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Research Article

Construction and Analysis of Smarandache Curves for Integral Binormal Curves in Euclidean 3-Space

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Abstract

This paper presents a detailed geometric analysis of Smarandache curves generated from integral binormal curves within three-dimensional Euclidean space. We provide a complete derivation of the Frenet apparatus encompassing tangent, normal, and binormal vectors, alongside curvature and torsion functions for four distinct types of Smarandache curves: TN, TB, NB, and TNB. Furthermore, we establish the necessary and sufficient criteria for these curves to be characterized as general helices or Salkowski curves. A significant outcome of our work is the demonstration that helical characteristics are transmitted from the original curve to its Smarandache derivatives. The theoretical framework is substantiated with numerical examples, including a circular helix and other spatial curves.

1. Introduction

The differential geometry of curves in three-dimensional Euclidean space, $\mathbb{E}^3$, remains a vibrant field of study due to its profound theoretical foundations and extensive practical applications in areas such as computer graphics, robotics, physics, and computer-aided design. A particularly intriguing area within this field involves exploring the relationships between different curves generated from a single original curve. The interplay between a curve and those derived from its geometric elements has been a focus of significant research interest [1–3].
Among these derived curves, Smarandache curves hold a special place due to their unique construction and properties. Defined by specific combinations of the tangent (T), normal (N), and binormal (B) vectors from the Frenet frame of a base curve, Smarandache curves offer novel perspectives on the original curve's geometry. They can reveal underlying properties, such as inherent helical structures or specific curvature constraints, that are not immediately apparent. Since their introduction by Smarandache [4], these curves have been extensively investigated across various geometric settings, including Euclidean space [5], Minkowski space [6, 7], and other specialized contexts [8–11].
The Frenet-Serret apparatus provides the essential framework for analyzing curves locally. This moving orthonormal frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and the accompanying formulae for curvature (κ) and torsion (τ) describe how a curve twists and turns through space. The curvature quantifies the deviation from a straight path, while the torsion measures the departure from a planar one. The Frenet-Serret equations precisely describe the evolution of this frame along the curve, making them indispensable for understanding a curve's local behavior [12].
Parallel to the study of Smarandache curves, the concept of integral curves has gained considerable attention. These curves are formed by integrating one of the vector fields of the original curve's Frenet frame. For instance, the integral binormal curve is defined by integrating the binormal vector field, while the integral normal curve arises from integrating the normal vector field [13]. These integral curves inherit and transform the geometric properties of the original curve, leading to new families of curves with potentially useful characteristics. Their applications span various domains, including dynamical systems and geometric modeling [14–16].
Previous research has explored Smarandache curves in relation to different frames. For example, studies have examined them using the Darboux frame [8] and the Bishop frame [9], yielding valuable insights into their geometric relationships. Other relevant works can be found in [17–19].

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This paper specifically addresses the construction and comprehensive analysis of Smarandache curves associated with integral binormal curves in $\mathbb{E}^3$. Our primary contributions include the detailed derivation of the full Frenet apparatus for four types of Smarandache curves (TN, TB, NB, TNB) constructed from an integral binormal curve. We obtain explicit analytical expressions for their tangent, normal, and binormal vectors, as well as their curvature and torsion functions. Moreover, we establish clear conditions under which these derived curves qualify as general helices or Salkowski curves. A central result confirms that the property of being a general helix is preserved from the original curve to its Smarandache counterparts, underscoring a deep geometric connection.

Beyond its theoretical contributions to differential geometry, this work bridges abstract mathematical concepts with practical application. By meticulously characterizing these curves and their properties, we provide a valuable toolkit for applications requiring sophisticated geometric modeling, such as advanced CAD systems, motion planning algorithms in robotics, and theoretical models in physics. This study not only expands the theoretical framework of curve theory but also opens new avenues for future research in more complex geometric spaces.

The structure of this paper is as follows: Section 2 reviews essential preliminary concepts, including the Frenet frame, Smarandache curves, and integral curves. Section 3 presents our main results, detailing the construction and analysis of the different Smarandache curves from an integral binormal curve. Section 4 provides illustrative examples with a circular helix. Finally, Section 5 concludes the paper and discusses potential future work.

2. Preliminaries

This section provides a concise overview of the fundamental concepts required for understanding the geometric properties of curves in three-dimensional Euclidean space, $\mathbb{E}^3$. These concepts include the structure of $\mathbb{E}^3$ itself, the definition of curves and their parameterization, the Frenet-Serret frame and its associated equations, the definitions of Smarandache and integral curves, and the specific characteristics of general helices and Salkowski curves.

