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On A Four-Step Exponentially Fitted Scheme for The Solution of Stiff Differential Systems

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Abstract. Many problems that are often encountered in fields like engineering, mechanics, electronic, astrophysics, chemistry and control theory, yield initial value problems involving systems of ordinary differential equations which exhibit a phenomenon which has come to be known as stiffness. In this work, a new four-step exponentially-fitted predictor-corrector method involving the second derivative for solving system of stiff differential equations is constructed using a combination of the extended backward differentiation formula and the technique of exponential fitting. The constructed method is well-suited for systems with pronounced stiffness. The stability property of the constructed scheme is also considered. To investigate the accuracy of the constructed method, three standard numerical examples with pronounced stiffness are considered. A comparison of the results obtained by implementing the proposed methods on the numerical problems compared with those of existing standard method show that the constructed method is efficient and accurate for solving stiff systems of ordinary differential equations.

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1. Introduction

Many problems that are often encountered in fields like engineering, mechanics, electronic, astrophysics, chemistry and control theory, yield initial value problems involving systems of ordinary differential equations which exhibit a phenomenon which has come to be known as *'stiffness'*.

Definition 1.1. The linear system $\mathbf{y}' = A\mathbf{y} + \phi(x)$ is said to be stiff if

(i): $Re\ \lambda_t < 0, t = 1, 2, \cdots, m,$ and
(ii): $\max_{t=1,2,\cdots,m} |Re\ \lambda_t| >> \min_{t=1,2,\cdots,m} |Re\ \lambda_t|$,

where $\lambda_t, t = 1, 2, \cdots, m$ are the eigenvalues of A. The ratio

$$\left[\max_{t=1,2,\cdots,m} |Re\ \lambda_t|\right] : \left[\min_{t=1,2,\cdots,m} |Re\ \lambda_t|\right]$$

is called the stiffness ratio [11].

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Non-linear systems $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ exhibit stiffness if the eigenvalues of the Jacobian $\partial\mathbf{f}/\partial\mathbf{y}$ behave in a similar fashion. The eigenvalues are no longer constant but depend on the solution, and therefore vary with x. Accordingly we say that the system $\mathbf{y}' = \mathbf{f}(x, \mathbf{y})$ is stiff in an interval I of x, if, for $x \in I$, the eigenvalues $\lambda_t(x)$ of $\partial\mathbf{f}/\partial\mathbf{y}$ satisfy *(i)* and *(ii)* above.

Early attempts to tackle the problem of stiffness with the use of classical methods and techniques encountered very substantial difficulties. The difficulty of solving stiff initial value problems was clearly identified in early 1950s when Curtiss and Hirschfelder [7] published one of the first papers in which the problem of stiffness was stated. Subsequently, a whole lot of methods and algorithms have been proposed for solving problems that exhibit stiffness. The author in [4] introduced a class of extended backward differentiation formulae suitable for the integration of stiff systems of autonomous initial value problems. In a later work [5], the author proposed classes of predictor-corrector method involving the second derivative using the extended backward differentiation formulae.

Exponential fitting are numerical methods which are very robust for the integration of differential equations whose Jacobian has large imaginary eigenvalues [6, 12]. Obtaining the solution of stiff differential equations using exponentially fitted formula was first proposed by Liniger and Willoughby [12]. Integration formulae (*which are shown to be A-stable*) containing free parameters were derived and these parameters were chosen so that a given function $\exp(q)$ where q is real, satisfies the integration formulae exactly.

In the work of Jackson and Kenue [10], the authors derived 2-step fourth-order exponentially fitted formulae involving the second derivative that are also shown to be A-stable. Based on the concept proposed by Cash [4–6], Okunuga [13, 14], in his works derived composite 2-step methods of order four which contain a*'built-in'* local error estimate. The methods derived by Okunuga [13, 14] gave better accuracy compared with those of Cash [5, 6]. Abhulimen [1, 2], using the idea of Okunuga [13, 14] in his works derived a 3-step composite scheme of order six and an exponentially fitted two-step third derivative methods of order eight respectively.

