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Strongly g^*-closed sets and Strongly $T^*_{\frac{1}{2}}$ spaces in bitopological spaces

H. S. Al-Saadi *

Abstract

This paper introduces with a new class of sets called ij-strongly g^*-closed. We prove that this lies between the class of τ_j-closed sets and the class of ij-g^*-closed sets. Also we find some basic properties and applications of ij-strongly g^*-closed sets. We also introduce and study a new class of spaces, namely ij-$ST^*_{\frac{1}{2}}$ spaces.

Keywords: ij-g^*-closed sets, ij-strongly g^*-closed sets, ij-$T^*_{\frac{1}{2}}$ spaces, ij-$ST^*_{\frac{1}{2}}$ spaces.

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1. Introduction

In 1970, Levine [14] introduced generalized closed sets. Kelly [9] initiated the study of bitopological spaces in 1963. A nonempty set X equipped with two topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X, τ_1, τ_2). Fukutake [7] introduced the concept of g-closed set in bitopological spaces and several toplogists [3, 4, 6, 11] generalized many of the results in topological spaces to bitopological spaces. John and Sundarum [8] introduced and studied the concepts of g^*-closed set, $T^*_{\frac{1}{2}}$-space and g^*-continuity for bitopological spaces. The aim of this paper is to introduce the concepts of strongly g^*-closed set, $ST^*_{\frac{1}{2}}$-space and $S^*T_{\frac{1}{2}}$-space for a bitopological space and to study about their properties.

*Mathematics Department, Faculty of Applied Sciences, Umm Al-Qura University, Makkah 21955, P. O Box 11155, Saudi Arabia
Email : hasa112@hotmail.com

2. Preliminaries

Let (X,τ) be a topological space. For a subset A of X, $\tau - cl(A), \tau - int(A)$ represent the closure of A with respect to τ, the interior of A with respect to τ, respectively and complement of A in X is denoted by A^c.

2.1. Definition. A subset A of a topological space (X,τ) is called

(i) a generalized closed set (briefly g-closed set) [14] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(ii) a generalized open set (briefly g-open set) [14] if A^c is g-closed in X.
(iii) a pre open (pre closed) set [14] if $A \subseteq int(cl(A))(cl(int(A)) \subseteq A)$.
(iv) a semi-open (semi-closed) set [13] if $A \subseteq cl(int(A))(int(cl(A)) \subseteq A)$.
(v) a semi-pre open (semi-pre closed) set [1] if $A \subseteq cl(int(cl(A)))(int(cl(int(A))) \subseteq A)$.
(vi) a regular open (regular closed) set [16] if $A = int(cl(A))(A = cl(int(A)))$.

2.2. Definition. The intersection of all semi-pre closed sets containing A is called the semi-pre closure of A and it is denoted by $\tau - spcl(A)$.

Throughout this paper, X and Y always represent nonempty bitopological spaces (X,τ_1,τ_2) on which no separation axioms are assumed unless explicitly mentioned. We denote the family of all g-open subsets of X with respect to the topology τ_i by $\tau_i - GO(X)$ and the family of all τ_j-closed sets is denoted by F_j. By (i,j) we mean the pair of topologies (τ_i,τ_j).

2.3. Definition. A subset A of a bitopological space (X,τ_1,τ_2) is called

(i) ij-semi-open (ij-semi-closed) set [4] if $A \subseteq j\text{-}cl(i\text{-}int(A))$ $(j\text{-}int(i\text{-}cl(A)) \subseteq A)$
(ii) ij-preopen (ij-preclosed) set [10] if $A \subseteq j\text{-}int(i\text{-}cl(A))$ $(j\text{-}cl(i\text{-}int(A)) \subseteq A)$
(iii) ij-semi-preopen set [12] if $A \subseteq i\text{-}cl(j\text{-}int(i - cl(A)))$
(iv) ij-regular open (ij-regular closed) set [15] if $A = j\text{-}int(i\text{-}cl(A))$ $(A = j\text{-}cl(i\text{-}int(A))$
(v) ij-δ-closed set [3] if $A = ij\text{-}\delta cl(A)$ where $ij\text{-}\delta cl(A) = \{x \in X : i\text{-}int(j\text{-}cl(U)) \cap A \neq \phi, U \in \tau_i$ and $x \in U\}$.
(vi) ij-θ-closed set [5] if $A = ij\text{-}\theta cl(A)$ where $ij\text{-}\theta cl(A) = \{x \in X : j\text{-}cl(U) \cap A \neq \phi, U \in \tau_i$ and $x \in U\}$