The three-dimensional Euclidean space, $\mathbb{E}^3$, is the standard geometric setting for our study. It is equipped with the standard Euclidean metric, which defines the distance between two points $\mathbf{P_1} = (x_1, y_1, z_1)$ and $\mathbf{P_2} = (x_2, y_2, z_2)$ as:

$$d(\mathbf{P_1}, \mathbf{P_2}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The inner product (dot product) of two vectors $\mathbf{U} = (u_1, u_2, u_3)$ and $\mathbf{V} = (v_1, v_2, v_3)$ in $\mathbb{E}^3$ is given by:

$$\langle \mathbf{U}, \mathbf{V} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

This inner product allows for the calculation of angles between vectors and the lengths of vectors. The cross product in $\mathbb{E}^3$ is defined as:

$$\mathbf{U} \times \mathbf{V} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1),$$

yielding a vector orthogonal to both $\mathbf{U}$ and $\mathbf{V}$.

A curve in $\mathbb{E}^3$ is typically represented as a continuous vector-valued function $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$, where I is an interval. The function $\gamma(t) = (x(t), y(t), z(t))$ provides the coordinates of points on the curve. A curve is termed *regular* if its velocity vector, $\gamma'(t)$, is non-zero for all t in its domain, ensuring a well-defined tangent direction at every point [**?**, 20, 21].

A *unit-speed* (or arc-length) parameterization is often employed to simplify geometric calculations. A curve is parameterized by arc length s if $\|\gamma'(s)\| = 1$ for all s. The arc length from a fixed point t_0 to a point t is given by:

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| du.$$

For a unit-speed curve $\gamma(s)$, the Frenet frame is an orthonormal moving frame consisting of three vectors:

- Tangent Vector: $\mathbf{T}(s) = \gamma'(s)$.
- Normal Vector: $\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$.
- Binormal Vector: $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$.

The evolution of this frame along the curve is governed by the Frenet-Serret equations:

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}, \tag{2.1}$$

where $\kappa(s)$ is the curvature and $\tau(s)$ is the torsion. Here the prime ($'$) denotes differentiation with respect to the arc length parameter s.

For a regular curve that is not necessarily parameterized by arc length, the Frenet frame is defined as:

$$\mathbf{T} = \frac{\dot{\gamma}}{||\dot{\gamma}||}, \quad \mathbf{N} = \frac{\dot{\mathbf{T}}}{||\dot{\mathbf{T}}||}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}. \tag{2.2}$$

The curvature κ and torsion τ are given by:

$$\kappa = \frac{||\dot{\gamma} \times \ddot{\gamma}||}{||\dot{\gamma}||^3}, \quad \text{and} \quad \tau = \frac{\det(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{||\dot{\gamma} \times \ddot{\gamma}||^2}. \tag{2.3}$$

Here the dot $(\cdot)$ denotes differentiation with respect to the arbitrary parameter t.
Smarandache curves are defined by taking specific linear combinations of the Frenet frame vectors of a given unit-speed curve $\gamma(s)$:

$$\eta(s) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N}), \qquad \text{(TN-Smarandache curve)}$$
$$\beta(s) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{B}), \qquad \text{(TB-Smarandache curve)}$$
$$\zeta(s) = \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B}), \qquad \text{(NB-Smarandache curve)}$$
$$\xi(s) = \frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{N}+\mathbf{B}). \qquad \text{(TNB-Smarandache curve)}$$

These curves often exhibit interesting geometric properties that provide new insights into the original curve.
Integral curves are constructed by integrating a vector field related to the original curve. Specifically, for a curve $\alpha(s)$:

- The *integral binormal curve* is defined as $\omega(s) = \int \mathbf{B}_\alpha(s)ds$.
- The *integral normal curve* is defined as $\mu(s) = \int \mathbf{N}_\alpha(s)ds$.

These curves inherit and transform the geometric properties (like curvature and torsion) of the original curve $\alpha(s)$ [13].
A curve $\gamma(s)$ is called a *general helix* if the ratio of its torsion to its curvature is constant, i.e., $\frac{\tau}{\kappa} = C$ for some constant C [12]. A *Salkowski curve* is a special type of curve characterized by having constant curvature but non-constant torsion [22].

3. Main Results

In this section, we define the Frenet apparatus of the Smarandache integral curve, curvatures, and torsion of the TB, TN, NB, and TNB Smarandache curves of the integral binormal curve. Furthermore, we provide the condition that these integral curves be either a general helix or a Salkowski curve.

Definition 3.1. *[23] Let γ be a curve in $\mathbb{E}^3$.We say a curve $\omega(s)$ is an integral binormal curve (B_γ-integral) of $\gamma(s)$ if $\omega(s)$ is given by*

$$\omega(s) = \int B_\gamma \,\mathrm{d}s, \qquad B_\gamma = \omega'(s). \tag{3.1}$$

Proposition 3.2. *[23] If $\omega(s)$ is a B_γ-integral curve of a curve γ, $T_\omega, N_\omega, B_\omega$ is the Frenet frame of the curve ω and $T_\gamma, N_\gamma, B_\gamma$ is the Frenet frame of γ. From Equation (3.1), the $T_\omega, N_\omega, B_\omega, \kappa_\omega$ and τ_ω can be written in terms of the curve γ as:*

$$T_\omega = B_\gamma, \;\; N_\omega = -N_\gamma, \;\; B_\omega = T_\gamma, \;\; \kappa_\omega = \tau_\gamma, \quad \tau_\omega = k_\gamma.$$

3.1. The TN-Smarandache curves of the integral binormal curve

We define the Frenet frame and curvatures of the TN-Smarandache curve derived from the integral binormal curve, and provide the condition that this curve be either a general helix or a Salkowski curve.