Recently, Ehigie et. al. [8], derive a class of 2-step exponentially fitted predictor-corrector method involving the second derivative using the extended backward differentiation formula. The method of Ehigie et. al. [8] was constructed as an hybrid of extended backward differentiation formula of Cash [4,5] and the exponential fitting of Okunuga [13,14].

Following [3, 8, 13, 14], we construct a class of 4-step exponentially fitted predictor-corrector method involving the second derivative. The qualitative properties of the constructed method are investigated. Numerical experiments on prominent stiff problem that show the accuracy of the constructed method compare with the methods of Abhulimen [3], Okunuga [14] and Ehigie et. al. [8] are also presented.

2. Derivation of The Proposed Scheme

Consider the initial value problem for a first-order system, which we may write as

$$\left.\begin{array}{ll} y_1'(x) = f_1(x, y_1, y_2, \cdots, y_r); & y_1(a) = \eta_1 \\ y_2'(x) = f_2(x, y_1, y_2, \cdots, y_r); & y_2(a) = \eta_2 \\ \vdots & \vdots \\ y_r'(x) = f_r(x, y_1, y_2, \cdots, y_r); & y_r(a) = \eta_r \end{array}\right\} \tag{2.1}$$

Introducing the vector notation, $\mathbf{y} = [y_1, y_2, \cdots, y_r]^T$, $\mathbf{f} = [f_1, f_2, \cdots, f_r]^T = \mathbf{f}(x, \mathbf{y})$ and $\eta = [\eta_1, \eta_2, \cdots, \eta_r]^T$, we may write the initial value problem (2.1) in the form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(a) = \eta$$

The general multiderivative multistep scheme for solving (2.1) is given by

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = \sum_{j=1}^{t} h^j \sum_{k=0}^{u} \gamma_{j,k} f_{n+k}^{(j-1)} \tag{2.2}$$

where $y_{n+i} \equiv y(x_{n+i})$ and $f_{n+k}^{(j-1)} \equiv f^{(j-1)}(x_{n+k})$.
Setting $s = 4,\ t = 2$ *and* $u = 4$ in (2.2), the following is obtained:

$$\begin{aligned} y_{n+4} + \alpha_3 y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = {} & h\left(\beta_0 y_n' + \beta_1 y_{n+1}' + \beta_2 y_{n+2}' + \beta_3 y_{n+3}' + \beta_4 y_{n+4}'\right) \\ & + h^2\left(\gamma_0 y_n'' + \gamma_1 y_{n+1}'' + \gamma_2 y_{n+2}'' + \gamma_3 y_{n+3}'' + \gamma_4 y_{n+4}''\right) \end{aligned}$$

By setting $\alpha_3 = 0,\ \alpha_2 = 0,\ \alpha_1 = 0,\ \alpha_0 = -1,\ \beta_1 = 0,\ \beta_3 = 0,\ \gamma_1 = 0,\ \gamma_3 = 0$, in (2.3), it reduces to:

$$y_{n+4} - y_n = h\left(\beta_0 y'_n + \beta_2 y'_{n+2} + \beta_4 y'_{n+4}\right) + h^2\left(\gamma_0 y''_n + \gamma_2 y''_{n+2} + \gamma_4 y''_{n+4}\right) \tag{2.3}$$

