t)$	0.01099	149.20	0.00492	0.01490	136.30	0.00282	0.01881	128.76	0.00071	0.02271	123.82	0.00140
$\hat{F}^{(1)}_{MJ_6}(t)$	0.01142	143.64	0.00523	0.01533	132.52	0.00354	0.01923	125.91	0.00185	0.02314	121.54	0.00016
$\hat{F}^{(1)}_{MJ_7}(t)$	0.00964	170.22	0.00186	0.01354	149.97	0.00231	0.01745	138.78	0.00448	0.02136	131.69	0.01065
$\hat{F}^{(1)}_{MJ_8}(t)$	0.01983	82.73	0.00806	0.02374	85.57	0.00849	0.02764	87.61	0.00893	0.03155	89.14	0.00936
$\hat{F}^{(1)}_{MJ_9}(t)$	0.00960	170.79	0.00172	0.01351	150.33	0.00252	0.01742	139.04	0.00676	0.02132	131.89	0.01100
$\hat{F}^{(1)}_{MJ_{10}}(t)$	0.02232	73.50	0.18444	0.02623	77.45	0.24609	0.03013	80.37	0.30774	0.03404	82.62	0.36939
$\hat{F}^{(1)}_{MJ_{11}}(t)$	0.02405	69.22	0.00540	0.02934	69.22	0.00589	0.03396	71.32	0.00601	0..03806	73.89	0.00620
$\hat{F}^{(1)}_{MJ_{12}}(t)$	0.02415	67.94	0.02065	0.02850	72.41	0.02250	0.03196	75.78	0.02434	0.03586	78.42	0.02619
$\hat{F}^{(1)}_{MJ_{13}}(t)$	0.01125	145.81	0.00514	0.01516	134.00	0.00329	0.01906	127.03	0.00144	0.02297	122.44	0.00040
$\hat{F}^{(1)}_{MJP}(t)$	0.00837	**196.00**	0.07121	0.01228	**165.45**	0.07121	0.01618	**149.65**	0.07121	0.02008	**140.00**	0.07121

Table 13. Values of MSE, PRE and ARB, of $\hat{F}^{(2)}_{Re_j}(t)$ for different values of k of Population II

	k=2			k=3			k=4			k=5		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator												
$\hat{F}^{(2)}_{MJ_0}(t)$	0.01641	100.00	-	0.02031	100.00	-	0.02422	100.00	-	0.02812	100	-
$\hat{F}^{(2)}_{Re_1}(t)$	0.03085	53.18	0.04182	0.03709	54.77	0.04849	0.04332	55.90	0.05516	0.04956	56.74	0.06183
$\hat{F}^{(2)}_{Re_2}(t)$	0.02090	78.49	0.01000	0.02554	79.53	0.01162	0.03018	80.26	0.01323	0.03481	80.89	0.01484
$\hat{F}^{(2)}_{Re_3}(t)$	0.02064	79.48	0.01000	0.02524	80.49	0.01162	0.02983	81.18	0.01323	0.03443	81.69	0.01484
$\hat{F}^{(2)}_{Re_4}(t)$	0.02096	78.28	0.01085	0.02560	79.34	0.01260	0.03025	80.06	0.01434	0.03489	80.60	0.01609
$\hat{F}^{(2)}_{Re_5}(t)$	0.02056	79.81	0.00978	0.02514	80.80	0.01135	0.02972	81.49	0.01293	0.03430	81.89	0.01451
$\hat{F}^{(2)}_{Re_6}(t)$	0.00691	237.95	0.03207	0.00916	221.84	0.03665	0.01140	212.43	0.04123	0.01364	206.11	0.04581
$\hat{F}^{(2)}_{Re_7}(t)$	0.00672	243.95	0.01575	0.00897	226.35	0.01787	0.01122	215.81	0.01999	0.01347	208.78	0.02211
$\hat{F}^{(2)}_{Re_8}(t)$	0.00926	177.08	0.00511	0.01198	169.49	0.00606	0.01470	164.71	0.00702	0.01742	161.42	0.00797
$\hat{F}^{(2)}_{Re_9}(t)$	0.00713	230.13	0.03826	0.00940	216.16	0.04377	0.01166	207.62	0.04929	0.01393	201.87	0.05480
$\hat{F}^{(2)}_{Re_{10}}(t)$	0.00998	164.31	0.00605	0.01283	158.36	0.00712	0.01567	154.56	0.00820	0.01851	151.93	0.00928
$\hat{F}^{(2)}_{MJP}(t)$	0.00670	**244.80**	0.08538	0.00894	**227.25**	0.09056	0.01117	**216.75**	0.11374	0.01341	**209.76**	0.12792

Table 14. Values of MSE, PRE and ARB, of $\hat{F}^{(1)}_{Re_j}(t)$ for different values of k of Population II

	k=2			k=3			k=4			k=5		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator												
$\hat{F}^{(2)}_{MJ_0}(t)$	0.01641	100.00	-	0.02031	100.00	-	0.02422	100.00	-	0.02812	100	-
$\hat{F}^{(1)}_{Re_1}(t)$	0.02852	57.53	0.03516	0.03242	62.65	0.03516	0.03632	66.66	0.03516	0.04023	69.90	0.03516
$\hat{F}^{(1)}_{Re_2}(t)$	0.02017	81.34	0. 00897	0.02408	84.36	0. 00897	0.02798	86.55	0. 00897	0.03188	88.19	0. 00897
$\hat{F}^{(1)}_{Re_3}(t)$	0.01995	82.22	0.00839	0.02386	85.13	0.00839	0.02776	87.22	0.00839	0.03167	88.80	0.00839
$\hat{F}^{(1)}_{Re_4}(t)$	0.02022	81.15	0.00910	0.02412	84.20	0.00910	0.02803	86.40	0.00910	0.03193	88.07	0.00910
$\hat{F}^{(1)}_{Re_5}(t)$	0.01988	82.52	0.00820	0.02379	85.39	0.00820	0.02769	87.45	0.00820	0.03160	89.00	0.00820
$\hat{F}^{(1)}_{Re_6}(t)$	0.00857	191.36	0.02750	0.01248	162.77	0.00038	0.01638	147.80	0.00038	0.02029	138.60	0.00038
$\hat{F}^{(1)}_{Re_7}(t)$	0.00838	195.71	0.01363	0.01229	165.28	0.02750	0.01619	149.54	0.02750	0.02010	139.91	0.02750
$\hat{F}^{(1)}_{Re_8}(t)$	0.01045	156.97	0.00416	0.01436	141.47	0.00416	0.01826	132.60	0.00416	0.004752217	126.86	0.00416
$\hat{F}^{(1)}_{Re_9}(t)$	0.00877	187.12	0.03274	0.01267	160.27	0.03274	0.01658	146.07	0.03274	0.02049	137.29	0.03274
$\hat{F}^{(1)}_{Re_{10}}(t)$	0.01105	148.49	0.00497	0.01495	135.82	0.00497	0.01886	128.04	0.00497	0.02277	123.53	0.00497
$\hat{F}^{(1)}_{MJP}(t)$	0.00837	**196.00**	0.07121	0.01228	**165.45**	0.07121	0.01618	**149.65**	0.07121	0.02008	**140.00**	0.07121