Definition 3.3. *If γ is a curve in E^3. The TN-Smarandache curve of the integral binormal curve is defined as:*

$$\Omega_1(s) = \frac{1}{\sqrt{2}}(T_\omega + N_\omega). \tag{3.2}$$

Theorem 3.4. *Let $\gamma(s)$ be a curve. If $\Omega_1(s)$ is a TN-Smarandache curve of the integral binormal curve of γ, then the Frenet vector fields, the curvature, and the torsion of Ω_1 are given by:*

$$\begin{aligned}
T_{\Omega_1} &= \frac{\kappa_\gamma T_\gamma - \tau_\gamma N_\gamma - \tau_\gamma B_\gamma}{\sqrt{\kappa_\gamma^2 + 2\tau_\gamma^2}},\\
N_{\Omega_1} &= \frac{1}{\sqrt{(\kappa_\gamma^2+2\tau_\gamma^2)\Pi}}\left(\kappa_\gamma\tau_\gamma(\kappa_\gamma^2+2\tau_\gamma^2) - 2\kappa_\gamma^2\tau_\gamma\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)T_\gamma + \left((\kappa_\gamma^2+\tau_\gamma^2)(\kappa_\gamma^2+2\tau_\gamma^2) - \kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)N_\gamma\\
&\quad - \frac{1}{\sqrt{(\kappa_\gamma^2+2\tau_\gamma^2)\Pi}}\left(\tau_\gamma^2(\kappa_\gamma^2+2\tau_\gamma^2) + \kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)B_\gamma,\\
B_{\Omega_1} &= \frac{\left(\tau_\gamma\left(\kappa_\gamma^2+2\tau_\gamma^2\right)\right)T_\gamma + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)' N_\gamma + \left(\kappa_\gamma(\kappa_\gamma^2+2\tau_\gamma^2) - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)B_\gamma}{\sqrt{\Pi}},\\
\kappa_{\Omega_1} &= \sqrt{\frac{2\Pi}{(\kappa_\gamma^2+2\tau_\gamma^2)^3}},\\
\tau_{\Omega_1} &= \frac{\sqrt{2}\left[(\kappa_\gamma^2+2\tau_\gamma^2)[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma - \tau_\gamma\kappa_\gamma^2\left(\frac{\tau_\gamma}{\kappa_\gamma}\right)'] + 3\kappa_\gamma^2[\kappa_\gamma\kappa_\gamma' + \tau_\gamma\tau_\gamma']\left(\frac{\tau_\gamma}{\kappa_\gamma}\right)'\right]}{\Pi},
\end{aligned} \tag{3.3}$$

where

$$\Pi = \tau_\gamma^2(\kappa_\gamma^2+2\tau_\gamma^2)^2+\kappa_\gamma^4\left(\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)^2+\left(\kappa_\gamma(\kappa_\gamma^2+2\tau_\gamma^2)-\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)^2.$$

Proof. Let Ω_1 be a TN-Smarandache curve of the integral binormal curve of γ, then by taking the derivatives of Equation (3.2) three times and using Equation (2.1), we get:

$$\begin{aligned}
\Omega_1' &= \frac{1}{\sqrt{2}}(\kappa_\gamma T_\gamma - \tau_\gamma N_\gamma - \tau_\gamma B_\gamma),\\
\Omega_1'' &= \frac{1}{\sqrt{2}}(\kappa_\gamma' + \tau_\gamma\kappa_\gamma)T_\gamma + (\kappa_\gamma^2+\tau_\gamma^2-\tau_\gamma')N_\gamma - (\tau_\gamma'+\tau_\gamma^2)B_\gamma),\\
\Omega_1''' &= \frac{1}{\sqrt{2}}((\kappa_\gamma''+2\tau_\gamma'\kappa_\gamma+\tau_\gamma\kappa_\gamma'-\kappa_\gamma(\kappa_\gamma^2+\tau_\gamma^2))T_\gamma+(-\tau_\gamma''+3(\tau_\gamma\tau_\gamma'+\kappa_\gamma\kappa_\gamma')+\tau_\gamma(\kappa_\gamma^2+\tau_\gamma^2))N_\gamma+(-\tau_\gamma''-3\tau_\gamma\tau_\gamma'+\tau_\gamma(\kappa_\gamma^2+\tau_\gamma^2))B_\gamma),
\end{aligned}$$