Equation (2.3) is the predictor scheme for the proposed method, while the proposed corrector scheme takes the form

$$y_{n+4} - y_n = h\left(\lambda_0 y'_n + \lambda_2 y'_{n+2} + \lambda_4 y'_{n+4} + \lambda_5 y'_{n+5}\right) + h^2\left(\rho_0 y''_n + \rho_2 y''_{n+2} + \rho_4 y''_{n+4}\right) \tag{2.4}$$

where $\beta_0,\ \beta_2,\ \beta_4,\ \gamma_0,\ \gamma_2,\ \gamma_4,\ \lambda_0,\ \lambda_2,\ \lambda_4,\ \lambda_5,\ \rho_0,\ \rho_2,\ \rho_4$ are constants to be determined. Equations (2.3) and (2.4) combine is the proposed predictor-corrector method. In order to obtain the constants of (2.3), we set $\beta_4 = a$, which is the free parameter associated with the predictor scheme, (2.3) is then expanded in Taylor's series and the resulting system of equation is given below

$$\begin{aligned}
4 - a - \beta_0 - \beta_2 &= 0\\
8 - 4a - 2\beta_2 - \gamma_0 - \gamma_2 - \gamma_4 &= 0\\
\frac{32}{3} - 8a - 2\beta_2 - 2\gamma_2 - 4\gamma_4 &= 0\\
\frac{32}{3} - \frac{32a}{3} - \frac{4\beta_2}{3} - 2\gamma_2 - 8\gamma_4 &= 0\\
\frac{128}{15} - \frac{32a}{3} - \frac{2\beta_2}{3} - \frac{4\gamma_2}{3} - \frac{32\gamma_4}{3} &= 0
\end{aligned}$$

Solving the above system of equation, we have

$$\beta_0 = \frac{28}{15} - a,\ \beta_2 = \frac{32}{15},\ \gamma_0 = \frac{8}{9} - \frac{2a}{3},\ \gamma_2 = \frac{112}{45} - \frac{8a}{3},\ \gamma_4 = \frac{16}{45} - \frac{2a}{3}$$

Thus, the predictor integrator is

$$\begin{aligned}
y_{n+4} - y_n = {} & h\left(\left(\frac{28}{15} - a\right)y'_n + \frac{32}{15}y'_{n+2} + a y'_{n+4}\right)\\
& + h^2\left(\left(\frac{8}{9} - \frac{2a}{3}\right)y''_n + \left(\frac{112}{45} - \frac{8a}{3}\right)y''_{n+2} + \left(\frac{16}{45} - \frac{2a}{3}\right)y''_{n+4}\right)
\end{aligned} \tag{2.5}$$

Applying (2.5) to the test function

$$y' = \lambda y,\quad y(x_0) = y_0,\quad \lambda h = q \tag{2.6}$$

we have

$$\frac{y_{n+4}}{y_n} = \frac{1 + \left(\frac{28}{15} - a\right)q + \left(\frac{8}{9} - \frac{2a}{3}\right)q^2 + e^{2q}\left(\left(\frac{112}{45} - \frac{8a}{3}\right)q^2 + \frac{32}{15}q\right)}{1 - aq - \left(\frac{16}{45} - \frac{2a}{3}\right)q^2} = \bar{R}(q) \tag{2.7}$$

from (2.7), the free parameter a, of the predictor integrator is obtained as

$$a(q) = \frac{45 - 45e^{4q} + 84q + 96qe^{2q} + 40q^2 + 112q^2e^{2q} + 16q^2e^{4q}}{15q\left(3 - 3e^{4q} + 2q + 8qe^{2q} + 2qe^{4q}\right)} \tag{2.8}$$

Similarly, the corrector formula (2.4) with $\lambda_5 = r$ set as the free parameter, gives the set of coefficient equations to be determined as follows:

