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Quenching for A Nonlinear Diffusion Equation with Singular Boundary Outfluxes

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Abstract. In this paper, we study a nonlinear diffusion equation $(\phi(u))_t = u_{xx}$, $0 < x < a$, $t > 0$ with singular boundary outfluxes $u_x(0,t) = u^{-p}(0,t)$, $u_x(a,t) = -u^{-q}(a,t)$. Firstly, we get the quenching occurs in a finite time at the boundary $x = a$ under certain conditions. Finally, we show the time derivative blows up at the quenching time and we also establish results on quenching time and rate for certain nonlinearities.

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1. Introduction

In this paper, we study the quenching behavior of the following nonlinear diffusion equation with singular boundary outfluxes:

$$\begin{cases} (\phi(u))_t = u_{xx},\ 0 < x < a,\ 0 < t < T, \\ u_x(0,t) = u^{-p}(0,t),\ u_x(a,t) = -u^{-q}(a,t),\ 0 < t < T, \\ u(x,0) = u_0(x),\ 0 \le x \le a, \end{cases} \tag{1.1}$$

where $\phi(s)$ is an appropriately smooth and strictly monotone increasing function with $\phi(0) = 0$ and $\phi''(s) > 0$. p, q are positive constants, $T \le \infty$ and the initial function $u_0(x)$ is a non-negative smooth function satisfying the compatibility conditions

$$u_0'(0) = u_0^{-p}(0),\ u_0'(a) = -u_0^{-q}(a).$$

Our main purpose is to examine the quenching behavior of the solutions of problem (1.1) having two singular heat sources.

Definition 1.1. The solution of problem (1.1) is said to quench if there exists a finite time $T = T(u_0) < \infty$ such that

$$\lim_{t \to T^-} \min\{u(x,t) : 0 \le x \le a\} \to 0.$$

From now on, we denote the quenching time of problem (1.1) with T.

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In the literature, there are a few studies about quenching problems with a singular boundary outflux [1,2,4–9]. In [2] Fila and Levine considered a heat equation with a singular boundary outflux:

$$\begin{aligned} &u_t = u_{xx},\ 0 < x < 1,\ 0 < t < T,\\ &u_x(0,t) = 0,\ u_x(1,t) = -u^{-\beta}(1,t),\ 0 < t < T,\\ &u(x,0) = u_0(x) > 0,\ 0 \le x \le 1. \end{aligned}$$

They showed that $x = 1$ is the unique quenching point in finite time, under certain hypotheses on u_0. Further, they obtained a lower bound for quenching time T, $T \ge u_0^{2(\beta+1)}(1)/(2\beta(\beta+1))$, and the quenching rate estimate $(T-t)^{1/2(\beta+1)}$. Deng and Xu [1] considered a problem with a nonlinear boundary outflux at one side:

$$\begin{cases} (\phi(u))_t = u_{xx},\ 0 < x < 1,\ 0 < t < T,\\ u_x(0,t) = 0,\ u_x(1,t) = -u^{-\beta}(1,t),\ 0 < t < T,\\ u(x,0) = u_0(x) > 0,\ 0 \le x \le 1, \end{cases} \tag{1.2}$$

where $0 < \beta < \infty$ and $\phi(u)$ is a monotone increasing function with $\phi(0) = 0$. They showed that u quenches in finite time T and the only quenching point is $x = 1$. Further, they also gave the quenching rate estimate. In [5], Selcuk and Ozalp considered a problem with a nonlinear boundary outflux at one side:

$$\begin{aligned} &u_t = u_{xx} + (1-u)^{-p},\ 0 < x < 1,\ 0 < t < T,\\ &u_x(0,t) = 0,\ u_x(1,t) = -u^{-q}(1,t),\ 0 < t < T,\\ &u(x,0) = u_0(x) > 0,\ 0 \le x \le 1, \end{aligned}$$

where $p, q > 0$. They showed that u quenches in finite time T, the only quenching point is $x = 1$ and the time derivative blows up at the quenching time under certain conditions. Further, they also gave the quenching rate estimate as $(T-t)^{1/2(\beta+1)}$. In [9], Zhi and Mu discussed the equation (1.2) with the Neumann boundary conditions $u_x(0,t) = u^{-q}(0,t)$, $u_x(1,t) = 0$, $t > 0$. They showed that the only quenching point is $x = 1$ and gave the quenching rate.