therefore we have

$$\begin{aligned}
\Omega_1'\times\Omega_1'' &= \frac{1}{2}\left[\tau_\gamma(\kappa_\gamma^2+2\tau_\gamma^2)T_\gamma+\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'N_\gamma+\kappa_\gamma(\kappa_\gamma^2+2\tau_\gamma^2)-\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'B_\gamma\right],\\
\det(\Omega_1',\Omega_1'',\Omega_1''') &= \frac{1}{2\sqrt{2}}\left[(\kappa_\gamma^2+2\tau_\gamma^2)[\kappa_\gamma''\tau_\gamma-\tau_\gamma''\kappa_\gamma-\tau_\gamma\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)']+3\kappa_\gamma^2[\kappa_\gamma\kappa_\gamma'+2\tau_\gamma\tau_\gamma']\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right].
\end{aligned}$$

By Equations (2.2) and (2.3), we can conclude the results. □

Theorem 3.5. *Let Ω_1 be a TN-Smarandache curve of the integral binormal curve of γ. If γ is a general helix, then Ω_1 is a general helix.*

Proof. Let Ω_1 be the Smarandache curve of the integral binormal curve of γ. Then, according to Equation (3.3), we obtain:

$$\frac{\tau_{\Omega_1}}{\kappa_{\Omega_1}} = \frac{(\kappa_\gamma^2+2\tau_\gamma^2)^3\left[(\kappa_\gamma^2+\tau_\gamma^2)[\kappa_\gamma''\tau_\gamma-\tau_\gamma''\kappa_\gamma-\tau_\gamma\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)']+3\kappa_\gamma^2[\kappa_\gamma\kappa_\gamma'+\tau_\gamma\tau_\gamma']\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right]}{\Pi^{\frac{3}{2}}},$$

by considering that γ is a general helix, i.e. $\frac{\tau_\gamma}{\kappa_\gamma}$ is constant. Consequently, Ω_1 is also a general helix. □

Corollary 3.6. *If Ω_1 is a TN-Smarandache curve of the integral binormal curve of γ. If τ_γ and κ_γ are constants, then Ω_1 is a Salkowski curve.*

3.2. The TB-Smarandache curve of the integral binormal curve

We define the Frenet apparatus of this curve by applying its Frenet frame and curvatures of the TB-Smarandache curve of the integral, and provide the condition that this curve be either a general helix or a Salkowski curve.

Definition 3.7. *If $\gamma(s)$ is a curve in E^3. The TB-Smarandache curve of the integral binormal curve is defined as:*

$$\Omega_2(s) = \frac{1}{\sqrt{2}}(T_\omega+B_\omega). \tag{3.4}$$

Theorem 3.8. *Let γ be a curve. If Ω_2 is a TB-Smarandache curve of the integral binormal curve of γ, then the Frenet vector fields, the curvature, and the torsion of Ω_2 are given by:*

$$\begin{aligned}
T_{\Omega_2} &= N_\gamma,\\
N_{\Omega_2} &= \frac{-\kappa_\gamma T_\gamma+\tau_\gamma B_\gamma}{\sqrt{\kappa_\gamma^2+\tau_\gamma^2}},\\
B_{\Omega_2} &= \frac{\tau_\gamma T_\gamma+\kappa_\gamma B_\gamma}{\sqrt{\kappa_\gamma^2+\tau_\gamma^2}},\\
\kappa_{\Omega_2} &= \sqrt{\frac{2(\kappa_\gamma^2+\tau_\gamma^2)}{(\kappa_\gamma-\tau_\gamma)^2}},\\
\tau_{\Omega_2} &= \frac{\sqrt{2}\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'}{(\kappa_\gamma-\tau_\gamma)(\kappa_\gamma^2+\tau_\gamma^2)}.
\end{aligned} \tag{3.5}$$

Proof. Let Ω_2 be a TB-Smarandache curve of the integral binormal curve of γ, then by taking the derivatives of Equation (3.4) three times, and using Equation (2.1), we get:

$$\Omega_2' = \frac{1}{\sqrt{2}}(\kappa_\gamma - \tau_\gamma)N_\gamma,$$
$$\Omega_2'' = \frac{1}{\sqrt{2}}[-\kappa_\gamma(\kappa_\gamma - \tau_\gamma)T_\gamma + (\kappa_\gamma' - \tau_\gamma')N_\gamma + \tau_\gamma(\kappa_\gamma - \tau_\gamma)B_\gamma],$$
$$\Omega_2''' = \frac{1}{\sqrt{2}}[(-\kappa_\gamma'(\kappa_\gamma - \tau_\gamma) - 2\kappa_\gamma(\kappa_\gamma' - \tau_\gamma')T_\gamma + (\kappa_\gamma'' - \tau_\gamma'') - (\kappa_\gamma - \tau_\gamma)(\kappa_\gamma^2 + \tau_\gamma^2)N_\gamma + \tau_\gamma'(\kappa_\gamma - \tau_\gamma) + 2\tau_\gamma(\kappa_\gamma' - \tau_\gamma')B_\gamma].$$

