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The General Form of Normal Quasi-Differential Operators for First Order and Their Spectrum

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Abstract. In this work, the general form of all normal quasi-differential operators for first order in the weighted Hilbert spaces of vector-functions on right semi-axis in term of boundary conditions has been found. Later on, spectrum set of these operators will be investigated.

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1. Introduction

It is known that a densely defined closed operator N in any Hilbert space is called formally normal if $D(N) \subset D(N^*)$ and $\|Nf\| = \|N^*f\|$ for all $f \in D(N)$, where N^* is the adjoint to the operator N. If a formally normal operator has no formally normal extension, then it is called maximal formally normal operator. If a formally normal operator N satisfied the condition $D(N) = D(N^*)$, then it is called a normal operator [1].

Generalization of J. von Neumann's theory to the theory of normal extensions of formally normal operators in Hilbert space has been done by E. A. Coddington in work [1]. And also the first results in the area of normal extension of unbounded formally normal operators in a Hilbert space are due to Y. Kilpi [6–8] and R. H. Davis [2]. Some applications of this theory to two-point regular type first order differential operators in Hilbert space of vector functions can be found in [5] (also see references therein).

In this work, in the third section all normal extensions of the minimal formally normal operator generated by a linear quasi-differential expression in weighted Hilbert space of vector-functions defined in right half-infinite interval are described. Furthermore, the spectrum of such extensions is investigated.

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2. Statement of the Problem

Let H be a separable Hilbert space and $a \in \mathbb{R}$. And also assumed that $\alpha : (a, \infty) \to (0, \infty)$, $\alpha \in C(a, \infty)$ and $\alpha^{-1} \in L^1(a, \infty)$. In the weighted Hilbert space $L^2_\alpha(H, (a, \infty))$ of $H-$ valued vector-functions defined on the right semi-axis consider the following linear quasi-differential expression with operator coefficient for first order in a form

$$l(u) = (\alpha u)'(t) + Au(t),$$

where $A : D(A) \subset H \to H$ is a selfadjoint operator with condition $A \geq E$, where $E : H \to H$ is an identity operator.

By a standard way the minimal L_0 and maximal L operators corresponding to quasi-differential expression $l(\cdot)$ in $L^2_\alpha(H, (a, \infty))$ can be defined (see [4, 5]). In this case the minimal operator L_0 is formally normal, but it is not maximal in $L^2_\alpha(H, (a, \infty))$.

The main purpose of this work is to describe of all normal extensions of the minimal operator L_0 in terms of boundary conditions in $L^2_\alpha(H, (a, \infty))$. Moreover, structure of the spectrum of these extensions will be surveyed.

3. The General Form of The Normal Extensions

In this section the general form of all normal extensions of the minimal operator L_0 in $L^2_\alpha(H, (a, \infty))$ will be investigated.

In a similar way the minimal operator L_0^+ generated by quasi-differential-operator expression

$$l^+(v) = -(\alpha v)'(t) + Av(t)$$

can be defined in $L^2_\alpha(H, (a, \infty))$ (see [4, 5]).

In this case the operator $L^+ = (L_0)^*$ in $L^2_\alpha(H, (a, \infty))$ is called the maximal operator generated by $l^+(\cdot)$.

It is clear that

$$L_0 \subset L, \ L_0^+ \subset L^+.$$

In this case the following assertion is true.

Lemma 3.1. *If $\widetilde{L}$ is any normal extension of the minimal operator L_0 in $L^2_\alpha(H, (a, \infty))$, then*

$$\alpha D(\widetilde{L}) \subset W^1_{2,\alpha}(H, (a, \infty)), \ AD(\widetilde{L}) \subset L^2_\alpha(H, (a, \infty)),$$

where $W^1_{2,\alpha}(H, (a, \infty))$ is a weighted Sobolev space.

Proof. In this case for any $u \in D(\widetilde{L}) = D(\widetilde{L}^*)$ we have

$$\begin{aligned} \widetilde{L}u &= (\alpha u)'(t) + Au(t) \in L^2_\alpha(H, (a, \infty)), \\ \widetilde{L}^*u &= -(\alpha u)'(t) + Au(t) \in L^2_\alpha(H, (a, \infty)). \end{aligned}$$

From these relations we have $(\alpha u)' \in L^2_\alpha(H, (a, \infty))$ and $Au \in L^2_\alpha(H, (a, \infty))$.