Table 15. MSE, PRE and ARB values of $\hat{F}^{(1)}_{Re_j}(t_y)$ for different values of m of Population I

	m=50			m=60			m=70		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(1)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00217	100.00	-	0.00168	100.00	-
$\hat{F}^{(1)}_{Re_1}(t_y)$	0.00073	389.71	0.00048	0.00055	390.80	0.00036	0.00043	392.13	0.00028
$\hat{F}^{(1)}_{Re_2}(t_y)$	0.00136	210.26	0.00203	0.00103	210.42	0.00154	0.00080	210.62	0.00120
$\hat{F}^{(1)}_{Re_3}(t_y)$	0.00234	122.10	0.00076	0.00178	122.11	0.00057	0.00138	122.13	0.00044
$\hat{F}^{(1)}_{Re_4}(t_y)$	0.00096	299.10	0.00245	0.00072	299.61	0.00186	0.00056	300.23	0.00145
$\hat{F}^{(1)}_{Re_5}(t_y)$	0.00252	113.27	0.00049	0.00192	113.28	0.00038	0.00149	113.29	0.00029
$\hat{F}^{(1)}_{Re_6}(t_y)$	0.00171	167.20	0.00161	0.00130	167.27	0.00122	0.00101	167.35	0.00095
$\hat{F}^{(1)}_{Re_7}(t_y)$	0.00209	136.66	0.00110	0.00159	136.69	0.00084	0.00123	136.72	0.00065
$\hat{F}^{(1)}_{Re_8}(t_y)$	0.00137	208.34	0.00202	0.00104	208.50	0.00173	0.00081	208.69	0.00119
$\hat{F}^{(1)}_{Re_9}(t_y)$	0.00114	250.78	0.00227	0.00086	251.07	0.00133	0.00067	251.43	0.00132
$\hat{F}^{(1)}_{Re_{10}}(t_y)$	0.00235	121.54	0.00074	0.00179	121.55	0.00056	0.00138	121.57	0.00044
$\hat{F}^{(1)}_{MJP}(t_y)$	0.00009	**3031.09**	0.00203	0.00007	**3223.24**	0.00155	0.00005	**3492.27**	0.00121

Table 16. MSE, PRE and ARB values of $\hat{F}^{(2)}_{Re_j}(t_y)$ for different values of m of Population I

	m=50			m=60			m=70		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(2)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00217	100.00	-	0.00168	100.00	-
$\hat{F}^{(2)}_{Re_1}(t_y)$	0.00157	181.76	0.00027	0.00125	173.14	0.00019	0.00103	163.74	0.00014
$\hat{F}^{(2)}_{Re_2}(t_y)$	0.00195	146.66	0. 00203	0.00152	142.61	0. 00154	0.00122	138.00	0. 00120
$\hat{F}^{(2)}_{Re_3}(t_y)$	0.00254	112.37	0.00046	0.00195	111.53	0.00033	0.00152	110.53	0.00023
$\hat{F}^{(2)}_{Re_4}(t_y)$	0.00170	167.57	0.00148	0.00135	160.96	0.00105	0.00110	153.60	0.00075
$\hat{F}^{(2)}_{Re_5}(t_y)$	0.00265	107.67	0.00030	0.00203	107.17	0.00021	0.00158	106.57	0.00015
$\hat{F}^{(2)}_{Re_6}(t_y)$	0.00216	132.30	0.00097	0.00167	129.75	0.00070	0.00133	126.78	0.00050
$\hat{F}^{(2)}_{Re_7}(t_y)$	0.00239	119.49	0.00067	0.00184	118.09	0.00048	0.00145	116.43	0.00034
$\hat{F}^{(2)}_{Re_8}(t_y)$	0.00196	146.10	0.00122	0.00153	142.11	0.00087	0.00122	137.56	0.00062
$\hat{F}^{(2)}_{Re_9}(t_y)$	0.00182	157.36	0.00137	0.00143	152.05	0.00098	0.00115	146.08	0.00070
$\hat{F}^{(2)}_{Re_{10}}(t_y)$	0.00255	112.08	0.00045	0.00195	111.26	0.00032	0.00153	110.29	0.00023
$\hat{F}^{(2)}_{MJP}(t_y)$	0.00120	**237.34**	0.00121	0.00100	**219.04**	0.00086	0.00084	**200.32**	0.00062

Table 17. MSE, PRE and ARB values of $\hat{F}^{(1)}_{Re_j}(t_y)$ for different values of W_M of Population I