2.4. Definition. Let (X,τ_1,τ_2) be a bitopological space. A subset A of X is called

(i) ij-g-closed [7] if $j\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$
(ii) ij-gs-closed [11] if $ji\text{-}scl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$
(iii) ij-gsp-closed [6] if $ji\text{-}spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i$
(iv) ij-rg-closed [2] if $j\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in τ_i
(v) ij-g^*-closed [8] if $j\text{-}cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau_i\text{-}GO(X)$

2.5. Definition. A bitopological space (X,τ_1,τ_2) is said to be

(i) ij-$T^*_{\frac{1}{2}}$ [8] if every ij-g^*-closed set is j-closed.
(ii) ij-${}^*T_{\frac{1}{2}}$ [8] if every ij-g-closed set is ij-g^*-closed.

2.6. Theorem. *[8] Every ij-g^*-closed set is ij-g-closed.*

2.7. Theorem. *[8] A bitopological space (X,τ_1,τ_2) is an ij-$T^*_{\frac{1}{2}}$-space if and only if every singleton space $\{x\}$ is either τ_j-open or τ_i-g-closed.*

3. Basic Properties of Pairwise Strongly g^*-closed set

We will start with the following definition

3.1. Definition. A subset A of a bitopological space (X, τ_1, τ_2) is called ij-strongly g^*-closed (briefly ij-Sg^*-closed) if j-cl(i-$int(A)) \subseteq U$ whenever $A \subseteq U$ and U is τ_i-g-open set.

The complement of an ij-Sg^*-closed set is called ij-Sg^*-open. If $A \subset X$ is 12-Sg^*-closed and 21-Sg^*-closed, then A is called pairwise Sg^*-closed.

We denote the family of all ij-Sg^*-closed sets in (X, τ_1, τ_2) by ij-SD^*.

3.2. Theorem. *Every j-closed set is ij-Sg^*-closed set.*

Proof. Let $A \subset X$ be a j-closed set and $A \subset U$ where U is τ_i-g-open set of X. Since A is j-closed, j-$cl(A) = A$ for every subset A of X. Therefore, j-cl(i-$int(A)) \subseteq U$ and hence A is ij-Sg^*-closed set. □

3.3. Remark. An ij-Sg^*- closed set need not be τ_j-closed as shown in the following example.

3.4. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Let $A = \{a, c\}$, then A is 12-Sg^*-closed but not 2-closed in (X, τ_1, τ_2)

3.5. Theorem. *Every ij-g^*-closed set is ij-Sg^*-closed.*

Proof. Let A be an ij-g^*-closed set and U be any τ_i-g-open set containing A. Since j-$cl(i-int(A)) \subset j$-$cl(A)$ and A is ij-g^*-closed, j-$cl(A) \subset U$ for every subset A of (X, τ_1, τ_2). Therefore, j-cl(i-$int(A)) \subset U$ and hence A is ij-Sg^*-closed. □

3.6. Remark. An ij-Sg^*-closed set need not be ij-g^*-closed set as shown in the following example.

3.7. Example. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{b\}, \{b, c\}\}$. Then the subset $\{b\}$ is 12-Sg^*-closed but not 12-g^*-closed set.

3.8. Theorem. *Every ij-Sg^*-closed set is ij-gs-closed set.*

Proof. Let A be an ij-Sg^*-closed set and U be a τ_i- open set such that $A \subseteq U$. Since every τ_i-open set is τ_i-g-open, and A is ij-Sg^*-closed, we have j-cl(i-$int(A)) \subseteq U$. Therefore, $A \cup (j$-cl(i-$int(A))) \subseteq A \cup U$. Hence ji-$scl(A) \subseteq U$, and so A is ij-gs-closed set in X. □

3.9. Remark. An ij-gs-closed set need not be ij-Sg^*-closed as shown in the following example.

3.10. Example. Let $X = \{a, b, c\}, \tau_1 = \{X, \phi, \{a\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{a, b\}\}$. Then the subset $\{a, c\}$ is 12-gs-closed but not 12-Sg^*-closed set.