Therefore $\alpha D(\widetilde{L}) \subset W^1_{2,\alpha}(H, (a, \infty))$ and $AD(\widetilde{L}) \subset L^2_\alpha(H, (a, \infty))$.

The minimal operator M_0 generated by following differential expression

$$m(u) = -i(\alpha u)'$$

in $L^2_\alpha(H, (a, \infty))$ is a symmetric. And also a operator $M = M_0^*$ in $L^2_\alpha(H, (a, \infty))$ it will be indicated a maximal operator corresponding to differential expression $m(\cdot)$.

Lemma 3.2. *The deficiency indices of the minimal operator M_0 in $L^2_\alpha(H, (a, \infty))$ are in form*

$$(n_+(M_0), n_-(M_0)) = (dimH, dimH).$$

Proof. For this consider the following differential equation

$$-i(\alpha u_\pm)'(t) \pm iu_\pm(t) = 0, u \in D(M).$$

Then

$$u_\pm(t) = \frac{1}{\alpha(t)} exp\left(\pm \int_a^t \frac{1}{\alpha(s)} ds\right) f, \ f \in H.$$

Consequently,

$$\begin{aligned}
\|u_{\pm}\|^2_{L^2_\alpha(H,(a,\infty))} &= \int_a^\infty \|u_{\pm}(t)\|^2_H dt \\
&= \int_a^\infty \| \frac{1}{\alpha(t)} exp\left(\pm \int_a^t \frac{1}{\alpha(s)} ds \right) f\|^2_H \alpha(t) dt \\
&= \int_a^\infty \frac{1}{\alpha(t)} exp\left(\pm 2 \int_a^t \frac{1}{\alpha(s)} ds \right) dt \|f\|^2_H \\
&= \int_a^\infty exp\left(\pm 2 \int_a^t \frac{1}{\alpha(s)} ds \right) d\left(\int_a^t \frac{1}{\alpha(s)} ds \right) \|f\|^2_H \\
&= \frac{\pm 1}{2} \left(exp\left(\pm 2 \int_a^\infty \frac{1}{\alpha(s)} ds \right) - 1 \right) \|f\|^2_H < \infty.
\end{aligned}$$

This shows that deficiency indices of the minimal operator M_0 in $L^2_\alpha(H,(a,\infty))$ have the form

$$(n_+(M_0), n_-(M_0)) = (dimH, dimH).$$

For the description of all selfadjoint extensions of the minimal operator M_0 in $L^2_\alpha(H,(a,\infty))$ we must be construct space of boundary values of M_0.

Definition 3.3 ([3]). Let $\mathfrak{H}$ be any Hilbert space and $S : D(S) \subset \mathfrak{H} \to \mathfrak{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathfrak{H}$ having equal finite or infinite deficiency indices. A triplet $(\mathbf{H}, \gamma_1, \gamma_2)$, where $\mathbf{H}$ is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into $\mathbf{H}$, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$(S^* f, g)_{\mathfrak{H}} - (f, S^* g)_{\mathfrak{H}} = (\gamma_1(f), \gamma_2(g))_{\mathbf{H}} - (\gamma_2(f), \gamma_1(g))_{\mathbf{H}}$$

while for any $F_1, F_2 \in \mathbf{H}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 3.4. *The triplet* (H, γ_1, γ_2),

$$\begin{aligned}
\gamma_1 : D(M) \to H,\ \gamma_1(u) &= \frac{1}{\sqrt{2}} ((\alpha u)(\infty) - (\alpha u)(a)) \quad and \\
\gamma_2 : D(M) \to H,\ \gamma_2(u) &= -\frac{1}{i\sqrt{2}} ((\alpha u)(\infty) + (\alpha u)(a)),\ u \in D(M)
\end{aligned}$$

is a space of boundary values of the minimal operator M_0 *in* $L^2_\alpha(H,(a,\infty))$.