$$\begin{aligned}
4 - r - \lambda_0 - \lambda_2 - \lambda_4 &= 0\\
8 - 5r - 2\lambda_2 - 4\lambda_4 - \rho_0 - \rho_2 - \rho_4 &= 0\\
\frac{32}{3} - \frac{25r}{2} - 2\lambda_2 - 8\lambda_4 - 2\rho_2 - 4\rho_4 &= 0\\
\frac{32}{3} - \frac{125r}{6} - \frac{4\lambda_2}{3} - \frac{32\lambda_4}{3} - 2\rho_2 - 8\rho_4 &= 0\\
\frac{128}{15} - \frac{625r}{24} - \frac{2\lambda_2}{3} - \frac{32\lambda_4}{3} - \frac{4\rho_2}{3} - \frac{32\rho_4}{3} &= 0\\
\frac{256}{45} - \frac{625r}{24} - \frac{4\lambda_2}{15} - \frac{128\lambda_4}{15} - \frac{2\rho_2}{3} - \frac{32\rho_4}{3} &= 0
\end{aligned}$$

solving the above system of equations we have,

$$\lambda_0 = \frac{1792-2295r}{1920},\ \lambda_2 = \frac{1}{240}(512-375r),\ \lambda_4 = \frac{14}{15}+\frac{225r}{128},\ \rho_0 = \frac{4}{15}-\frac{45r}{64},\ \rho_2 = -\frac{75r}{16},\ \rho_4 = -\frac{4}{15}-\frac{225r}{64}$$

Thus our corrector integrator is

$$\begin{aligned} y_{n+4}-y_n &= h\left(\left(\frac{1792-2295r}{1920}\right)y'_n + \frac{1}{240}(512-375r)y'_{n+2} + \left(\frac{14}{15}+\frac{225r}{128}\right)y'_{n+4} + ry'_{n+5}\right) \\ &+h^2\left(\left(\frac{4}{15}-\frac{45r}{64}\right)y''_n - \frac{75r}{16}y''_{n+2} - \left(\frac{4}{15}+\frac{225r}{64}\right)y''_{n+4}\right) \end{aligned} \tag{2.9}$$

Applying (2.9) to the test function (2.6), we have

$$\frac{y_{n+4}}{y_n} = \frac{1+\left(\frac{1792-2295}{1920}r\right)q + \left(\frac{4}{15}-\frac{45}{64}r\right)q^2 + \left(\frac{1}{240}(512-375r)q - \frac{75}{16}rq^2\right)\bar{R}(q)^{1/2} + rq\bar{R}(q)^{5/4}}{1-\left(\frac{14}{15}+\frac{225}{128}r\right)q + \left(\frac{4}{15}+\frac{225r}{64}\right)q^2} = R(q) \tag{2.10}$$

from (2.10), the free parameter r, of the corrector integrator is obtained as

$$r(q) = -\frac{128\left(-15+15e^{4q}-14q-32qe^{2q}-14qe^{4q}-4q^2+4q^2e^{4q}\right)}{15q\left(153+200e^{2q}-225e^{4q}-128e^{5q}+90q+600qe^{2q}+450qe^{4q}\right)} \tag{2.11}$$

Equation (2.10) is the proposed composite integration formula for stiff problems which allow exponential fittings. We shall denote this scheme by *"EF4SPC"*

3. Stability of The Proposed Composite Integration Scheme

Definition 3.1. A numerical method is said to be A-stable if its region of absolute stability (RAS), contains the whole of the left-half of the complex plane [11].

In order to investigate the stability of the proposed method, we need to determine the range of values of the free parameters a and r as given by (2.8) and (2.11) respectively in the left-half $h\lambda$-plane $(-\infty, 0]$. Now, we need to find the conditions that $a(q)$ and $r(q)$ satisfy respectively such that $|\bar{R}(q)| < 1$ and $|R(q)| < 1$.