We can proceed analogously as in [3] and [5] to prove that the positive local solution $u(x,t)$ of (1.1) exists, and $u(x,t) \in C^{2,1}([0,a] \times [0,T_0))$ for some $T_0 > 0$. Here, we consider a quenching problem with two singular boundary outflux terms $u^{-p}(0,t)$, $-u^{-q}(a,t)$. Motivated by problem (1.2), we investigate the quenching behavior of problem (1.1). This paper is arranged as follows. In Section 2, we show that under certain conditions quenching occurs in finite time, the only quenching point is $x = a$. Finally, we show u_t blows up at the quenching time and we also establish results on quenching time and quenching rate for certain nonlinearities.

2. Quenching on The Boundary and Blow-Up of u_t

In this section, we assume that $u_0''(x) \le 0$ in $[0,a]$. First, we rewrite problem (1.1) into the following form

$$\begin{cases} u_t = B(u)u_{xx},\ 0 < x < a,\ 0 < t < T,\\ u_x(0,t) = u^{-p}(0,t),\ u_x(a,t) = -u^{-q}(a,t),\ 0 < t < T,\\ u(x,0) = u_0(x),\ 0 \le x \le a, \end{cases}$$

where $B(u) = 1/\phi'(u)$.

Lemma 2.1. *Assume that $u_0''(x) \le 0$ in $[0,a]$. Then $u_t(x,t) < 0$ in $(0,a) \times (0,T_0)$.*

Proof. Let $w(x,t) = u_t(x,t)$. Then, $w(x,t)$ satisfy

$$\begin{aligned} &w_t = B(u)w_{xx} + B'(u)u_{xx}w,\ 0 < x < a,\ 0 < t < T_0,\\ &w_x(0,t) = -pu^{-p-1}(0,t)w(0,t),\ w_x(a,t) = qu^{-q-1}(a,t)w(a,t),\ 0 < t < T_0,\\ &w(x,0) = B(u_0(x))u_0''(x). \end{aligned}$$

From the Maximum Principle, it follows that $w < 0$ and hence $u_t(x,t) < 0$ in $(0,a)\times(0,T_0)$. (See Lemma 2.1 in [1]). □

Theorem 2.2. *If $u_0''(x) \le 0$ in $[0,a]$, then there exists a finite time T, such that the solution u of problem (1.1) quenches at time T.*

Proof. Assume that if u_0 is an upper solution. Then we get

$$\omega = u^{-q}(a,0) + u^{-p}(0,0) > 0.$$

Introduce a mass function; $m(t) = \int_0^a \phi(u(x,t))dx, 0 < t < T$. Then we get

$$m'(t) = -u^{-q}(a,t) - u^{-p}(0,t) \le -\omega,$$

by Lemma 2.1. Thus, $m(t) \le m(0) - \omega t$; which means that $m(T_0) = 0$ for some $T_0, (0 < T \le T_0)$ and so u quenches in finite time. □

Theorem 2.3. *If $u_0''(x) \le 0$ in $[0,a]$, then quenching occurs only at the boundary $x = a$.*

Proof. Since $u_x(0,t) = u^{-p}(0,t) > 0, u_x(a,t) = -u^{-q}(a,t) < 0$ and $B(u)u_{xx} = u_t < 0$ in $(0,a)\times(0,T)$. Then, u_x is a decreasing function. Also, we assume $\varepsilon \in (0,a)$, $u_x(x,t) > 0$ in $[0,a-\varepsilon)\times(0,T)$, $u_x(a-\varepsilon,t) = 0$ and $u_x(x,t) < 0$ in $(a-\varepsilon,a]\times(0,T)$.

It is easily shown that u does not reach zero in $x \in [0,a-\varepsilon]$ since $u_x(x,t) \ge 0$ in $[0,a-\varepsilon]\times(0,T)$. Let $M > 0$, $\varepsilon_1 \in (0,a)$, $\varepsilon_1 < \varepsilon$. We assume $u_x(x,t) = -M < 0$ in $(a-\varepsilon,a]\times(0,T)$. Integrating both sides with respect to x from $a-\varepsilon_1$ to a, we have

$$u(a-\varepsilon_1,t) = u(a,t) + M\varepsilon_1 > 0.$$

So u does not reach zero in $(a-\varepsilon,a)$. The theorem is proved. □

3. Results for Certain Nonlinearities

In this section, we discuss the special case $\phi(u) = u^m$ $(m > 0)$. That is, we consider the following problem:

$$\begin{cases} (u^m)_t = u_{xx},\ 0 < x < a,\ 0 < t < T, \\ u_x(0,t) = u^{-p}(0,t),\ u_x(a,t) = -u^{-q}(a,t),\ 0 < t < T, \\ u(x,0) = u_0(x),\ 0 \le x \le a, \end{cases} \tag{3.1}$$