Therefore, we have

$$\Omega_2' \times \Omega_2'' = \frac{1}{2}(\kappa_\gamma - \tau_\gamma)^2[\tau_\gamma T_\gamma + \kappa_\gamma B_\gamma],$$
$$\det(\Omega_2', \Omega_2'', \Omega_2''') = \frac{1}{2\sqrt{2}}\kappa_\gamma^2(\kappa_\gamma - \tau_\gamma)^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'.$$

With Equations (2.2) and (2.3), we can conclude the results.

□

Theorem 3.9. *Let Ω_2 be a TB-Smarandache curve of the integral binormal curve of γ. If γ is a general helix, then Ω_2 is a general helix.*

Proof. Let Ω_2 be a TB-Smarandache curve of the integral binormal curve of γ, then according to Equation (3.5), we have:

$$\frac{\tau_{\Omega_2}}{\kappa_{\Omega_2}} = \frac{\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'}{(\kappa_\gamma^2 + \tau_\gamma^2)^{\frac{3}{2}}},$$

by considering that γ is a general helix and $\frac{\tau_\gamma}{\kappa_\gamma}$ is constant. Consequently, Ω_2 is also a general helix.

□

Corollary 3.10. *If Ω_2 is a TB-Smarandache curve of the integral binormal curve of γ. If τ_γ and κ_γ are constants, then Ω_2 is a Salkowski curve.*

3.3. The *NB*-Smarandache curves of the integral binormal curve

We define the Frenet apparatus of this curve by applying its Frenet frame and curvatures of the *NB*-Smarandache curve of the integral, and provide the condition that this curve be either a general helix or a Salkowski curve.

Definition 3.11. *If γ is a curve in E^3. The NB-Smarandache curve of the integral binormal curve is defined as:*

$$\Omega_3(s) = \frac{1}{\sqrt{2}}(N_\omega + B_\omega). \tag{3.6}$$

Theorem 3.12. *Let $\gamma(s)$ be a curve. If Ω_3 is a NB-Smarandache curve of the integral binormal curve of γ, then the Frenet vector fields, the curvature, and the torsion of Ω_3 are given by:*

$$T_{\Omega_3} = \frac{\kappa_\gamma T_\gamma + \kappa_\gamma N_\gamma - \tau_\gamma B_\gamma}{\sqrt{2\kappa_\gamma^2 + \tau_\gamma^2}},$$
$$N_{\Omega_3} = \frac{1}{\sqrt{(2\kappa_\gamma^2 + \tau_\gamma^2)\phi}}\left(-\kappa_\gamma^2(2\kappa_\gamma^2 + \tau_\gamma^2) - \kappa_\gamma^2\tau_\gamma\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)T_\gamma + \left((\kappa_\gamma^2 + \tau_\gamma^2)(2\kappa_\gamma^2 + \tau_\gamma^2) - \kappa_\gamma^2\tau_\gamma\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)N_\gamma$$
$$+ \left(\tau_\gamma\kappa_\gamma(2\kappa_\gamma^2 + \tau_\gamma^2) - 2\kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)B_\gamma,$$
$$B_{\Omega_3} = \frac{1}{\sqrt{2}}\frac{\left(\tau_\gamma\left(2\kappa_\gamma^2 + \tau_\gamma^2\right) - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)T_\gamma + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'N_\gamma + \kappa_\gamma(2\kappa_\gamma^2 + \tau_\gamma^2)B_\gamma}{\sqrt{\phi}},$$
$$\kappa_{\Omega_3} = \sqrt{\frac{2\phi}{(2\kappa_\gamma^2 + \tau_\gamma^2)^3}},$$
$$\tau_{\Omega_3} = \frac{\sqrt{2}\left[(2\kappa_\gamma^2 + \tau_\gamma^2)[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma + \kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'] + 3\kappa_\gamma^2[\kappa_\gamma\kappa_\gamma' + \tau_\gamma\tau_\gamma']\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right]}{\phi}, \tag{3.7}$$

where

$$\phi = \kappa_\gamma^2(2\kappa_\gamma^2 + \tau_\gamma^2)^2 + \kappa_\gamma^4\left(\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)^2 + \left(\tau_\gamma(2\kappa_\gamma^2 + \tau_\gamma^2) - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)^2.$$