Proof. For any $u,\ v \in D(M)$

$$\begin{aligned}
(Mu, v)_{L^2_\alpha(H,(a,\infty))} - (u, Mv)_{L^2_\alpha(H,(a,\infty))} &= (-i(\alpha u)', v)_{L^2_\alpha(H,(a,\infty))} - (u, -i(\alpha v)')_{L^2_\alpha(H,(a,\infty))} \\
&= \int_a^\infty (-i(\alpha u)'(t), v(t))_H \alpha(t) dt - \int_a^\infty (u(t), -i(\alpha v)'(t))_H \alpha(t) dt \\
&= -i \left[\int_a^\infty ((\alpha u)'(t), (\alpha v)(t))_H dt + \int_a^\infty ((\alpha u)(t), (\alpha v)'(t))_H dt \right] \\
&= -i \int_a^\infty ((\alpha u)(t), (\alpha v)(t))'_H dt \\
&= -i \left[((\alpha u)(\infty), (\alpha v)(\infty))_H - ((\alpha u)(a), (\alpha v)(a))_H \right] \\
&= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H.
\end{aligned}$$

Now for any given elements $f,\ g \in H$ find the function $u \in D(M)$ such that

$$\gamma_1(u) = \frac{1}{\sqrt{2}}((\alpha u)(\infty) - (\alpha u)(a)) = f \text{ and } \gamma_2(u) = -\frac{1}{i\sqrt{2}}((\alpha u)(\infty) + (\alpha u)(a)) = g.$$

From this it is obtained that

$$(\alpha u)(\infty) = \frac{1}{\sqrt{2}}(f - ig) \text{ and } (\alpha u)(a) = \frac{-1}{\sqrt{2}}(f + ig).$$

If choose the functions $u(\cdot)$ in following forms

$$u(t) = \frac{1}{\alpha(t)}(1 - e^{a-t})(f - ig)/\sqrt{2} + \frac{1}{\alpha(t)}e^{a-t}(-f - ig)/\sqrt{2},$$

then it is clear that $u \in D(M)$ and $\gamma_1(u) = f,\ \gamma_2(u) = g$.

Theorem 3.5. *If $\widetilde{M}$ is a selfadjoint extension of the minimal operator M_0 in $L^2_\alpha(H,(a,\infty))$, then it generates by the differential-operator expression $m(\cdot)$ and boundary condition*

$$(\alpha u)(\infty) = W(\alpha u)(a),$$

where $W : H \to H$ is a unitary operator. Moreover, the unitary operator W in H is determined uniquely by the extension $\widetilde{M}$, i.e. $\widetilde{M} = M_W$ and vice versa.

Proof. It is known that each selfadjoint extensions of the minimal operator M_0 are described by differential-operator expression $m(\cdot)$ with boundary condition

$$(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0,$$

where $V : H \to H$ is a unitary operator. So from Lemma 3.4 we have

$$(V - E)((\alpha u)(\infty) - (\alpha u)(a)) + (V + E)(-((\alpha u)(\infty) + (\alpha u)(a))) = 0.$$

Hence it is obtained that

$$(\alpha u)(\infty) = -V(\alpha u)(a).$$

Choosing $W = -V$ in last boundary condition we have

$$(\alpha u)(\infty) = W(\alpha u)(a).$$

Now we describe the general form of all normal extensions of minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$.

Theorem 3.6. *Let $A^{1/2}W^1_{2,\alpha}(H,(a,\infty)) \subset W^1_2(H,(a,\infty))$. Each normal extension $\widetilde{L}$, $L_0 \subset \widetilde{L} \subset L$ of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$ generates by the quasi-differential-operator expression $l(\cdot)$ with boundary condition*

$$(\alpha u)(\infty) = W(\alpha u)(a),$$

where W and $A^{1/2}WA^{-1/2}$ are unitary operators in H. The unitary operator W is determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$.

On the contrary, the restriction of the maximal operator L to the linear manifold of vector-functions $(\alpha u) \in W^1_{2,\alpha}(H,(a,\infty))$ that satisfy mentioned above condition for some unitary operator W, where $A^{1/2}WA^{-1/2}$ also unitary operator in H, is a normal extension of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$.