	W_M=0.25			W_M=0.30			W_M=0.35		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(1)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00313	100.00	-	0.00324	100.00	-
$\hat{F}^{(1)}_{Re_1}(t_y)$	0.00073	389.71	0.00048	0.00080	391.27	0.00053	0.00083	391.83	0.00055
$\hat{F}^{(1)}_{Re_2}(t_y)$	0.00136	210.26	0.00203	0.00149	210.49	0.00223	0.00154	210.58	0.00231
$\hat{F}^{(1)}_{Re_3}(t_y)$	0.00234	122.10	0.00076	0.00256	122.12	0.00083	0.00265	122.13	0.00086
$\hat{F}^{(1)}_{Re_4}(t_y)$	0.00096	299.10	0.00245	0.00104	299.83	0.00269	0.00108	300.09	0.00278
$\hat{F}^{(1)}_{Re_5}(t_y)$	0.00252	113.27	0.00049	0.00276	113.28	0.00054	0.00286	113.28	0.00056
$\hat{F}^{(1)}_{Re_6}(t_y)$	0.00171	167.20	0.00161	0.00187	167.30	0.00176	0.00194	167.34	0.00183
$\hat{F}^{(1)}_{Re_7}(t_y)$	0.00209	136.66	0.00110	0.00229	136.70	0.00121	0.00237	136.72	0.00125
$\hat{F}^{(1)}_{Re_8}(t_y)$	0.00137	208.34	0.00202	0.00150	208.56	0.00221	0.00155	208.65	0.00229
$\hat{F}^{(1)}_{Re_9}(t_y)$	0.00114	250.78	0.00227	0.00125	251.20	0.00249	0.00129	251.35	0.00258
$\hat{F}^{(1)}_{Re_{10}}(t_y)$	0.00235	121.54	0.00074	0.00258	121.56	0.00081	0.00267	121.56	0.00084
$\hat{F}^{(1)}_{MJP}(t_y)$	0.00009	**3031.09**	0.00203	0.00009	**3314.18**	0.00224	0.00009	**3427.33**	0.00232

Table 18. MSE, PRE and ARB values of $\hat{F}^{(2)}_{Re_j}(t_y)$ for different values of W_M of Population I

	W_M=0.25			W_M=0.30			W_M=0.35		
	MSE	PRE	ARB	MSE	PRE	ARB	MSE	PRE	ARB
Estimator									
$\hat{F}^{(2)}_{MJ_0}(t_y)$	0.00286	100.00	-	0.00313	100.00	-	0.00324	100.00	-
$\hat{F}^{(2)}_{Re_1}(t_y)$	0.00157	181.76	0.00027	0.00184	169.66	0.00027	0.00196	165.68	0.00027
$\hat{F}^{(2)}_{Re_2}(t_y)$	0.00195	146.66	0. 00123	0.00222	140.93	0. 00123	0.00233	138.97	0. 00123
$\hat{F}^{(2)}_{Re_3}(t_y)$	0.00254	112.37	0.00046	0.00282	111.17	0.00046	0.00293	110.74	0.00046
$\hat{F}^{(2)}_{Re_4}(t_y)$	0.00170	167.57	0.00148	0.00198	158.24	0.00148	0.00209	155.13	0.00148
$\hat{F}^{(2)}_{Re_5}(t_y)$	0.00265	107.68	0.00030	0.00293	106.96	0.00030	0.00304	106.70	0.00030
$\hat{F}^{(2)}_{Re_6}(t_y)$	0.00216	132.30	0.00097	0.00243	128.67	0.00097	0.00254	127.41	0.00097
$\hat{F}^{(2)}_{Re_7}(t_y)$	0.00239	119.49	0.00067	0.00266	117.49	0.00067	0.00278	116.79	0.00067
$\hat{F}^{(2)}_{Re_8}(t_y)$	0.00196	146.10	0.00122	0.00223	140.45	0.00122	0.00234	138.52	0.00122
$\hat{F}^{(2)}_{Re_9}(t_y)$	0.00182	157.36	0.00137	0.00209	149.86	0.00137	0.00220	147.33	0.00137
$\hat{F}^{(2)}_{Re_{10}}(t_y)$	0.00255	112.08	0.00045	0.00282	110.91	0.00045	0.00293	110.50	0.00045
$\hat{F}^{(2)}_{MJP}(t_y)$	0.00120	**237.34**	0.00121	0.00148	**211.94**	0.00121	0.00159	**204.07**	0.00121

From Tables 7–8, it is observed that $MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right)$ and $MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)$ increases at increasing rate of (k) for the given data set. Percentage Relative Efficiencies of $\left(\hat{F}^{(2)}_{MJ_{(i=1-3,5,9,10,12,P)}}(t_y)\right)$ and $\left(\hat{F}^{(1)}_{MJ_{(i=2,4-9,11,13)}}(t_y)\right)$ increases by increasing values of (k) and decreases for $\left(\hat{F}^{(2)}_{MJ_{(i=6,8,11,13)}}(t_y)\right)$, $\left(\hat{F}^{(1)}_{MJ_{(i=1,3,10,12,P)}}(t_y)\right)$, but having same values for $\left(\hat{F}^{(2)}_{MJ_{(i=4,7)}}(t_y)\right)$. The PREs of ratio type of estimators having more values in comparison with product type of estimators as there is positive correlation between study and auxiliary variables here, but our proposed estimator $\hat{F}^{(.)}_{MJP}(t_y)$ is more efficient from all other suggested estimators considered here. As for as ARBs of $\hat{F}^{(2)}_{MJ_{(i=1-P)}}(t_y)$ and for $\hat{F}^{(1)}_{MJ_{(i=1,3-13)}}(t_y)$ increases at increasing rates of inverse sampling rate (k) but having same values for $\hat{F}^{(1)}_{MJ_{(i=2,P)}}(t_y)$.

From Tables 9–10, we examined that $MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right)$ and $MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)$ increases at increasing rate of (k) for the given data set. By increasing values of k the PREs of $\left(\hat{F}^{(2)}_{Re_{(j=1-P)}}(t_y)\right)$ increases, and decreases for $\left(\hat{F}^{(1)}_{Re_{(j=1-P)}}(t_y)\right)$. The ARBs of $\left(\hat{F}^{(2)}_{Re_{(j=1-P)}}(t_y)\right)$ increases and having same values for $\left(\hat{F}^{(1)}_{Re_j}(t_y)\right)$ at increasing rates of inverse sampling rates.