To achieve this, we must show that *a(q)* and *r(q)* both have finite limits

$$\lim_{q\to-\infty} a(q) = \frac{4}{3} \qquad \textit{and} \qquad \lim_{q\to 0} a(q) = \frac{14}{15}$$

Therefore, for $q \in (-\infty, 0]$, $a \in \left(\frac{14}{15}, \frac{4}{3}\right)$. Similarly,

$$\lim_{q\to-\infty} r(q) = \frac{256}{675} \qquad \textit{and} \qquad \lim_{q\to 0} r(q) = \frac{2048}{23625}$$

Also, for $q \in (-\infty, 0]$, $r \in \left(\frac{2048}{23625}, \frac{256}{675}\right)$. By plotting the graphs of $a(q)$ and $r(q)$ over the range $q \in (-\infty, 0]$, we see that as q decreases, parameters a and r are monotonic increasing as given in *Figure 1*. This shows that the porposed *"EF4SPC"* method is A-stable for values of a and r in the intervals $\left(\frac{14}{15}, \frac{4}{3}\right)$ and $\left(\frac{2048}{23625}, \frac{256}{675}\right)$ respectively.

To obtain the stability function of the proposed method, there is need to unite the stability functions of both the predictor integrator (2.5) and the corrector integrator (2.9). Applying the predictor (2.5) to the test problem (2.6) and simplifying the result, we obtain the stability function $\bar{R}(q)$ of (2.5) as given by (2.7). Now, to obtain the desired stability function, the corrector integrator (2.9) is applied to the test problem (2.6) and using the facts that

$$\left(\frac{y_{n+2}}{y_n}\right) = \bar{R}(q)^{1/2} \qquad \textit{and} \qquad \left(\frac{y_{n+5}}{y_n}\right) = \bar{R}(q)^{5/4},$$

we obtain

$$\begin{aligned} \frac{y_{n+4}}{y_n} &= R(q) \\ &= \frac{1+\left(\frac{1792-2295}{1920}r\right)q + \left(\frac{4}{15}-\frac{45}{64}r\right)q^2 + \left(\frac{1}{240}(512-375r)q - \frac{75}{16}rq^2\right)\bar{R}(q)^{1/2} + rq\bar{R}(q)^{5/4}}{1-\left(\frac{14}{15}+\frac{225}{128}r\right)q + \left(\frac{4}{15}+\frac{225r}{64}\right)q^2} \end{aligned}$$

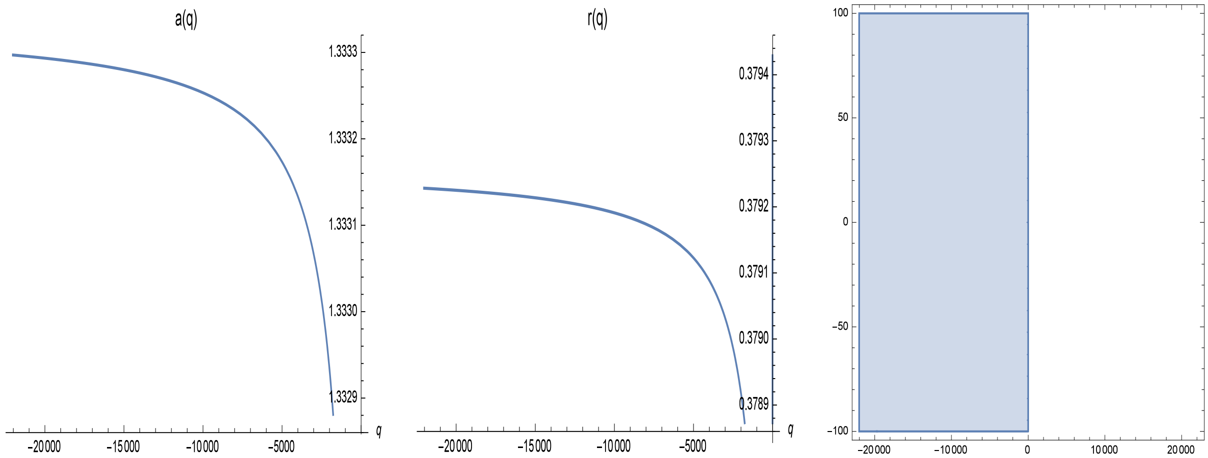

FIGURE 1. Graphs of the free parameters (*a(q) and r(q)*) and truncated region of absolute stability of the proposed method