Proof. Let Ω_3 be a *NB*-Smarandache curve of the integral binormal curve of γ, then by taking the derivatives of Equation (3.6) three times and using Equation (2.1), we get:

$$\Omega_3' = \frac{1}{\sqrt{2}}(\kappa_\gamma T_\gamma + \kappa_\gamma N_\gamma - \tau_\gamma B_\gamma),$$
$$\Omega_3'' = \frac{1}{\sqrt{2}}((\kappa_\gamma' - \kappa_\gamma^2)T_\gamma + (\kappa_\gamma^2 + \tau_\gamma^2 + \kappa_\gamma')N_\gamma + (-\tau_\gamma' + \tau_\gamma\kappa_\gamma)B_\gamma),$$
$$\Omega_3''' = \frac{1}{\sqrt{2}}([\kappa_\gamma'' - 3\kappa_\gamma'\kappa_\gamma - \kappa_\gamma(\kappa_\gamma^2+\tau_\gamma^2)]T_\gamma + [\kappa_\gamma'' + 3(\tau_\gamma\tau_\gamma' + \kappa_\gamma\kappa_\gamma') - \kappa_\gamma(\kappa_\gamma^2+\tau_\gamma^2)]N_\gamma + [-\tau_\gamma'' + 2\tau_\gamma\kappa_\gamma' + \kappa_\gamma\tau_\gamma' + \tau_\gamma(\kappa_\gamma^2+\tau_\gamma^2)]B_\gamma).$$

Therefore, we have

$$\Omega_3' \times \Omega_3'' = \frac{1}{2}\left[\left(\tau_\gamma(2\kappa_\gamma^2+\tau_\gamma^2) - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)T_\gamma + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'N_\gamma + \left(\kappa_\gamma(2\kappa_\gamma^2+\tau_\gamma^2)\right)B_\gamma\right],$$
$$\det(\Omega_3', \Omega_3'', \Omega_3''') = \frac{1}{2\sqrt{2}}\left[(2\kappa_\gamma^2+\tau_\gamma^2)\left[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma + \kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right] + 6\kappa_\gamma^3\kappa_\gamma'\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right].$$

By Equations (2.2) and (2.3), we can conclude the results. □

Theorem 3.13. *Let Ω_3 be a NB-Smarandache curve of the integral binormal curve of γ. If γ is a general helix, then Ω_3 is a general helix.*

Proof. Let Ω_3 be a *NB*-Smarandache curve of the integral binormal curve of γ, then according to Equation (3.7), we have:

$$\frac{\tau_{\Omega_3}}{\kappa_{\Omega_3}} = \frac{\left[(2\kappa_\gamma^2+\tau_\gamma^2)^{\frac{5}{2}}\left[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma + \kappa_\gamma^3\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right] + 6\kappa_\gamma^3\kappa_\gamma'(2\kappa_\gamma^2+\tau_\gamma^2)^{\frac{3}{2}}\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right]}{\phi^{\frac{3}{2}}},$$

by considering that γ is a general helix, i.e $\frac{\tau_\gamma}{\kappa_\gamma}$ is constant. Consequently, Ω_3 is also a general helix. □

Corollary 3.14. *If Ω_3 is an NB-Smarandache curve of the integral binormal curve of γ. If τ_γ and κ_γ are constants, then Ω_3 is a Salkowski curve.*

3.4. The *TNB*-Smarandache curves of the integral binormal curve

We define the Frenet apparatus of this curve by applying its Frenet frame and curvatures of the *TNB*-Smarandache curve of the integral, and provide the condition that this curve be either a general helix or a Salkowski curve.

Definition 3.15. *If γ is a curve in E^3. The TNB-Smarandache curve of the integral binormal curve is defined as:*

$$\Omega_4(s) = \frac{1}{\sqrt{3}}(T_\omega + N_\omega + B_\omega). \tag{3.8}$$

Theorem 3.16. *Let γ be a curve. If Ω_4 is a TNB-Smarandache curve of the integral binormal curve of γ, then the Frenet vector fields, the curvature, and the torsion of Ω_4 are given by:*

$$T_{\Omega_4} = \frac{\kappa_\gamma T_\gamma + (\kappa_\gamma - \tau_\gamma)N_\gamma - \tau_\gamma B_\gamma}{\sqrt{Q}},$$
$$N_{\Omega_4} = \frac{1}{\sqrt{QA}}(-\kappa_\gamma(\kappa_\gamma-\tau_\gamma)Q + (\kappa_\gamma - 2\tau_\gamma)\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'T_\gamma + (\kappa_\gamma^2+\tau_\gamma^2)Q - \kappa_\gamma^2(\kappa_\gamma+\tau_\gamma)\left(\frac{\tau_\gamma}{k_\gamma}\right)'N_\gamma$$
$$+ \tau_\gamma(\kappa_\gamma-\tau_\gamma)Q + \kappa_\gamma^2(\tau_\gamma - 2\kappa_\gamma)\left(\frac{\tau_\gamma}{k_\gamma}\right)'B_\gamma),$$
$$B_{\Omega_4} = \frac{\left[Q\tau_\gamma - \kappa_\gamma^2\left(\frac{\tau_\gamma}{\kappa_\gamma}\right)'\right]T_\gamma + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'N_\gamma + \left[Q\kappa_\gamma - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right]B_\gamma}{\sqrt{A}},$$
$$\kappa_{\Omega_4} = \sqrt{\frac{3A}{Q^3}},$$
$$\tau_{\Omega_4} = \frac{\sqrt{3}H}{A}, \tag{3.9}$$