As is well known, $0 < m < 1$ corresponds to the porous medium case, $m > 1$ refers to the fast diffusion case, and when $m = 1$ the equation in (3.1) is reduced to the heat equation.

Definition 3.1. μ is called an upper solution of the problem (3.1) if μ satisfies the following conditions:

$$\begin{array}{l} m\mu^{m-1}\mu_t - \mu_{xx} \ge 0,\ 0 < x < a,\ 0 < t < T, \\ \mu_x(0,t) \le \mu^{-p}(0,t),\ \mu_x(a,t) \ge -\mu^{-q}(a,t),\ 0 < t < T, \\ \mu(x,0) \ge u_0(x),\ 0 \le x \le a. \end{array}$$

It is a lower solution when the inequalities are reversed.

Remark 3.2. Let $\widetilde{u}$ and $\widehat{u}$ be a positive upper solution and a nonnegative lower solution of problem (3.1) in $[0,a]\times[0,T)$, respectively. Then, we get $\widetilde{u} \ge \widehat{u}$ in $[0,a]\times[0,T)$. (see Lemma 3.2. in [4])

Theorem 3.3. *If $m \ge \frac{2q}{q+1}$ and $a \le 1$, then $x = a$ is a quenching point.*

Proof. Let $\max\limits_{x\in[0,a]} u_0(x) = M_0 \ge 0$. Define

$$\mu(x,t) = \left(\frac{(1+q)\left(a^2 - x^2 + \tau - t\right)}{2}\right)^{1/(1+q)} \quad \text{in } [0,a]\times[0,\tau],$$

where $\tau = 2M_0^{q+1}/(q+1)$. We have

$$m\mu^{m-1}\mu_t - \mu_{xx} = \frac{m}{2}\left(\frac{(1-q)\left(a^2 - x^2 + \tau - t\right)}{2}\right)^{\frac{(-2q-1)}{(q+1)}+m} - \frac{x^2 q}{q+1}\left(\frac{(1-q)\left(a^2 - x^2 + \tau - t\right)}{2}\right)^{\frac{(-2q-1)}{(q+1)}} \ge 0$$

for $m \ge \frac{2q}{q+1}$, $a \le 1$ and $x \in (0,a), t \in (0,\tau]$. Further,

$$\begin{array}{rcl} \mu_x(0,t) & = & 0, \\ \mu_x(a,t) & \ge & -\mu^{-q}(a,t) \end{array}$$

for $a \le 1$ and $t \in (0, \tau]$. Furthermore,

$$\mu(x,0) = \left(\frac{(1+q)\left(a^2 - x^2 + \tau\right)}{2} \right)^{1/(q+1)} \ge \left(\frac{(1+q)\,\tau}{2} \right)^{1/(q+1)} = M_0,$$

for $x \in [0, a]$. Thus, $\mu(x,t)$ is an upper solution of problem (3.1). In addition, for $t = \tau$ and $x = a$, we get

$$\mu(a, \tau) = 0.$$

Hence, we have

$$u(a,\tau) \le \mu(a,\tau) = 0$$

by Remark 3.2. Thus, $x = a$ is a quenching point. □

Corollary 3.4. *If $\max\limits_{x\in[0,a]} u_0(x) = M_0, m \ge \frac{2q}{q+1}$ and $a \le 1$, then $x = a$ is a quenching point, and a lower bound for the quenching time is*

$$T \le 2M_0^{q+1}/(q+1)$$

from Theorem 3.3. Also, we get a quenching rate

$$u(a,T) \le C_1\,(T-t)^{1/(q+1)}$$

where $C_1 = \left(\frac{q+1}{2}\right)^{1/(q+1)}$.

Theorem 3.5. *Assume that $u_0''(x) \le 0$ in $[0, a]$ and m is an even number. Then u_t blows up at the quenching time and there exists a positive constant C_2 such that*

$$u(a,T) \ge C_2\,(T-t)^{1/(q+1)}$$

for t sufficiently close to T.