3.11. Theorem. *Every ij-Sg^*-closed set is ij-rg-closed set.*

Proof. It follows from the fact that every τ_i-g-closed set is ij-rg-closed set . □

3.12. Remark. An ij-rg-closed set need not be ij-Sg^*-closed as shown in the following example.

3.13. Example. Let $X = \{a, b, c\}$,$\tau_1 = \{X, \phi, \{a\}$,$\{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \phi, \{a\}$, $\{b, c\}, \{a, c\}\}$. Then the set $A = \{a, b\}$ is 12-rg-closed but not 12-Sg^*-closed set.

3.14. Theorem. *Every ij-Sg^*-closed set is ij-gsp-closed set.*

Proof. Let A be an ij-Sg^*-closed set and U be an τ_i- open set such that $A \subseteq U$. Since every τ_i-open set is τ_i-g-open, and A is ij-Sg^*-closed, we have i-$int(j$-$cl(i$-$int(A))) \subseteq U$. Therefore, $A \cup (i$-$int(j$-$cl(i$-$int(A)))) \subseteq A \cup U$. Hence ji-$spcl(A) \subseteq U$, and so A is ij-gsp-closed set in X. □

3.15. Remark. An ij-gsp-closed set need not be ij-Sg^*-closed as shown in the following example.

3.16. Example. Let $X = \{a, b, c\}$,$\tau_1 = \{X, \phi, \{b\}, \{b, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Let $A = \{b\}$, then A is 12-gsp-closed but not 12-Sg^*-closed.

3.17. Theorem. *Every ij-δ-closed set is an ij-Sg^*-closed.*

Proof. It follows from the definition. □

3.18. Remark. An ij-Sg^*-closed set need not to be ij-δ-closed as shown in the following example.

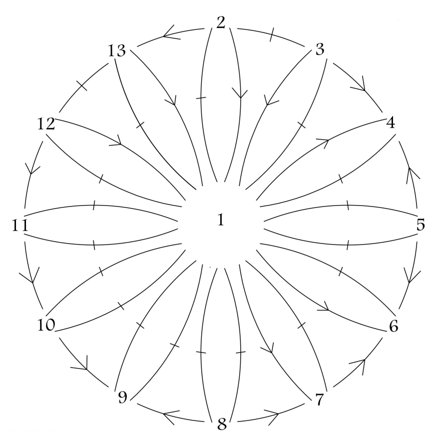
3.19. Example. Let $X = \{a, b, c\}$,$\tau_1 = \{X, \phi, \{a\}, \{a, c\}\}$ and $\tau_2 = \{X, \phi, \{a\}, \{b, c\}\}$. Let $A = \{a\}$. Then A is 12-Sg^*-closed but not 12-δ-closed.

3.20. Theorem. *Every ij-θ-closed set is an ij-Sg^*-closed set .*

Proof. The proof of the theorem is immediate from the definition. □

3.21. Example. An ij-Sg^*-closed set need not be ij-θ-closed as shown in the following example.
Let $X = \{a, b, c\}$, $\tau_1 = \{x, \phi, \{c\}, \{a, c\}\}$ and $\tau_2 = \{x, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. If $A = \{c\}$, then A is 12-Sg^*-closed but not 12-θ-closed.

3.22. Remark. The following diagram shows the relationships of ij-Sg^*-closed set with the other known existing set, $A \to B$ represent, A implies B but not conversely.