From Tables 11–12, we observed that $MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)$ decreases, as compared to $MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right)$, at fixed values of (k), and increases, at increasing rate of (k) for Population II. Efficiencies increases for $\left(\hat{F}^{(1)}_{MJ_{(i=1,3,8,10-12)}}(t_y)\right)$ and $\left(\hat{F}^{(2)}_{MJ_{(i=1,3,8,10-12)}}(t_y)\right)$ by increases (k) and decreases for $\left(\hat{F}^{(1)}_{MJ_{(i=2,4,7,9,13,P)}}(t_y)\right)$, $\left(\hat{F}^{(2)}_{MJ_{(i=2,4-7,9,13,P)}}(t_y)\right)$. Here, efficiencies of product type of estimators having more values as compared to ratio type of estimators because of negative correlation between study and auxiliary variables here, but our proposed estimator $\hat{F}^{(.)}_{MJP}(t_y)$ is more efficient from all suggested estimators. As for as ARB of $\hat{F}^{(1)}_{MJ_{(i=1-6,8,10-P)}}(t_y)$ and $(\hat{F}^{(2)}_{MJ_{(i=1,3,7-11)}}(t_y)$ increases, decreases for $\hat{F}^{(1)}_{MJ_{(i=7,9)}}(t_y)$ and $(\hat{F}^{(2)}_{MJ_{(i=4-6,12,13)}}(t_y)$ but having same values for $\hat{F}^{(2)}_{MJ_{(i=2,P)}}(t_y)$.

From Tables 15–16, we studied that $MSE\left(\hat{F}^{(2)}_{MJ_i}(t_y)\right)$ and $MSE\left(\hat{F}^{(1)}_{MJ_i}(t_y)\right)$ decreases by increasing values of (m) for the given data set. For increasing the sample size (m), the Percentage Relative Efficiencies of $\left(\hat{F}^{(2)}_{Re_{(j=0-P)}}(t_y)\right)$ increases, and decreases for $\left(\hat{F}^{(1)}_{Re_{(j=0-P)}}(t_y)\right)$. The ARBs decreases under both the situations of $\left(\hat{F}^{(2)}_{Re_j}(t_y)\right)$ and $\left(\hat{F}^{(1)}_{Re_j}(t_y)\right)$ by increasing the sample size (m) respectively.

From Tables 17–18, we see that $MSE\left(\hat{F}^{(1)}_{Re_j}(t_y)\right)$ increases, as compared to the $MSE\left(\hat{F}^{(2)}_{Re_j}(t_y)\right)$, at fixed values of (W_M) for given Population, but increases under both the situations at increasing rates of (W_M). PREs of $\left(\hat{F}^{(2)}_{Re_{(j=0-P)}}(t_y)\right)$ increases and decreases for $\left(\hat{F}^{(1)}_{Re_{(j=0-P)}}(t_y)\right)$ at increasing rates of (W_M). The ARBs of $\left(\hat{F}^{(2)}_{Re_{(j=1-P)}}(t_y)\right)$ increases and having same values for $\left(\hat{F}^{(1)}_{Re_j}(t_y)\right)$ at increasing values of (W_M). Overall our proposed class of estimators $\hat{F}^{(.)}_{MJP}(t_y)$ is more efficient than all other suggested estimators considered here.

6.1. Comparison through percentage loss in efficiency. To judge the effect of non-response in simple random sampling, we obtain the percent relative loss in efficiency of proposed estimator $\hat{F}^{(2)}_{Re_j}(t_y)$ with respect to the same situation as discussed earlier but without of non-response. For this we modify $\hat{F}^{(1)}_{Re_j}(t_y)$ as $\hat{F}^{(')}_{Re_j}(t_y)$ for estimating population distribution function. The *MSE* of $\hat{F}^{(')}_{Re_j}(t_y)$ is given by

$$(6.1)\qquad MSE\left(\hat{F}^{(')}_{Re_j}(t_y)\right) = F_Y^2(t_y)\left[V_{20} + \left(\frac{1}{4}\psi_j^2 V_{02} - \psi_j V_{11}^{*'}\right)\right].$$

The Percentage Relative Loss in Precision (PRLP) of $\hat{F}^{(')}_{Re_j}(t_y)$ with respect to $\hat{F}^{(1)}_{MJ_0}(t_y)$ is given by

$$(6.2)\qquad PRLP\left(\hat{F}^{(')}_{Re_j}(t_y)\right) = \frac{MSE\left(\hat{F}^{(1)}_{MJ_0}(t_y)\right) - MSE\left(\hat{F}^{(')}_{Re_j}(t_y)\right)}{MSE\left(\hat{F}^{(1)}_{MJ_0}(t_y)\right)} \times 100$$

Table 19. PRE, PRLP, of $\hat{F}^{(.)}_{Re_j}(t_y)$ at different values of W_M, m and k

W_M	m	k	Case	$\hat{F}^{(')}_{MJ_0}(t_y)$	$\hat{F}^{(')}_{Re_1}(t_y)$	$\hat{F}^{(')}_{Re_2}(t_y)$	$\hat{F}^{(')}_{Re_3}(t_y)$	$\hat{F}^{(')}_{Re_4}(t_y)$	$\hat{F}^{(')}_{Re_5}(t_y)$	$\hat{F}^{(')}_{Re_6}(t_y)$	$\hat{F}^{(')}_{Re_7}(t_y)$	$\hat{F}^{(')}_{Re_8}(t_y)$	$\hat{F}^{(')}_{Re_9}(t_y)$	$\hat{F}^{(')}_{Re_{10}}(t_y)$	$\hat{F}^{(')}_{MJ_P}(t_y)$
0.25	50	2	PRE	100.00	389.71	210.86	122.10	299.10	113.27	167.20	136.66	208.34	250.78	121.54	3031.09
			PRLP	38.88	83.86	70.70	49.89	79.20	46.01	63.29	55.19	70.43	75.33	49.66	96.74
		3	PRE	100.00	397.76	211.01	122.16	301.45	113.30	167.52	136.78	209.06	252.13	121.60	4184.14
			PRLP	55.99	88.38	78.90	63.92	85.02	61.12	73.57	67.73	78.71	82.24	63.75	97.66
0.30	60	2	PRE	100.00	392.40	210.66	122.13	300.36	113.29	167.37	136.73	208.73	251.50	121.57	3553.50
			PRLP	48.04	86.28	75.09	57.40	82.32	54.10	68.80	61.90	74.86	79.03	57.20	97.23
		3	PRE	100.00	397.44	211.40	122.19	302.69	113.32	167.68	136.85	209.44	252.84	121.63	5228.86
			PRLP	64.90	90.73	83.17	71.22	88.06	68.99	78.92	74.27	83.02	85.83	71.09	98.13
0.35	70	2	PRE	100.00	394.33	210.95	122.16	301.25	113.30	167.49	136.77	209.00	252.01	121.59	4050.60
			PRLP	54.52	87.99	78.20	62.72	84.53	59.83	72.69	66.66	78.00	81.64	62.55	97.58
		3	PRE	100.00	399.17	211.65	122.21	303.48	113.33	167.78	136.88	209.69	253.30	121.64	6222.85
			PRLP	70.57	92.23	85.89	75.87	89.99	74.00	82.32	78.42	85.76	88.12	75.76	98.43

From Table 15, it is observed that, if non-response is considered, there is loss in precision. The percent relative loss in precision increases by increasing the values of k and m respectively. It is also observed that for fixed values of k and m with the increasing values of W_M, PRLP also increase, so more values of k and m, taken more loss in precision is to be observed due to presence of non-response.