Proof. If $\widetilde{L}$ is any normal extension of the minimal operator L_0 in $L^2_\alpha(H,(a,\infty))$, then

$$Re(\widetilde{L}) = \overline{A \otimes E},\ Re(\widetilde{L}) : D(\widetilde{L}) \to L^2_\alpha(H,(a,\infty)),$$

$$Im(\widetilde{L}) = \overline{E \otimes -i\frac{d}{dt}(\alpha)},\ Im(\widetilde{L}) : D(\widetilde{L}) \to L^2_\alpha(H,(a,\infty)),$$

where the symbol $\otimes$ denotes a tensor product, are selfadjoint extensions of $Re(L_0)$ and $Im(L_0)$ in $L^2_\alpha(H,(a,\infty))$, respectively. Then the extension $Im(\widetilde{L})$ is generated by quasi-differential expression $m(\cdot)$ and boundary condition

$$(\alpha u)(\infty) = W(\alpha u)(a),$$

where W is a unitary operators in H such that it determined uniquely by the extension $\widetilde{L}$, i.e. $\widetilde{L} = L_W$ [3].

On the other hand since the extension $\widetilde{L}$ is a normal operator, then for every $u \in D(\widetilde{L})$ the following equality holds

$$\left(Re(\widetilde{L})u, Im(\widetilde{L})u\right)_{L^2_\alpha(H,(a,\infty))} = \left(Im(\widetilde{L})u, Re(\widetilde{L})u\right)_{L^2_\alpha(H,(a,\infty))}.$$

In other words, for every $u \in D(\widetilde{L})$ we have

$$((\alpha u)', Au)_{L^2_\alpha(H,(a,\infty))} + (Au, (\alpha u)')_{L^2_\alpha(H,(a,\infty))} = 0.$$

From last relation and condition of theorem

$$A^{1/2} W^1_{2,\alpha}(H,(a,\infty)) \subset W^1_2(H,(a,\infty))$$

we have

$$\left((\alpha A^{1/2}u)', \alpha A^{1/2}u\right)_{L^2(H,(a,\infty))} + \left(\alpha A^{1/2}u, (\alpha A^{1/2}u)'\right)_{L^2(H,(a,\infty))} = 0,$$

that is, for every $u \in D(\widetilde{L})$

$$\int_a^\infty \left(\alpha A^{1/2}u, \alpha A^{1/2}u\right)'_H dt = \|\left(\alpha A^{1/2}u\right)(\infty)\|^2_H - \|\left(\alpha A^{1/2}u\right)(a)\|^2_H = 0.$$

Hence there exists a isometry operator V in H, such that

$$A^{1/2}(\alpha u)(\infty) = VA^{1/2}(\alpha u)(a),$$

that is,

$$(\alpha u)(\infty) = A^{-1/2}VA^{1/2}(\alpha u)(a),\ u \in D(\widetilde{L}).$$

Since the unitary operator W in H uniquely is determined by the extension $\widetilde{L}$, then from last equation it is obtained that

$$A^{-1/2}VA^{1/2} = W,$$

that is,

$$V = A^{1/2}WA^{-1/2}$$

is unitary in H.

On the other hand, a sufficient part of this theorem can be easily to check.

Hence the proof of theorem is completed.

4. Spectrum of The Normal Extensions

Here the spectrum of the normal extension of the minimal operator L_0 generated by linear quasi-differential expression $l(\cdot)$ with corresponding boundary condition in Theorem 3.6 in $L^2_\alpha(H,(a,\infty))$ will be investigated.

Firstly let us prove the following results.

Theorem 4.1. *The spectrum of any normal extension L_W in $L^2_\alpha(H,(a,\infty))$ of the minimal operator L_0 has a form*

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \left(\int_a^\infty \frac{ds}{\alpha(s)}\right)^{-1} (ln|\mu|^{-1} + 2n\pi i - iarg\mu),\ n \in \mathbb{Z},\ \mu \in \sigma\left(W^* exp\left(-A\int_a^\infty \frac{ds}{\alpha(s)}\right)\right)\right\}.$$

*Proof.*Consider a problem for the spectrum for the any normal extension L_W, that is

$$(\alpha u)'(t) + Au(t) = \lambda u(t) + f(t), \lambda \in \mathbb{C},\ Re\lambda = \lambda_r \geq 1,\ u,\ f \in L^2_\alpha(H,(a,\infty))$$

with boundary condition

$$(\alpha u)(\infty) = W(\alpha u)(a),$$

where W and $A^{1/2}WA^{-1/2}$ are the unitary operators in H.