The function *R(q)* unites the predictor integrator (2.5) and the corrector integrator(2.9) is the desired stability function of the proposed *"EF4SPC"* method. The absolute stability region of the proposed *"EF4SPC"* method is the whole of the left-half plane $x \in [-\infty, 0] \times y \in [-\infty, \infty]$ as seen from Figure 1. Therefore, our composite integration formula (2.5, 2.9), is A-stable within the range of values specified for the choices of parameters $a(q)$ and $r(q)$. Also, the proposed scheme is absolutely stable for $q \in (-\infty, 0]$

Remark: The predictor integrator of the proposed method is of order 5, and the corrector integrator of the proposed method is of order 6, hence, the proposed "EF4SPC" method is of order 6.

Proof: Suppose that the exact solution at the point x_{n+4} is given by $y(x_{n+4}) = y(x_n + 4h)$, then, it can easily be verified that for the predictor:

$$y_{n+4} - y(x_{n+4}) = \left(\frac{16a}{45} - \frac{224}{675}\right) h^6 y^{(6)}(x) + O(h^7)$$

If $a \neq 14/15$, then the integrator is of order 5. Moreover, since $\lim_{q \to 0} a(q) = \frac{14}{15}$ and $q \neq 0$, then, the integrator of the proposed method has order 5. In a similar manner, for the corrector, it can easily be seen that:

$$y_{n+4} - y(x_{n+4}) = \left(\frac{128}{4725} - \frac{5r}{16}\right) h^7 y^{(7)}(x) + O(h^8)$$

the corrector integrator is of order 6 if $r \neq \frac{2048}{23625}$. Hence our proposed method *"EF4SPC"* is of order 6. Therefore, we conclude that our composite integration formula (2.10) is A-stable within the range of values specified for the choices of parameters a and r. Also, the proposed scheme is absolutely stable for $q \in (-\infty, 0]$.

4. Computational Analysis

The aim of the computational analysis carried out in this section is to investigate the accuracy and efficiency of the proposed *"EF4SPC"* method compared with some existing methods. The proposed scheme is implemented on three standard problems that have been studied in the literatures [1–3, 6, 8, 10, 13, 14]. Since our proposed method is of order 6, then it becomes logical to compare it with methods of orders greater or equal to 6. We shall compare the proposed *"EF4SPC"* method with the following schemes: the sixth-order scheme $F^{(6)}$ derived by [3]; the methods of order seven *AB7*, and nine *NM9* proposed by [2].

For all computations, we used the steplengths that were used in the literatures [1–3, 8, 13, 14] in order to have a proper comparison.

4.1. **Problem 1.** Consider the linear problem

$$\begin{aligned} y_1'(x) &= -y_1 + 95y_2, \quad y_1(0) = 1 \\ y_2'(x) &= -y_1 - 97y_2, \quad y_2(0) = 1 \qquad x \in [0, 1] \end{aligned}$$

with exact solution

$$\frac{1}{47}e^{-96t}\left(95e^{94t}-48\right), \qquad \frac{1}{47}e^{-96t}\left(48-e^{94t}\right).$$

The eigenvalues of the Jacobian matrix are $\lambda_1 = -2$ and $\lambda_2 = -96$.

TABLE 1. The absolute error of the proposed *"EF4SPC"* method compared with some existing methods at *x=1* on problem 1

Step	Method	$y(1)$ ($\lvert error\rvert$)	$z(1)\times 10^{-2}$ ($\lvert error\rvert$)
0.0625	AB7	0.27354004 (4.0×10^{-5})	−0.28796321 (6.0×10^{-5})
	NM9	0.27354004 (7.9×10^{-5})	−0.28794740 (8.3×10^{-7})
	F^6	27354004 (3.2×10^{-10})	−0.28794748 (2.4×10^{-10})
	"EF4SPC"	27354004 (0.6×10^{-16})	−0.28794741 (0.7×10^{-18})
0.03125	NM9	0.27354004 (3.7×10^{-5})	−0.28794744 (4.0×10^{-5}))
	F^6	0.27355005 (1.2×10^{-10})	−0.28794741 (8.1×10^{-10})
	"EF4SPC"	0.27355004(0.2×10^{-14})	−0.28794741 (0.2×10^{-16})
Exact Solution at x=1		$y(1) = 0.27355004$	$z(1) = -0.28794741\times 10^{-2}$