7. Cost of the survey

Following Hansen and Hurwitz [11], for attaining the better precision at minimum cost, we consider here the case, for determining the number of questionnaires to be sent out and the personal interviews to take in follow ups for non-responses to the mail questionnaires. For this, we assume that questionnaires are sent to 30 people, randomly drawn from 120 countries of a given data set. Further assume that 50 percent or 15 respond, and from other 15 which are non-respondents, 10 percent or 3 visited for insuring some representation of the class of non-respondents. An unbiased estimate is given by:

$$\hat{F}_X^{(*)}(t_x) = w_R \hat{F}_X^{(1)}(t_x) + w_M \hat{F}_X^{(2r)}(t_x), \tag{7.1}$$

where $w_i = m_i/m$ for $i = R, M$.
$N = 120$ = Total number of people in population;
m =30 = Total mailed out questionnaires;
$\hat{F}_X^{(1)}(t_x)$= The average of people to the mailed out questionnaires;
$m_R = 15$ = The number of respondents;
$\hat{F}_X^{(2r)}(t_x)$ = The average number of peoples for personal interviewing;
m_M= 3 = Not reply through mailed questionnaires which are personally interviewed.
It is noted that the actual processed sample size would be 15 +3 = 18. The sample variance of $\hat{F}_X^{(*)}(t)$, is given by

$$V\left(\hat{F}_X^{(*)}(t_x)\right) = \begin{bmatrix} \dfrac{N-\hat{m}}{\hat{m}(N-1)} F_X(t_x)(1-F_X(t_x)) \\ + \dfrac{W_M\,(k-1)}{m} \dfrac{N_M}{N_M-1} F_X^{(2)}(t_x)\left(1-F_X^{(2)}(t_x)\right) \end{bmatrix}, \tag{7.2}$$

where $F_X(t_x)(1-F_X(t_x))$ is the variance of whole population and $F_X^{(2)}(t_x)\left(1-F_X^{(2)}(t_x)\right)$ is the variance from not respondents. N_M is the number of people in the population having no response; r is personally visited numbers; and $k = \frac{m_M}{r}$, m_M is the number of non-respondents in the sample. By using Equation (7.2), we can see that there are wide range of different sample sizes which will give us the same reliability and finally we reached at that point the sample size m alone give us the poor indicator for sampling reliability. For example, assume that $F_X(t_x)(1-F_X(t_x)) = F_X^{(2)}(t_x)\left(1-F_X^{(2)}(t_x)\right)$ and that N and N_M are so large that is $\frac{N}{N-1}$ and $\frac{N_M}{N_M-1}$ tends to one. Further assume that the accuracy we required is such that, the average value of (ϵ = standard error) would be given by $m = 30$ people when $W_R = 100\%$. If questionnaires were mailed to a random sample of m people with $W_R = 100\%$, the variance of the auxiliary variable $F_X(t_x)$ estimated from sample would be

$$V\left(\hat{F}_X^{(*)}(t_x)\right) = \frac{N-m}{m(N-1)} F_X(t_x)(1-F_X(t_x)), \tag{7.3}$$

Thus,

$$\epsilon^2 = \frac{N-30}{30(N-1)} F_X(t_x)(1-F_X(t_x)). \tag{7.4}$$

By substituting different numerical values at different response rate of mailed returns along with personal interviews in (7.2), we see that, although m which differs in size but each one will give us same reliability.

Table 20. Some Cost under different sample sizes at $W_R = 50\%$ for Situation I.

m	m_R	m_M	$r = \frac{m_M}{k}$	Scheduled Tabulated $(m_R + r)$	Cost=$c_0 m + c_1 m_R + c_2 m_M$
16	8	8	8	16	**1040**
20	10	10	5	15	**800**
26	13	13	5	18	**890**
30	15	15	4	19	**850**
40	20	20	4	24	**1000**
60	30	30	3	33	**1200**

Let $c_0 =$ Rs 5= Overhead cost,
$c_1 =$ Rs 20 = Cost per unit for responding stratum, and
$c_2 = R_s 100, =$ Cost per unit for non-responding stratum.
Generally c_2 having more values than c_1, as extra effort is required for making contact with non-respondents and obtained responses from them.
From Table 20, Column 5 shows that for different sample sizes m each one give us the same required precision. For instance, sending 20 questionnaires, obtaining (10 by mail and 05 by personal interviewing) give us the same (ϵ) as for sending out 60 questionnaires and obtaining a total of 33 questionnaire (30 by mail and 03 by visited personally). Therefore at some point it would be at a loss to put extra money for having additional mail returns. Sensibly, it will be better to spend extra effort for those which are non-respondents. Column 6 gives us the total cost for each of the sample sizes under the unit cost. Since in Table 20, for different schedules tabulated all give us same precision, so logically it will be better for us to choose that particular value of m which would give us minimum cost. Consequently, by sending 20 schedules, 10 of them are returned by mail (at 50 per cent response rate) and 5 are personally interviewed which were non-respondents.
Now instead of suggested procedure for Table 20, we obtain an optimum number of schedules mailed out and choose personal interviews accordingly, the optimum values of m and r are given by:

$$m^{(opt)} = \hat{m}\left\{1 + (k-1) W_M\right\}, \tag{7.5}$$

and

$$r = \frac{m_M}{k} \tag{7.6}$$

where, $k = \sqrt{\frac{c_2 W_R}{c_0 + c_1 W_R}}$
and

$$\hat{m} = \frac{NT_1}{(N-1)\epsilon^2 + T_1}, \tag{7.7}$$

where
$T_1 = F_Y(t_y)(1 - F_Y(t_y)) + F_X(t_x)(1 - F_X(t_x)) - 2\left(\frac{N_{11}N_{22} - N_{12}N_{21}}{N^2}\right)$,
W_R is the response rate obtained through mailed questionnaire, $W_R = 1 - W_M$. Expressions in (7.5) and (7.6) are obtained under the assumptions that $F_X(t)(1 - F_X(t)) =$