1. ij-Sg^*-closed 2. ij-θ-closed 3. ij-g^*-closed 4. ij-rg-closed 5. ij-g-closed 6. ij-gs-closed 7. ij-gsp-closed 8. ij-pr-closed 9. ij-sp-closed 10. ji-semi-closed 11. ij-α-closed 12. ij-δ-closed 13. j-closed

3.23. Theorem. *Let (X, τ_1, τ_2) be a bitopological space. If A is τ_i- open and ij-Sg^*-closed subset of X then it is τ_j-closed.*

Proof. Let A be a subset of X. If it is τ_i-open and ij-Sg^*- closed, then j-$cl(A) \subseteq j$-$cl(i$-$int(A)) \subseteq A$. Since $A \subseteq j$-$cl(A)$, we have $A = j$-$cl(A)$. Thus A is τ_j-closed. □

3.24. Corollary. *If A is τ_i-open and ij-Sg^*-closed in X, then it is both ji-regular open and ij-regular closed in X.*

Proof. Suppose a subset A of X is both τ_i-open and ij-Sg^*-closed. Then $A = i$-$int(j$-$cl(A))$. Since A is τ_i-open and τ_j-closed, A is ji-regular open. Therefore A is τ_i-open and j-$cl(i$-$int(A)) = j$-$cl(A)$. As A is τ_j-closed and j-$cl(i$-$int(A)) = A$, A is ij-regular closed. □

3.25. Corollary. *If A is both τ_i- open and ij-Sg^*-closed, then it is ij-rg-closed.*

Proof. This follows from Corollary 3.24. □

3.26. Theorem. *If A is both ij-Sg^*-closed and ij-semi-open in a bitopological space (X, τ_1, τ_2), then it is ij-g^*-closed.*

Proof. Let A be both ij-Sg^*-closed and ij-semi open in X such that $A \subset U$ where U is an τ_i-g-open. Since A is ij-Sg^*-closed, j-$cl(i$-$int(A)) \subseteq U$. Since A is ij-semi open, j-$cl(A) \subset U$. Thus A is ij-g^*-closed set. □

3.27. Corollary. *If A is both ij-Sg^*-closed and τ_i-open set, then it is ij-g^*-closed set.*

Proof. By the above theorem and the fact that every τ_i- open set is ij-semi-open the proof follows. □

4. Basic properties of ij-Sg^*-closed sets

4.1. Theorem. *If $A, B \in ij$-SD^*, then $A \cup B \in ij$-SD^*.*

Proof. Let A, B be ij-Sg^*-closed sets in X and let $A \cup B \subseteq U$ with U is τ_i-g-open set. Then $j - cl(i - int(A \cup B)) = j - cl(i - int(A)) \cup j - cl(i - int(B)) \subseteq U$ and hence $A \cup B$ is ij-Sg^*-closed set. □

4.2. Remark. The intersection of two ij-Sg^*-closed sets need not be ij-Sg^*-closed set as seen from the following example.

4.3. Example. If $X = \{a, b, c\}, \tau_1 = \{\phi, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$, then {a,b} and {b,c} are 12-Sg^*-closed sets but $\{a, b\} \cap \{b, c\} = \{b\}$ is not 12-Sg^*-closed.

4.4. Remark. In general 12-SD^* is not equal to 21-SD^*.

4.5. Example. In Example 2.24, 12-$SD^* = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and 21-$SD^* = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$, then 12-$SD^* \neq$ 21-SD^*.

4.6. Theorem. *If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2), then 21-$SD^* \subseteq$ 12-SD^*.*

Proof. The proof of the theorem is immediate from the definition. □

4.7. Remark. If 21-$SD^* \subseteq$ 12-SD^*, then τ_1 need not be contained in τ_2 as seen from the following example.

4.8. Example. Let X ={a,b,c}, $\tau_1 = \{\phi, \{b\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. Then 21-$SD^* = \{\{a\}, \{c\}, \{a, c\}\} \subseteq \{\{a\}, \{c\}, \{a, b\}, \{a, c\}\} =$ 12-SD^* but τ_1 is not contained in τ_2.