From the results shown in Table 1, our proposed method gave better accuracy compared with the existing method. Though our proposed method is of order 6, it performs better than methods of order 7 and order 9 derived by [2].

4.2. **Problem 2.** We also consider the Chemical Kinetic Problem, which is a non-linear stiff problem

$$\begin{aligned} y_1'(x) &= -0.013y_1 + 1000y_1y_3, \quad y_1(0)=1 \\ y_2'(x) &= 2500y_2y_3, \quad y_2(0)=1 \\ y_3'(x) &= 0.013y_1 - 1020y_1y_3 - 2500y_2y_3, \quad y_3(0)=0, \qquad x\in[0,1]. \end{aligned}$$

The eigenvalues of the linearized system are given as $\lambda_1 = 0$, $\lambda_2 = -0.00928572$ and $\lambda_3 = -3500.003714$. For this problem, the results of our proposed *"EF4SPC"* method is compared with the results obtained by the eighth-order method of Abhulimen [2].

TABLE 2. The absolute error of the proposed *"EF4SPC"* method compared with some existing methods at *x=1* on problem 2

Step	Method	$y_1(1)$ ($\lvert error\rvert$)	$y_2(1)$ ($\lvert error\rvert$)	$y_3(1)$ ($\lvert error\rvert$)
0.0625	AB8	0.5884667145 (1.8×10^{-4})	1.0090563343 (1.8×10^{-4})	−2.7919757498 (5.2×10^{-4})
	"EF4SPC"	0.59166648 (0.2×10^{-14})	1.00924 (0.2×10^{-14})	−2.76777747 (0.5×10^{-14})
0.1	AB8	0.5882826902 (2.2×10^{-8})	1.0092403584 (2.2×10^{-9})	−2.7914604809 (6.3×10^{-9})
	"EF4SPC"	0.59166648 (0.1×10^{-14})	1.00924 (0.1×10^{-14})	−2.76777747 (0.3×10^{-14})
Exact Solution at x=1		$y_1(1) = 0.59166648$	$y_2(1) = 1.00924$	$y_3(1) = -2.76777747$

From Table 3, our proposed method gave very accurate solutions compared with the method of Abhulimen [2].

4.3. **Problem 3.** Another problem considered in this paper is the stiff problem from [9],

$$\begin{aligned} y_1'(t) &= -10^4y_1 + 100y_2 - 10y_3 + y_4; \quad y_1(0)=1 \\ y_2'(t) &= -1000y_2 + 10y_3 - 10y_4; \quad y_2(0)=1 \\ y_3'(t) &= -y_3 + 10y_4; \quad y_3(0)=1 \\ y_4'(t) &= -0.1y_4; \quad y_4(0)=1 \end{aligned}$$

Table 3. The absolute error of the proposed *"EF4SPC"* method compared with the methods of of [3] and [2] at *x=1* on problem 3