$F_X^{(2)}(t)\left(1-F_X^{(2)}(t)\right)$ and $\frac{N}{N-1}=\frac{N_M}{N_M-1}\cong 1$. But when the above assumption is eliminated the optimum values of m and r are given by

$$(7.8)\qquad m^{(opt)}=\hat{m}\left\{1+(k-1)\,W_M\frac{T_2}{T_1}\right\},$$

where

$$T_2=F_Y^{(2)}(t_y)(1-F_Y^{(2)}(t_y))+F_X^{(2)}(t_x)(1-F_X^{(2)}(t_x))-2\left(\frac{N_{11}^{(2)}N_{22}^{(2)}-N_{12}^{(2)}N_{21}^{(2)}}{(N_2^{(2)})^2}\right).$$

and

$$(7.9)\qquad r=\frac{m_M}{k},$$

where $k=\sqrt{\frac{c_2W_R}{c_0+c_1W_R}\left\{\frac{(N-m)T_1}{NW_MT_2}-1\right\}}$.

Of course, at different response rate m_R and r varies for achieving the specified precision. In practice we do not know what will be the approximate response rate but for estimating optimum values and by using (7.8) and (7.9), there must be an approximate known value of W_R in advance.

For the case, when W_R is not known in advance, suggesting to design the survey for achieving at least definite specified precision at minimal cost, and parallel to these must know about total cost of the survey. Under such circumstances, it is possible for obtaining optimum values of m_R and k. For example, instead of using 50% response rate in Table 20, we compute optimum values of m_R and r at different values of W_R (10% to 90%) respectively for achieving same precision. The optimum values of Equations (7.5) and (7.6) along with their cost are given in Table 21.

Table 21. Comparison of minimum cost for various response rates of W_R

W_R	m	m_R	m_M	$r=\frac{m_M}{k}$	Optimum Cost	Cost of Strategy 1	Increase in Cost
0.10	35	4	31	26	2855	2910	**055**
0.20	42	8	34	23	2670	2710	**040**
0.30	44	13	31	19	2380	2430	**050**
0.40	44	18	26	15	2080	2190	**110**
0.50	42	21	21	12	1830	1950	**120**
0.60	40	24	16	9	1600	1710	**110**
0.70	38	27	11	6	1330	1470	**140**
0.80	35	28	7	4	1135	1230	**095**
0.90	33	30	3	2	965	990	**025**

Table 21, Column 6 gives us the optimum cost at specified response rate, but for unknown response rate, we give Strategy 1 by sending 30 questionnaires and follow up on all non-respondents whatever the value of W_R is. Therefore for specified response rate W_R, the cost for Strategy 1 will always be more than the optimum cost as shown in Table 21.

It is interesting to note that how costly this strategy is, as compared to the optimum cost method by using different response rates W_R. The comparison between Column 6 and Column 7 is presented in Column 8 which shows that at smaller response rates give low cost since less questionnaires have been received. For at least 30% response rate, increase in cost from 09% to 22% is to be expected for this strategy.

When an approximate value of W_R is not known in advance, the Strategy 2 is preferable as compared to Strategy 1. There are two steps involved in Strategy 2 as:

(i) Determine the maximum number of m, whatever the size for W_R,

(ii) Determine r for achieving the required precision and value of W_R is actually determined from the sample results. Hence r will change its value with the actual W_R;

In Table 21 as there are maximum number (44) questionnaires to be sent out for $W_R =$ 40% then by using formula, $r = \frac{m_M}{k}$, we get $r = 15$.

Table 22. Comparison between optimum cost for known values of W_R and Strategies 1 and 2.

W_R	m	m_R	m_M	$r = \frac{mM}{k}$	Cost of Strategy 2	Cost of Strategy 1	Optimum cost
0.10	44	4	40	33	3600	2910	**2855**
0.20	44	9	35	23	2700	2710	**2670**
0.30	44	13	31	19	2380	2430	**2380**
0.40	44	18	26	15	2080	2190	**2080**
0.50	44	22	22	12	1860	1950	**1830**
0.60	44	26	18	10	1740	1710	**1600**
0.70	44	31	13	7	1540	1470	**1330**
0.80	44	35	9	5	1420	1230	**1135**
0.90	44	40	4	2	1220	990	**965**

Table 22, show the number of mailed questionnaires and number of personal interviewed r at varying values of W_R for achieving the required precision, along with their total cost of the survey. The optimum costs are also given at known values of W_R. Of course at high value of W_R, the optimum cost of any survey will give us the small values accordingly.

Thus from the above discussions, we conclude that it is not necessarily found that an optimum value of m and r but an optimum procedure (Strategy) is also vital even when we have nothing in hand about values of W_R in advance and consequently it will give us in any case at least a precise procedure at slightly low cost.

8. Conclusion

In this article, we proposed an improved generalized class of ratio type exponential estimators $\hat{F}^{(.)}_{MJP}(t_y)$. Expressions for bias and MSE of the proposed class of estimators $\hat{F}^{(.)}_{MJP}(t_y)$ are compared with two suggested families of estimators theoretically and numerically under Situations I and II. From Tables 7–15, it is observed that proposed class of estimators $\hat{F}^{(.)}_{MJP}(t_y)$ is preferable in both the Situations and is recommended for precise estimation for population distribution function in the presence of non-response.

Acknowledgments

The authors are very grateful to the unknown referees and editor (Prof. Dr. Cem Kadilar) for their constructive and encouraging suggestions which led to the present improved version of the manuscript. The first author would like to thank to Dr. Abdul Haq (Department of Statistics, Quaid-i-Azam Univesity, Islamabad, Pakistan) for his valuable suggestions.

References

[1] Ahmed, M. S. and Abu-Dayyeh, W. (2001). Estimation of finite population distribution function using multivariate auxiliary information. *Statistics in Transition*, 3:501–507.