where

$$Q = (\kappa_\gamma^2+\tau_\gamma^2) + (\kappa_\gamma - \tau_\gamma)^2,$$
$$A = Q^2(\kappa_\gamma^2+\tau_\gamma^2) + 3\left(\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)^2 - 2Q\kappa_\gamma^2(\kappa_\gamma+\tau_\gamma)\left(\frac{\tau_\gamma}{k_\gamma}\right)',$$
$$H = \left[Q[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma] + 3[\kappa_\gamma\kappa_\gamma' + \tau_\gamma\tau_\gamma']\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)' + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'(\kappa_\gamma - \tau_\gamma)\left[Q + 3(\kappa_\gamma' - \tau_\gamma')\right]\right].$$

Proof. Let Ω_4 be a TNB-Smarandache curve of the integral binormal curve of γ, then by taking the derivatives of Equation (3.8) three times and using Equation (2.1), we get:

$$\Omega_4' = \frac{1}{\sqrt{3}}(\kappa_\gamma T_\gamma + (\kappa_\gamma - \tau_\gamma)N_\gamma - \tau_\gamma B_\gamma),$$
$$\Omega_4'' = \frac{1}{\sqrt{3}}(\kappa_\gamma' - \kappa_\gamma(\kappa_\gamma - \tau_\gamma))T_\gamma + (\kappa_\gamma^2 + \tau_\gamma^2 + \kappa_\gamma' - \tau_\gamma')N_\gamma + (-\tau_\gamma' + \tau_\gamma(\kappa_\gamma - \tau_\gamma))B_\gamma),$$
$$\Omega_4''' = \frac{1}{\sqrt{3}}([\kappa_\gamma'' - \kappa_\gamma'(\kappa_\gamma - \tau_\gamma) - 2\kappa_\gamma(\kappa_\gamma' - \tau_\gamma') - \kappa_\gamma(\kappa_\gamma^2 + \tau_\gamma^2)]T_\gamma + [\kappa_\gamma'' - \tau_\gamma'' + 3(\tau_\gamma\tau_\gamma' + \kappa_\gamma\kappa_\gamma') - (\kappa_\gamma - \tau_\gamma)(\kappa_\gamma^2 + \tau_\gamma^2)]N_\gamma$$
$$+ [-\tau_\gamma'' + \tau_\gamma'(\kappa_\gamma - \tau_\gamma) + 2\tau_\gamma(\kappa_\gamma' - \tau_\gamma') + \tau_\gamma(\kappa_\gamma^2 + \tau_\gamma^2)]B_\gamma),$$

therefore we have

$$\Omega_4' \times \Omega_4' = \frac{1}{3}\left[\left(\tau_\gamma Q - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)T_\gamma + \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)' N_\gamma + \left(\kappa_\gamma Q - \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'\right)B_\gamma\right],$$
$$\det(\Omega_4', \Omega_4'', \Omega_4''') = \frac{1}{3\sqrt{3}}[Q[\kappa_\gamma''\tau_\gamma - \tau_\gamma''\kappa_\gamma] + 3[\kappa_\gamma\kappa_\gamma' + \tau_\gamma\tau_\gamma']\kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)' \kappa_\gamma^2\left(\frac{\tau_\gamma}{k_\gamma}\right)'(\kappa_\gamma - \tau_\gamma)\left[Q + 3(\kappa_\gamma' - \tau_\gamma')\right]].$$

By Equations (2.2) and (2.3), we can conclude the results. □

Theorem 3.17. *Let Ω_4 be a TNB-Smarandache curve of the integral binormal curve of γ. If γ is a general helix, then Ω_4 is a general helix.*

Proof. Let Ω_4 be a TNB-Smarandache curve of the integral binormal curve of γ, then according to Equation (3.9), we have:

$$\frac{\tau_{\Omega_4}}{\kappa_{\Omega_4}} = \frac{HQ^{\frac{3}{2}}}{A^{\frac{3}{2}}}.$$

By considering that γ is a general helix, i.e. $\frac{\tau_\gamma}{\kappa_\gamma}$ is constant. Consequently, Ω_4 is also a general helix. □

Corollary 3.18. *If Ω_4 is a TNB-Smarandache curve of the integral binormal curve of γ. If τ_γ and κ_γ are constants, then Ω_4 is a Salkowski curve.*

4. Example

Let $\gamma(s)$ be a circular helix parameterized by arc length s as in Figure 4.1:

$$\gamma(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\frac{s}{c}\right),$$

where $c = \sqrt{a^2 + b^2}$, and a, b are constants.