4.9. Theorem. *For each $x \in X$, the singleton $\{x\}$ is τ_i-g-closed or $\{x\}^c$ is ij-Sg^*-closed.*

Proof. Suppose $\{x\}$ is not τ_i-g-closed, then $\{x\}^c$ will not be τ_1-g-open. Then X is the only τ_i-g-open set containing $\{x\}^c$ and j-$cl(i$-$int(\{x\}^c)) \subset X$. Hence $\{x\}^c$ is ij-Sg^*-closed set and $\{x\}$ is 12-g-open set. □

4.10. Theorem. *If A is ij-Sg^*-closed set of (X, τ_1, τ_2), then j-$cl(i$-$int(A)$-A does not contain a non-empty τ_i-g-closed set.*

Proof. Suppose that A is ij-Sg^*-closed, let F be an τ_i-g-closed set contained in j-$cl(i$-$int(A))$-A, i.e. $F \subseteq j$-$cl(i$-$int(A))$-A. This implies $F \subseteq j$-$cl(i$-$int(A)) \cap A^c$. Thus $F \subseteq j$-$cl(i$-$int(A))$ and so F^c is τ_i-g-open set such that A $\subseteq F^c$. Since A is ij-Sg^*-closed, then j-$cl(i$-$int(A)) \subseteq F^c$. Thus $F \subseteq (j$-$cl(i$-$int(A))^c$. Also we have $F \subseteq j$-$cl(i$-$int(A))$-A . Therefore $F \subseteq j$-$cl(i$-$int(A)) \cap (j$-$cl(i$-$int(A)))^c = \phi$ and so, $F = \phi$. □

The converse of the above theorem is not true as seen from the following example.

4.11. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_2 = \{\phi, \{b\}, X\}$. If $A = \{a\}$, then 2-cl(1-$int(A))$-$A = \{c\}$ does not contain any nonempty τ_1-g-closed set. But A is not 12-Sg^*-closed.

4.12. Corollary. *An ij-Sg^*- closed set A is ij-regular closed iff j-$cl(i$-$int(A))$-A is τ_i-g-closed and $A \subseteq j$-$cl(i$-$int(A))$.*

Proof. Let A be ij-regular closed set. Since j-$cl(i$-$int(A)) = A$, j-$cl(i$-$int(A))$- $A = \phi$ is ij-regular closed set and hence τ_i-g-closed set.
Conversely, suppose that j-$cl(i$-$int(A))$-A is τ_i-g-closed. Then j-$cl(i$-$int(A))$-A contains no nonempty τ_i-g-closed set. Therefore j-$cl(i$-$int(A))$-$A = \phi$ and so A is ij-regular closed. □

4.13. Theorem. *If A is an ij-Sg^*-closed set, then i-$cl(x_i) \cap A \neq \phi$ for each $x_i \in j$-$cl(i$-$int(A))$.*

Proof. Let $x_i \in j$-$cl(i$-$int(A))$ and A be an ij-Sg^*-closed set. Suppose that i-$cl(x_i) \cap A = \phi$. Then $A \subset X - \tau_i - cl(x_i)$ where X-τ_i-$cl(x_i)$ is τ_i-g-open set. Thus $x_i \in X$-τ_i-$cl(x_i)$ which is contradiction. □

4.14. Remark. The converse of the above Theorem is not true. The subset $A = \{a\}$ in Example 4.11 is not 12-Sg^*-closed. However τ_i-$cl(x_i) \cap A \neq \phi$ for each $x_i \in j$-$cl(i$-$int(A))$.

4.15. Theorem. *If A is an ij-Sg^*-closed set and $A \subseteq B \subseteq j$-$cl(i$-$int(A))$, then B is also an ij-Sg^*-closed.*

Proof. Let U be τ_i-open set such that $B \subseteq U$, then $A \subseteq U$. Since A is ij-Sg^*-closed set, then j-$cl(i$-$int(A)) \subseteq U$. Since $B \subseteq$j−cl(i−int(A)),*then*j-$cl(i$-$int(B)) \subseteq$j−cl(i−int(A))$\subseteq U$. Therefore, B is ij-Sg^*-closed. □

4.16. Theorem. *Let $A \subseteq Y \subseteq X$ and suppose that A is ij-Sg^*-closed in X. Then A is ij-Sg^*-closed relative to Y.*

Proof. Suppose that $A \subseteq Y \subseteq X$ and A is ij-Sg^*-closed, $A \subseteq U$ implies j-$cl(i$-$int(A)) \subseteq U$. Thus $Y \cap j$-$cl(i$-$int(A)) \subseteq Y \cap U$. Thus A is ij-Sg^*-closed relative to Y. □