[2] Chambers, R., Dunstan, A. H., and Hall, P. (1992). Properties of estimators of finite population distribution function. *Biometrika*, 79(3):577–582.

[3] Chambers, R. and Dunstan, R. (1986). Estimating distribution function from survey data. *Biometrika*, 72:597–604.

[4] Diana, G. and Perri, P. F. (1953). On probability mechanism to attain an economic balance between the resultant error of response and bias of non-response. *Journal of the American Statistical Association*, 48(264):743–772.

[5] Diana, G. and Perri, P. F. (2012). A class of estimators in two-phase sampling with sub-sampling the non-respondents. *Applied Mathematics and Computation*, 40:139–148.

[6] Dorfman, A. H. (1993). A comparison of design-based and model-based estimators fo the finite population distribution function. *Australian Journal of Statistics*, 35:29–41.

[7] Dunkelberg, W. C. and Goerge, S. D. (1973). Nonresponse bias and call backs in sample surveys. *Journal of Marketing Research*, 10(2):160–168.

[8] El-Badry, M. A. (1956). A sampling procedure for mailed questionnaire. *Journal of the American Statistical Association*, 51(274):209–227.

[9] Garcia, M. R. and Cebrian, A. A. (1998). Quantile interval estimation in finite population using a multivariate ratio estimator. *Metrika*, 47:203–213.

[10] Gross, S. T. (1980). Median estimation in sample surveys. *American Statistical Association*, pages 181–184.

[11] Hansen, M. H. and Hurwitz, W. N. (1946). The problem of non response in sample surveys. *Journal of the American Statistical Association*, 41:517–529.

[12] Hansen, M. H., Hurwitz, W. N., Marks, E. S., and Mauldin, P. K. (1951). Response errors in surveys. *Journal of the American Statistical Association*, 46(254):147–190.

[13] John, A. C. and Robert, N. F. (1947). Controlling bias in mail questionnaires. *Journal of the American Statistical Association*, 42(240):497–511.

[14] Khare, B. B. and Sinha, R. R. (2007). Estimation of ratio of the two population means using multi-auxiliary characters in the presence of non-response. *Narosa Publishing House, New Delhi, India*, 10(3):3–14.

[15] Khare, B. B. and Sinha, R. R. (2009). On class of estimators for population mean using multi-auxiliary characters in the presence of non-response. *Applied Mathematics and Computation*, 10(3):45–56.

[16] Khare, B. B. and Sinha, R. R. (2011). Estimation of population mean using multi-auxiliary characters with subsampling the nonrespondents. *Statistics in Transition*, 12(1):3–14.

[17] Khare, B. B. and Srivastava, S. (1993). Estimation of population mean using auxiliary characters in the presence of non-response. *National Academy Science letters, India*, 16(3):111–114.

[18] Khare, B. B. and Srivastava, S. (1995). Study of conventional and alternative two-phase sampling ratio, product and regression estimators in the presence of non-response. *Proceeding of the National Academy of Science, India*, 16A(II):195–203.

[19] Khare, B. B. and Srivastava, S. (1997). Transformed ratio type estimators for the population mean in the presence of non-response. *Communication in Statistics-Theory and Methods*, 26(5):1779–1791.

[20] Klein, M. (2000). Two altenatives to the schewhart $\bar{X}$ control chart. *Journal of Quality Technology*, 32(4):427–431.

[21] Kuk, A. Y. C. (1998). Estimating of distribution functions and medians under sampling with unequal probabilities. *Biometrika*, 75(1):97–103.

[22] Kuk, A. Y. C. and Mak, T. k. (1989). Median estimation in presence of auxiliary information. *Journal of the Royal Statistical Society*, 51:261–269.

[23] Kuk, Y. C. and Mak, T. k. (1993). A new method for estimating finite-population quantiles using auxiliary information. *The Canadian Journal of Statistics*, 21(1):29–38.

[24] Okafor, F. C. and Lee, H. (2000). Double sampling for ratio and regression estimation with sub-sampling the non-respondents. *Survey Methodology*, 26(2):183–188.

[25] Olkin, I. (1958). Multivariate ratio estimation for finite populations. *Boimetrika*, 45:154–165.

[26] Politz, A. and Simmons, W. (1949). An attempt to get the " not at homes" into the sample without callbacks. *Journal of the American Statistical Association*, 44(245):9–16.

[27] Rao, J. N. K., Kovar, J. G., and Mental, H. J. (1990). On estimating distribution functions and quantiles from survey data using auxiliary information. *Biometrika*, 77(2):365–375.

[28] Rao, P. S. R. S. (1986). Ratio estimation with sub sampling of non-respondents. *Survey Methodology*, 12:217–230.

[29] Rueda, M. and Arcos, A. (2004). Improving ratio-type quantile estimates in a finite population. *Statistical Papers*, 45:231–248.

[30] Rueda, M. M., Arcos, A., and Martinez, M. D. (2003). Difference estimators of quantiles in finite populations. *Biometrika*, 12(2):481–496.

[31] Rueda, M. M., Arcos, A., Martinez, M. D., and Roman, Y. (2004). Some improved estimators of finite population quantile using auxiliary information in sample survey. *Computational Statistics and data analysis*, 45:825–848.

[32] Sarndal, C. E., Swensson, B., and Wretman, J. (1992). *Model Assisted Survey Sampling. Springer– Verlag, New York.*

[33] Shabbir, J. and Nasir, S. (2013). On estimating the finite population mean using two auxiliary variables in two phase sampling in the presence of non response. *Communication in Statistics-Theory and Methods*, 42:4127–4145.

[34] Singh, H. P. and Kumar, S. (2009). A general procedure of estimating the population mean in the presence of non-response under double sampling using auxiliary information. *SORT*, 33(1):71–84.

[35] Singh, H. P. and Kumar, S. (2010a). Estimation of mean in presence of non-response using two phase sampling scheme. *Statistical Papers*, 51:559–582.

[36] Singh, H. P. and Kumar, S. (2010b). Improved estimation of population mean under two phase sampling with sub-sampling of the non-respondents. *Journal of Statistical Planning and Inference*, DOI:10.1016/.jspi.

[37] Singh, H. P., Sarjinder, S., and Kozak, M. (2008). A family of estimators of finite-population distribution function using auxiliary information. *Accta Applicandae Mathematicae*, 104:115–130.