Then it is clear that a general solution of above differential equation is in form

$$\begin{aligned} u(t;\lambda) &= \frac{1}{\alpha(t)}exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda \\ &+\frac{1}{\alpha(t)}\int_a^t \exp\left((\lambda E - A)\int_s^t \frac{d\tau}{\alpha(\tau)}\right)f(s)ds,\ f_\lambda \in H. \end{aligned}$$

In this case

$$\begin{aligned} &\|\frac{1}{\alpha(t)}exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda\|^2_{L^2_\alpha(H,(a,\infty))} \\ =& \int_a^\infty \|\frac{1}{\alpha(t)}exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda\|^2_H \alpha(t)dt \\ =& \int_a^\infty \left(\frac{1}{\alpha(t)}exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda, \frac{1}{\alpha(t)}exp\left((\lambda E - A)\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda\right)_H \alpha(t)dt \\ =& \int_a^\infty \frac{1}{\alpha(t)}exp\left(2\lambda_r \int_a^t \frac{ds}{\alpha(s)}\right)\left(exp\left(-A\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda, exp\left(-A\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda\right)dt \\ =& \int_a^\infty \frac{1}{\alpha(t)}exp\left(2\lambda_r \int_a^t \frac{ds}{\alpha(s)}\right)\|exp\left(-A\int_a^t \frac{ds}{\alpha(s)}\right)f_\lambda\|^2_H dt \\ \le& \int_a^\infty \frac{1}{\alpha(t)}exp\left(2\lambda_r \int_a^t \frac{ds}{\alpha(s)}\right)dt\|f_\lambda\|^2_H \\ =& \frac{1}{2\lambda_r}\left(exp\left(2\lambda_r \int_a^\infty \frac{ds}{\alpha(s)}\right)-1\right)\|f_\lambda\|^2_H < \infty \end{aligned}$$

and

$$\begin{aligned} &\|\frac{1}{\alpha(t)}\int_a^t \exp\left((\lambda E - A)\int_s^t \frac{d\tau}{\alpha(\tau)}\right)f(s)ds\|^2_{L^2_\alpha(H,(a,\infty))} \\ =& \int_a^\infty \|\frac{1}{\alpha(t)}\int_a^t \exp\left((\lambda E - A)\int_s^t \frac{d\tau}{\alpha(\tau)}\right)f(s)ds\|^2_H \alpha(t)dt \\ =& \int_a^\infty \frac{1}{\alpha(t)}\|\int_a^t \exp\left((\lambda E - A)\int_s^t \frac{d\tau}{\alpha(\tau)}\right)f(s)ds\|^2_H dt \\ =& \int_a^\infty \frac{1}{\alpha(t)}\|\int_a^t \exp\left(\lambda E\int_s^t \frac{d\tau}{\alpha(\tau)}\right)\left[\exp\left(-A\int_s^t \frac{d\tau}{\alpha(\tau)}\right)f(s)\right]ds\|^2_H dt \end{aligned}$$

$$
\begin{aligned}
&= \int_a^\infty \frac{1}{\alpha(t)} \|\int_a^t \exp\left((\lambda_r + i\lambda_i)E \int_s^t \frac{d\tau}{\alpha(\tau)}\right)\left[\exp\left(-A\int_s^t \frac{d\tau}{\alpha(\tau)}\right)\frac{1}{\alpha(s)}(\alpha(s)f(s))\right] ds\|_H^2 dt \\
&\leq \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^\infty \frac{1}{\alpha(s)} \exp\left(\lambda_r E \int_s^t \frac{d\tau}{\alpha(\tau)}\right) ds\right)\left(\int_a^\infty \alpha(s)\|f\|_H^2 ds\right) dt \\
&= \int_a^\infty \frac{1}{\alpha(t)} \left(\int_a^\infty \frac{1}{\alpha(s)} \exp\left(\lambda_r E \int_a^\infty \frac{d\tau}{\alpha(\tau)}\right) ds\right) \|f\|^2_{L^2_\alpha(H,(a,\infty))} \\
&= \exp\left(\lambda_r E \int_a^\infty \frac{d\tau}{\alpha(\tau)}\right)\left(\int_a^\infty \frac{ds}{\alpha(s)}\right)^2 \|f\|^2_{L^2_\alpha(H,(a,\infty))} < \infty.
\end{aligned}
$$