4.17. Theorem. *In a bitopological space (X, τ_1, τ_2), the inclusion i-$GO(X) \subseteq ij$-$RC(X)$ is true if every subset of X is an ij-Sg^*-closed set.*

Proof. Suppose that i-$GO(X) \subseteq ij$-$RC(X)$. Let A be a subset of X such that $A \subseteq U$, where $U \in i$-$GO(X)$. Then j-$cl(i$-$int(A)) \subseteq j$-$cl(i$-$int(U)) = U$ and hence A is ij-Sg^*-closed. □

5. ij- strongly $T^*_{\frac{1}{2}}$-space

We start with the following definition.

5.1. Definition. A bitopological space (X, τ_1, τ_2) is said to be an ij-$ST^*_{\frac{1}{2}}$-space if every ij-Sg^*-closed set is τ_j-closed.

5.2. Theorem. *If (X, τ_1, τ_2) is ij-$ST^*_{\frac{1}{2}}$-space, then it is an ij-$T^*_{\frac{1}{2}}$-space.*

Proof. The proof is straightforward since every ij-Sg^*-closed set is ij-g^*-closed set. □

5.3. Remark. The converse of the above theorem is not true as it can be see from the following example.

5.4. Example. Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then (X, τ_1, τ_2) is a 12-$T^*_{\frac{1}{2}}$-space but not a 12-$ST^*_{\frac{1}{2}}$-space.

5.5. Theorem. *For a bitopological space (X, τ_1, τ_2), the following conditions are equivalent.*

(i) (X, τ_1, τ_2) is an ij-$ST^*_{\frac{1}{2}}$-space.
(ii) Every singleton space $\{x\}$ is either τ_j-open or τ_i-g-closed.

Proof. (i)⟹ (ii) Let $x \in X$. Suppose $\{x\}$ is not τ_i-g-closed. Then $\{x\}^c$ is not τ_i-g-open set. Thus $\{x\}^c$ is an ij-Sg^*-closed by Theorem 4.9. Since (X, τ_1, τ_2) is ij-$ST^*_{\frac{1}{2}}$-space, $\{x\}^c$ is τ_j-closed set of X, i.e $\{x\}$ is τ_j-open set of (X, τ_1, τ_2).
(ii)⟹ (i) Let A be an ij-Sg^*-closed set of (X, τ_1, τ_2). Take $x \in j$-cl(i-$int(A)$). By (ii), $\{x\}$ is either τ_j-open or τ_i-g-closed.
case (i) Let $\{x\}$ be a τ_j-open. Since $x \in j$-cl(i-$int(A)$), $\{x\} \cap A \neq \phi$. This shows that $x \in A$.
case (ii) Let $\{x\}$ be a τ_j-g-open. If we assume that $x \notin A$, then we would have $x \in j$-cl(i-$int(A)$)-A, which can not happen according to Theorem 4.10. Hence $x \in A$.
So, in both cases we have that F is τ_j-closed. Hence (X, τ_1, τ_2) is an ij-$ST^*_{\frac{1}{2}}$-space. □

5.6. Remark. If (X, τ_1, τ_2) is 12-$ST^*_{\frac{1}{2}}$-space, then the space (X, τ_1) is not generally $ST^*_{\frac{1}{2}}$-space as shown in the following example.

5.7. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ_1, τ_2) is 12-$ST^*_{\frac{1}{2}}$-space but (X, τ_1) is not $ST^*_{\frac{1}{2}}$-space.

5.8. Remark. If both (X, τ_1) and (X, τ_2) are $ST^*_{\frac{1}{2}}$-space, then (X, τ_1, τ_2) is not generally 12-$ST^*_{\frac{1}{2}}$-space as shown in the following example.

5.9. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then both (X, τ_1) and (X, τ_2) are $ST^*_{\frac{1}{2}}$-spaces but (X, τ_1, τ_2) is not 12-$ST^*_{\frac{1}{2}}$-space.