[38] Singh, S. (2003). *Advanced sampling theory with applications.*

[39] Singh, S. and Joarder, A. H. (2002). Estimation of the distribution function and median in two phase sampling. *Pakistan Journal of Statistics*, 18(2):301–319.

[40] Tabasum, R. and Khan, I. A. (2006). Double sampling for ratio estimator for the population mean in the presence of non-response. *Assam Statistical Review*, 20:73–83.

[41] Wang, S. and Alan, H. D. (1996). A new estimator for the finite population distribution function. *Journal of the American Statistical Association*, 47:635–646.

[42] Woodruff, R. S. (1952). Confidence interval for medians and other position measures. *Biometrika*, 83(3):639–652.

Appendix A

For finding the $Cov\left(\hat{F}_Y^{(*)}(t_y), \hat{F}_X^{(*)}(t_x)\right)$, we have

$$(8.1) \qquad = Cov\left(w_R \hat{F}_Y^{(1)}(t_y) + w_M \hat{F}_Y^{(2r)}(t_{y_2}), w_R \hat{F}_X^{(1)}(t_x) + w_M \hat{F}_X^{(2r)}(t_{x_2})\right).$$

(8.1) can also be written as

$$(8.2) \qquad \begin{aligned} &= Cov\left\{\hat{F}_Y(t_y) + w_M\left(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})\right),\right. \\ &\qquad \left.\hat{F}_X(t_x) + w_M\left(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2})\right)\right\}. \end{aligned}$$

By applying covariance on (8.2), we have

$$(8.3) \qquad = \begin{bmatrix} Cov\left(\hat{F}_Y(t_y), \hat{F}_X(t_x)\right) + Cov\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), \hat{F}_X(t_x)\right) \\ + Cov\left(\hat{F}_Y(t_y), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) \\ + Cov\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2})\right) \end{bmatrix}.$$

Since
$Cov\left(\hat{F}_Y(t_y), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right)$
$= Cov\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), \hat{F}_X(t_x)\right) = 0.$
Hence (8.3) becomes

$$(8.4) \qquad = \begin{bmatrix} Cov\left(\hat{F}_Y(t_y), \hat{F}_X(t_x)\right) \\ + Cov\left(w_M \hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2}), w_M \hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2})\right) \end{bmatrix}.$$

Following Garcia and Cebrian [9], it is easy to obtain

$$Cov\left(\hat{F}_Y(t_y), \hat{F}_X(t_x)\right) = \frac{N-m}{N-1}\frac{m}{N^2}\left(\frac{N_{11}N_{22} - N_{12}N_{21}}{m^2}\right).$$

If we consider $N - 1 \cong N$, then we can write above as

$$(8.5) \qquad Cov\left(\hat{F}_Y(t_y), \hat{F}_X(t_x)\right) = \frac{1-f}{m}\left(\frac{N_{11}N_{22} - N_{12}N_{21}}{N^2}\right),$$

where $f = \frac{m}{N}$ and now consider last term of (8.3),
$Cov\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right)$

$$(8.6) \qquad = \begin{bmatrix} E_1 Cov_2\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) \\ + Cov_1 E_2\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) \end{bmatrix}.$$

Since,
$E_2\left(w_2(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_2(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) = 0.$
Hence (8.6), becomes

$$(8.7) \qquad = \left[E_1 Cov_2\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right)\right].$$

Now consider
$Cov_2\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right),$

$$= \begin{bmatrix} E_2 Cov_3\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) \\ + Cov_2 E_3\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right) \end{bmatrix}. \tag{8.8}$$

$$= \left[E_2 Cov_3\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right)\right]. \tag{8.9}$$

Consider $Cov_3\left(w_M(\hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2})), w_M(\hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2}))\right),$

$$= Cov_3\left(\hat{F}_Y^{(2r)}(t_{y_2}), \hat{F}_X^{(2r)}(t_{x_2})\right). \tag{8.10}$$

By taking r sample points out of m_M and after some simplifications, (**??**) becomes

$$Cov_3\left(\hat{F}_Y^{(2r)}(t_{y_2}), \hat{F}_X^{(2r)}(t_{x_2})\right) = \frac{(k-1)}{m_M - 1}\left(\frac{N_{11}^{(2)} N_{22}^{(2)} - N_{12}^{(2)} N_{21}^{(2)}}{(m_M^{(2)})^2}\right). \tag{8.11}$$

$E_2 Cov_3\left(w_M \hat{F}_Y^{(2r)}(t_{y_2}) - \hat{F}_Y^{(2)}(t_{y_2}), w_M \hat{F}_X^{(2r)}(t_{x_2}) - \hat{F}_X^{(2)}(t_{x_2})\right),$

$$= \frac{m_M^2(k-1)}{m^2(m_M - 1)} E_2\left[\left(\frac{\hat{N}_{11}^{(2)} \hat{N}_{22}^{(2)} - \hat{N}_{12}^{(2)} \hat{N}_{21}^{(2)}}{(m_M^{(2)})^2}\right)\right]. \tag{8.12}$$

Applying expectations on (8.12), we have

$$= \frac{(k-1)W_M}{m}\left(\frac{N_{11}^{(2)} N_{22}^{(2)} - N_{12}^{(2)} N_{21}^{(2)}}{N_M\,(N_M - 1)}\right). \tag{8.13}$$

For considering, $N_M - 1 \cong N_M$, Equation (8.13) becomes,

$$= \frac{W_M(k-1)}{m}\left(\frac{N_{11}^{(2)} N_{22}^{(2)} - N_{12}^{(2)} N_{21}^{(2)}}{(N_M^{(2)})^2}\right). \tag{8.14}$$

Using (8.5) and (8.14) in (8.2) we have following result given as

$$Cov\left(\hat{F}_Y^{(*)}(t_y), \hat{F}_X^{(*)}(t_x)\right) = \begin{bmatrix} \frac{(1-f)}{m}\left(\frac{N_{11}N_{22} - N_{12}N_{21}}{(N)^2}\right) \\ + \frac{W_M(k-1)}{m}\left(\frac{N_{11}^{(2)} N_{22}^{(2)} - N_{12}^{(2)} N_{21}^{(2)}}{(N_M^{(2)})^2}\right) \end{bmatrix}. \tag{8.15}$$