Hence for $u(\cdot, \lambda) \in L^2_\alpha(H, (a, \infty))$ for $\lambda \in \mathbb{C},\ \lambda_r \geq 1$.
In this case the boundary condition we get the following relation

$$
\left(exp\left(-\lambda \int_a^\infty \frac{ds}{\alpha(s)}\right) - W^* exp\left(-A \int_a^\infty \frac{ds}{\alpha(s)}\right)\right) f_\lambda = exp\left(-\lambda \int_a^\infty \frac{ds}{\alpha(s)}\right) W^* \int_a^\infty \exp\left((\lambda E - A)\int_s^\infty \frac{d\tau}{\alpha(\tau)}\right) f(s) ds.
$$

From this it is seen that in order to $\lambda \in \sigma(L_W)$ the necessary and sufficient condition is

$$
exp\left(-\lambda \int_a^\infty \frac{ds}{\alpha(s)}\right) = \mu \in \sigma\left(W^* exp\left(-A \int_a^\infty \frac{ds}{\alpha(s)}\right)\right).
$$

Therefore

$$
\lambda = \left(\int_a^\infty \frac{ds}{\alpha(s)}\right)^{-1} (ln|\mu|^{-1} + 2n\pi i - iarg\mu),\ n \in \mathbb{Z},\ \mu \in \sigma\left(W^* exp\left(-A \int_a^\infty \frac{ds}{\alpha(s)}\right)\right).
$$

Example 4.2. The spectrum of boundary value problem L_γ

$$
\begin{aligned}
&(t^\gamma u(t,x))' - \frac{\partial^2 u(t,x)}{\partial x^2} = (\lambda - 1)u(t,x) + f(t,x),\ t > 1,\ 0 < x < 1,\ \gamma > 1, \\
&u(t,0) = u(t,1) = 0,\ t > 1, \\
&(t^\gamma u)(1,x) = (t^\gamma u)(\infty, x), 0 < x < 1
\end{aligned}
$$

in $L^2_{t^\gamma}((1,\infty) \times (0,1))$ is in form

$$
\begin{aligned}
\sigma(L(\gamma)) &= \left\{\left(\int_1^\infty \frac{ds}{s^\gamma}\right)^{-1}\left((\tau + 1)\int_1^\infty \frac{ds}{s^\gamma} + 2n\pi i\right),\ n \in \mathbb{Z},\ \tau \in \sigma\left(-\frac{\partial^2}{\partial x^2}\right)\right\} \\
&= \left\{(\tau + 1) + 2n\pi i(\gamma - 1):\ n \in \mathbb{Z},\ \tau \in \sigma\left(-\frac{\partial^2}{\partial x^2}\right)\right\}.
\end{aligned}
$$

References

[1] Coddington, E.A., *Extension theory of formally normal and symmetric subspaces* , Mem. Amer. Math. Soc., **134**(1973), 1–80. 1

[2] Davis, R.H., Singular Normal Differential Operators, Tech. Rep., Dep. Math., California Univ., 1955. 1

[3] Gorbachuk, V.I., Gorbachuk, M.L., *On boundary value problems for a first-order differential equation with operator coefficients and the expansion in eigenvectors of this equation*, Soviet Math. Dokl., **14(1)**(1973), 244–248. 3.3, 3

[4] Hörmander, L., *On the theory of general partial differential operators*, Acta Mathematica, **94**(1955), 161–248. 2, 3

[5] Ismailov, Z.I., *Compact inverses of first-order normal differential operators*, J. Math., Anal. Appl. USA, **320(1)**(2006), 266–278. 1, 2, 3

[6] Kilpi, Y., *Über lineare normale transformationen in Hilbertschen raum*, Ann. Acad. Sci. Fenn. Math. Ser. AI, **154**(1953). 1

[7] Kilpi, Y., *Über das komplexe momenten problem*, Ann. Acad. Sci. Fenn. Math. Ser. AI, **236**(1957), 1–32. 1

[8] Kilpi, Y., *Über die anzahl der hypermaximalen normalen fort setzungen normalen transformationen*, Ann. Univ. Turkuensis. Ser. AI, **65**(1963). 1