5.10. Definition. A bitopological space (X, τ_1, τ_2) is said to be strongly pairwise $ST^*_{\frac{1}{2}}$-space if it is both 12-$ST^*_{\frac{1}{2}}$ and 21-$ST^*_{\frac{1}{2}}$-space.

5.11. Theorem. *If (X, τ_1, τ_2) is strongly pairwise $ST^*_{\frac{1}{2}}$-space, then it is strongly pairwise $T^*_{\frac{1}{2}}$-space.*

5.12. Remark. The converse of the above theorem is not true as it can be see from the following example.

5.13. Example. Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then (X, τ_1, τ_2) is both 12-$T^*_{\frac{1}{2}}$-space and 21-$T^*_{\frac{1}{2}}$−space, therefore it is strongly pairwise $T^*_{\frac{1}{2}}$-space. But (X, τ_1, τ_2) is not strongly pairwise $ST^*_{\frac{1}{2}}$-space.

5.14. Definition. A bitopological space (X, τ_1, τ_2) is said to be an ij-$S^*T_{\frac{1}{2}}$-space if every ij-g-closed set is ij-Sg^*-closed.

5.15. Theorem. *Every ij-$S^*T_{\frac{1}{2}}$-space is an ij-$T^*_{\frac{1}{2}}$-space.*

Proof. The proof follows since every ij-g^*-closed set is ij-Sg^*-closed. □

5.16. Remark. The converse of the above theorem is not true as it can be see from the following example.

5.17. Example. Let $X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. Then (X, τ_1, τ_2) is 12-$T^*_{\frac{1}{2}}$-space but not 12-$S^*T_{\frac{1}{2}}$-space.

5.18. Theorem. *Every ij-${}^*T_{\frac{1}{2}}$-space is an ij-$S^*T_{\frac{1}{2}}$-space.*

Proof. The proof follow by Theorem 2.7 and Theorem 5.5. □

5.19. Remark. The converse of the above theorem is not true as it can be seen from the following example.

5.20. Example. Let X, τ_1 and τ_2 be as in Example 5.13. Then (X, τ_1, τ_2) is 12-$ST^*_{\frac{1}{2}}$-space but not a 12-${}^*T_{\frac{1}{2}}$-space.

5.21. Remark. ij-$ST^*_{\frac{1}{2}}$ and ij-$S^*T_{\frac{1}{2}}$-spaces are independent as seen from the following two examples.

5.22. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then (X, τ_1, τ_2) is 12-$ST^*_{\frac{1}{2}}$-space but 12-$S^*T_{\frac{1}{2}}$- space.

5.23. Example. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, c\}, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Then (X, τ_1, τ_2) is 12-$S^*T_{\frac{1}{2}}$-space but not 12-$ST^*_{\frac{1}{2}}$-space.

5.24. Theorem. *A bitopological space (X, τ_1, τ_2) is an ij-$T^*_{\frac{1}{2}}$ space if and only if it is both ij-$ST^*_{\frac{1}{2}}$ and ij-$S^*T_{\frac{1}{2}}$−space.*

Proof. Suppose that (X, τ_1, τ_2) is an ij-$T^*_{\frac{1}{2}}$-space. Let A be ij-g^*-closed set of (X, τ_1, τ_2). Then by Theorem 3.5, A is an ij-Sg^*-closed set. Since (X, τ_1, τ_2) is an ij-$T^*_{\frac{1}{2}}$- space, A is τ_j-closed set. Hence X is ij-$ST^*_{\frac{1}{2}}$-space. Therefore, A is ij-g-closed set, by Theorem 2.6. Thus A is ij-Sg^*-closed set, by assumption. Hence (X, τ_1, τ_2) is ij-$ST^*_{\frac{1}{2}}$-space. In this case, (X, τ_1, τ_2) is ij-$S^*T_{\frac{1}{2}}$-space and so, ij-$ST^*_{\frac{1}{2}}$- space.

Conversely, suppose that (X, τ_1, τ_2) is both ij-$S^*T_{\frac{1}{2}}$ and ij-$ST^*_{\frac{1}{2}}$ − *space*. Then by Theorem 5.2 and Theorem 5.15, (X, τ_1, τ_2) is ij-$T^*_{\frac{1}{2}}$-space. □

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