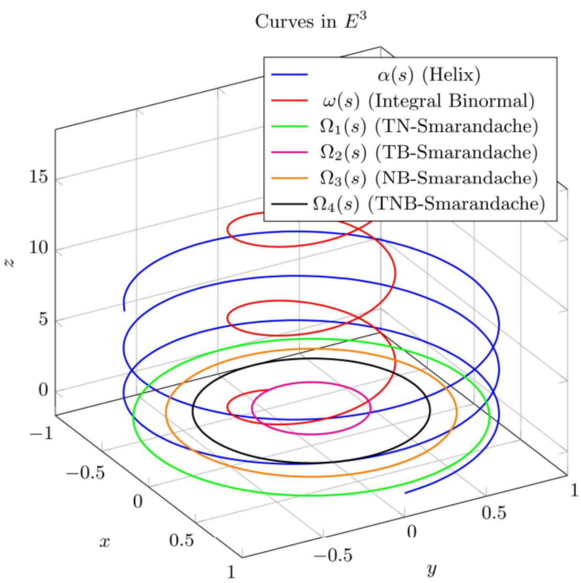

Figure 4.1: The curve $\alpha(s)$, the integral binormal curve $\omega(s)$ and the four types of Smarandache curves

The Frenet Frame $\{T, N, B\}$ of $\gamma(s)$ is given by:

$$T_\gamma(s) = \gamma'(s) = \left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right),$$

$$N_\gamma(s) = \frac{T'(s)}{\kappa} = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right),$$

$$B_\gamma(s) = T_\gamma(s) \times N_\gamma(s) = \left(\frac{b}{c}\sin\left(\frac{s}{c}\right), -\frac{b}{c}\cos\left(\frac{s}{c}\right), \frac{a}{c}\right).$$

The integral binormal curve $\omega(s)$ is defined as:

$$\omega(s) = \int B_\gamma(s)\,ds.$$

Then,

$$\omega(s) = \left(-\frac{b}{c}\cos\left(\frac{s}{c}\right), -\frac{b}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}s\right).$$

The four types of Smarandache curves are

1. TN-**Smarandache Curve** $\Omega_1(s)$**:**

$$\Omega_1(s) = \frac{1}{\sqrt{2}}(T_\omega + N_\omega) = \frac{1}{\sqrt{2}}\left[\left(-\cos\left(\frac{s}{c}\right) - \frac{b}{c}\sin\left(\frac{s}{c}\right),\ -\sin\left(\frac{s}{c}\right) + \frac{b}{c}\cos\left(\frac{s}{c}\right),\ -\frac{a}{c}\right)\right].$$

2. TB-**Smarandache Curve** $\Omega_2(s)$**:**

$$\Omega_2(s) = \frac{1}{\sqrt{2}}(T_\omega + B_\omega) = \frac{1}{\sqrt{2}}\left[\left(-\cos\left(\frac{s}{c}\right) - \frac{a}{c}\sin\left(\frac{s}{c}\right),\ -\sin\left(\frac{s}{c}\right) + \frac{a}{c}\cos\left(\frac{s}{c}\right),\ \frac{b}{c}\right)\right].$$

3. NB-**Smarandache Curve** $\Omega_3(s)$**:**

$$\Omega_3(s) = \frac{1}{\sqrt{2}}(N_\omega + B_\omega) = \frac{1}{\sqrt{2}}\left[\left(-\frac{a+b}{c}\sin\left(\frac{s}{c}\right),\ \frac{a+b}{c}\cos\left(\frac{s}{c}\right),\ \frac{b-a}{c}\right)\right].$$

4. TNB-**Smarandache Curve** $\Omega_4(s)$**:**

$$\Omega_4(s) = \frac{1}{\sqrt{3}}(T_\omega + N_\omega + B_\omega) = \frac{1}{\sqrt{3}}\left(-\cos\left(\frac{s}{c}\right) - \frac{a+b}{c}\sin\left(\frac{s}{c}\right),\ -\sin\left(\frac{s}{c}\right) + \frac{a+b}{c}\cos\left(\frac{s}{c}\right),\ \frac{b-a}{c}\right).$$

5. Conclusion

This study provides a complete geometric characterization of Smarandache curves derived from integral binormal curves in $\mathbb{E}^3$. We have explicitly derived the Frenet apparatus for the TN, TB, NB, and TNB types and established the conditions under which these curves form general helices or Salkowski curves. A key finding is the inheritance of helical properties from the original curve to its Smarandache counterparts.

These results extend the theoretical framework of curve theory and offer practical value for applications requiring precise geometric modeling, such as CAD and path planning in robotics. The work underscores the profound connection between a curve and the geometric objects constructed from its frame, providing a foundation for future research in more complex spaces.

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