Step	Method	Absolute error ($\left\|y_i^E(1)\right\|$)at t=1 for $y_i; i = 1,2,3,4$			
		$\left\|y_1^E(1)\right\|$	$\left\|y_2^E(1)\right\|$	$\left\|y_3^E(1)\right\|$	$\left\|y_4^E(1)\right\|$
0.05	AB7	3.2×10^{-2}	3.2×10^{-2}	3.3×10^{-1}	3.7×10^{-5}
	NM9	2.2×10^{-3}	3.5×10^{-2}	3.2×10^{-5}	3.2×10^{-6}
	F^6	3.5×10^{-5}	3.8×10^{-4}	3.5×10^{-7}	3.7×10^{-8}
	"EF4SPC"	0.2×10^{-16}	0.2×10^{-15}	0.2×10^{-13}	0.2×10^{-14}
0.1	AB7	2.5×10^{-2}	2.1×10^{-3}	2.4×10^{-3}	2.7×10^{-5}
	NM9	2.7×10^{-3}	2.4×10^{-3}	2.2×10^{-4}	2.5×10^{-6}
	F^6	2.9×10^{-5}	2.7×10^{-4}	2.6×10^{-6}	2.6×10^{-8}
	"EF4SPC"	0.3×10^{-16}	0.3×10^{-15}	0.3×10^{-13}	0.3×10^{-14}

Table 4. The absolute error of the proposed *"EF4SPC"* method compared with the method of [8] at *x=20* on problem 3

Step	Method	Absolute error ($\left\|y_i^E(20)\right\|$)at t=20 for $y_i; i = 1,2,3,4$			
		$\left\|y_1^E(20)\right\|$	$\left\|y_2^E(20)\right\|$	$\left\|y_3^E(20)\right\|$	$\left\|y_4^E(20)\right\|$
0.05	*"SDEBDF"*	5.31×10^{-12}	7.27×10^{-11}	5.90×10^{-9}	1.34×10^{-9}
	"EF4SPC"	0.62×10^{-16}	0.62×10^{-15}	0.68×10^{-13}	0.62×10^{-14}
0.1	*"SDEBDF"*	2.25×10^{-10}	2.29×10^{-9}	2.50×10^{-7}	2.06×10^{-8}
	"EF4SPC"	0.91×10^{-16}	0.92×10^{-15}	0.10×10^{-12}	0.91×10^{-14}

The eigenvalues of the system are given as $\lambda_1 = -0.1$, $\lambda_2 = -1$, $\lambda_3 = -1000$ and $\lambda_4 = -10000$. The analytical solution of problem 3 is given as

$$
\begin{aligned}
y_1(t) &= \frac{60651229e^{-10000t}}{61276991} + \frac{9989911e^{-1000t}}{899010090} + \frac{20579099e^{-t}}{2261464832} - \frac{23634663e^{-\frac{t}{10}}}{2363466565}\\
y_2(t) &= \frac{60651229e^{-10000t}}{61276991} + \frac{9989911e^{-1000t}}{899010090} + \frac{20579099e^{-t}}{2261464832} - \frac{23634663e^{-\frac{t}{10}}}{2363466565}\\
y_3(t) &= -\frac{e^{-10000t}}{4044502215687230} - \frac{e^{-1000t}}{3790038290900910} - \frac{19295083149241e^{-t}}{1908304926848} + \frac{21203388076089e^{-\frac{t}{10}}}{1908304926848}\\
y_4(t) &= -\frac{e^{-10000t}}{4044502215687230} - \frac{e^{-1000t}}{3790038290900910} - \frac{19295083149241e^{-t}}{1908304926848} + \frac{21203388076089e^{-\frac{t}{10}}}{1908304926848}.
\end{aligned}
$$

This problem is solved within the range $x \in [0, 20]$ with steplengths $h = 0.05$ and $h = 0.1$. For this problem, we compare the results (at $t = 1$) of our proposed method with the sixth-order scheme $F^{(6)}$ of [3]; the methods of order seven *AB7*, and nine *NM9* proposed by [2]. Similarly, at *x=20*, we compare the results of our method with those of [8].

5. Conclusion

Evidently, the newly derived scheme is more accurate as seen from the computational results presented above, since its absolute error is the least of all the methods presented in this paper. It therefore follows that the scheme is quite efficient. We therefore conclude that the proposed *EF4SPC* method is reliable, stable and with high accuracy in computation.

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