Proof. Define

$$J(x,t) = u_t + \delta u^{-q} \text{ in } [b_1, b_2] \times [\tau, T),$$

where $b_1 \in [0, a)$, $b_2 \in (b_1, a]$, $\tau \in (0, T)$ and δ is a positive constant to be specified later. Then, $J(x,t)$ satisfies

$$\begin{aligned} m(u_t)^{m-1} J_t - J_{xx} &= -m(u_t)^m \delta q u^{-q-1} - \delta q(q+1)u^{-q-2}u_x^2 + \delta q u^{-q-1} u_{xx} \\ &= -m\delta q u^{-q-1} u_t((u_t)^{m-1} - u^{m-1}) - \delta q(q+1)u^{-q-2}u_x^2 \\ &< 0 \end{aligned}$$

in $(b_1, b_2) \times [0, T)$. If δ is small enough, then $J(x, \tau) \le 0$ by Remark 3.2. Further, if δ is small enough,

$$\begin{aligned} J(b_1, t) &= u_t(b_1, t) + \delta u^{-q}(b_1, t) < 0, \\ J(b_2, t) &= u_t(b_2, t) + \delta u^{-q}(b_2, t) < 0, \end{aligned}$$

for $t \in (\tau, T)$. By the Maximum Principle, we obtain that $J(x,t) \le 0$ for $(x,t) \in [b_1, b_2] \times [0, T)$. We obtain $u_t \le -\delta u^{-q}(x,t)$ for $(x,t) \in [b_1, b_2] \times [\tau, T)$, i.e. $u_t \le -\delta u^{-q}(x,t)$ for $(x,t) \in [0, a] \times [\tau, T)$. Putting $x = a$, we get

$$u_t(a,t) \le -\delta u^{-q}(a,t),$$

for $t \in (\tau, T)$. Thus, we obtain

$$\lim_{t\to T^-} u_t(a,t) \le \lim_{t\to T^-} -\delta u^{-q}(a,t) = -\infty.$$

Integrating from t to T we get

$$u(a,T) \ge C_2\,(T-t)^{1/(q+1)}$$

where $C_2 = (\delta(q+1))^{1/(q+1)}$ and for t sufficiently close to T. The theorem is proved. □

Corollary 3.6. *Suppose that the assumptions of Theorem 3.3 and Theorem 3.5 hold. From Theorem 3.3, Corollary 3.4 and Theorem 3.5, near the quenching time T, the solution $u(x,t)$ of problem (3.1) has the following quenching rate estimate*

$$u(a,T) \sim (T-t)^{1/(q+1)}.$$

References

[1] Deng, K., Xu, M., *Quenching for a nonlinear diffusion equation with a singular boundary condition*, Z. Angew. Math. Phys., **50**(1999), 574–584. 1, 2

[2] Fila, M., Levine, H.A., *Quenching on the boundary*, Nonlinear Anal., **21**(1993), 795–802. 1

[3] Pao, C.V., *Quasilinear parabolic and elliptic equations with nonlinear boundary conditions*, Nonlinear Analysis, **66**(2007), 639–662. 1

[4] Selcuk, B., Ozalp, N., *Quenching behavior of semilinear heat equations with singular boundary conditions*, Electron. J. Diff. Equ., **2015(311)**(2015), 1–13. 1, 3.2

[5] Selcuk, B., Ozalp, N., *Quenching for a semilinear heat equation with a singular boundary outflux*, Int. J. Appl. Math, **29(4)**(2016), 451–464. 1, 1

[6] Yang, Y., *Quenching phenomenon for a non-Newtonian filtration equation with singular boundary flux*, Boundary Value Problems, (2015), 2015:233. 1

[7] Yang, Y., Yin, J., Jin, C., *Quenching phenomenon of positive radial solutions for p-Laplacian with singular boundary flux*, J. Dyn.Control Syst., **22**(2016), 653–660. 1

[8] Yang, Y., *Finite time quenching for a nonlinear diffusion equation with singular boundary flux*, Journal of Physics: Conference Series, **814(1)**(2017), 1–6. 1

[9] Zhi, Y., Mu, C., *The quenching behavior of a nonlinear parabolic equation with a nonlinear boundary outflux*, Appl. Math.Comput., **184**(2007), 624–630. 1, 1