On the duality of frames and fusion frames

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Abstract

Optimal frame bounds play a key role in many applications of frame theory, such as filter banks. In this paper, we study the relation between the bounds of a frame and its alternate dual and then present some approach to construct a family of Parseval frames. Also, we survey some problems on duals of fusion frames. In particular, we discuss on some essential differences between duals of ordinary frames and fusion frames. Finally, we characterize duals of some fusion frames.

1. Introduction and preliminaries

Although frames were first introduced by Duffin and Schaeffer in 1952 [12], today they have been developed rapidly in mathematics and have achieved successful applications in various areas of pure and applied sciences and engineering [2, 3, 4, 5, 6]. Recently, due to applications and theoretical goals, some generalizations of frames have been presented [1, 7, 14, 18, 19]. Fusion frames are one of the most important extension of frames. In fact, fusion frames are created to model sensor networks perfectly and provide a mathematical frame work to design and analyze applications under distributed processing requirements, which share many properties with discrete frames. However, they are very different in the duality.

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Let $\mathcal{H}$ be a separable Hilbert space and I be a countable index set. A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist the constants $0 < A \leq B < \infty$ such that

$$(1.1) \qquad A\|f\|^2 \leq \sum_{i\in I}|\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}).$$

The constants A and B are called *frame bounds* that are not unique. Supremum of all lower frame bounds is called the optimal lower frame bound and likewise, the optimal upper frame bound is defined as the infimum of all upper frame bounds. If $A = B$, we call $\{f_i\}_{i\in I}$ a *tight frame*, and in the case of $A = B = 1$ we call it a *Parseval frame*. We say that $\{f_i\}_{i\in I}$ is a *Bessel sequence* whenever in equation of (1.1), the right-hand side holds. Given a frame $F = \{f_i\}_{i\in I}$, the *frame operator* is defined by

$$S_F f = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

It is a bounded, invertible, and self-adjoint operator [11]. For a Bessel sequence $\{f_i\}_{i\in I}$, the *synthesis operator* $T_F : l^2 \to \mathcal{H}$ is defined by $T_F\{c_i\} = \sum_{i\in I} c_i f_i$. If $\{f_i\}_{i\in I}$ is a frame, then $S_F = T_F T_F^*$ where $T_F^* : \mathcal{H} \to l^2$ the adjoint of T, given by $T_F^* f = \{\langle f, f_i\rangle\}_{i\in I}$, is called the *analysis operator*. Moreover, the optimal bounds A and B for a frame $F = \{f_i\}_{i\in I}$ of $\mathcal{H}$ are given by

$$A = \|S_F^{-1}\|^{-1} = \|T_F^\dagger\|^{-2}, \quad B = \|S_F\| = \|T_F\|^2,$$

where $T_F^\dagger$ is the pseudo inverse of T_F, see [11].

Since the frame operator is invertible, we can reconstruct vectors in the space by

$$f = S_F^{-1} S_F f = \sum_{i\in I}\langle f, S_F^{-1} f_i\rangle f_i, \qquad (f \in \mathcal{H}).$$

The family $\{S_F^{-1} f_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, so called the *canonical dual* frame. In general, Bessel sequence $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called an *alternate dual* or simply a *dual* for Bessel sequence $\{f_i\}_{i\in I}$ if

$$(1.2) \qquad f = \sum_{i\in I}\langle f, g_i\rangle f_i, \qquad (f \in \mathcal{H}).$$

There are some characterizations of dual frames [10, 11]. For example there exists a one-to-one correspondence between duals a frame F and the left bounded inverses of T_F^*. Also, the following proposition describes a characterization of alternate dual frames.

1.1. Proposition. [2] *Let* $F = \{f_i\}_{i\in I}$ *be a frame for* $\mathcal{H}$ *and* $\{\delta_i\}_{i\in I}$ *the standard orthonormal basis of* ℓ^2. *Then* $\{g_i\}_{i\in I}$ *is a dual for* $\{f_i\}_{i\in I}$ *if and only if* $g_i = S_F^{-1} f_i + \psi^* \delta_i$ *for some operator* $\psi \in B(\mathcal{H}, \ell^2)$ *such that* $T_F \psi = 0$.

The structure of this paper is as follows: In Section 2, we discuss on the relation between the optimal bounds of a frame and its duals also we present some approaches for constructing of Parseval frames. Section 3 is devoted to survey some properties of dual fusion frames and characterizes duals of some fusion frames in Hilbert spaces.

2. Dual frame bounds

The relation between optimal bounds of a frame and its canonical dual is a well known dependency [11]. In fact, if $F = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with the optimal bounds A and B, respectively. The canonical dual $\{S^{-1} f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ with the optimal bounds $1/B$ and $1/A$, respectively. In this section, we survey the relationship between optimal bounds of a frame and its alternate dual.

2.1. Proposition. *Let $F = \{f_i\}_{i \in I}$ be a tight frame with the optimal bound A, and let $\{g_i\}_{i \in I}$ be a tight dual of $\{f_i\}_{i \in I}$ with the optimal bound $1/A$. Then $\{g_i\}_{i \in I}$ is the canonical dual of F.*

Proof. Since $\{g_i\}_{i \in I}$ is a dual of $\{f_i\}_{i \in I}$, by the Proposition 1.1 there exists a mapping $\psi \in B(\mathcal{H}, l^2)$ such that $g_i = S_F^{-1} f_i + \psi^* \delta_i$ where $\{\delta_i\}_{i \in I}$ is the standard orthonormal basis of l^2. Therefore

$$
\begin{aligned}
\frac{1}{A} f &= \sum_{i \in I} \langle f, g_i \rangle g_i = \sum_{i \in I} \langle f, S_F^{-1} f_i + \psi^* \delta_i \rangle (S_F^{-1} f_i + \psi^* \delta_i) \\
&= S_F^{-1} \sum_{i \in I} \langle f, S_F^{-1} f_i \rangle f_i + \sum_{i \in I} \langle f, \psi^* \delta_i \rangle S_F^{-1} f_i \\
&\quad + \sum_{i \in I} \langle f, S_F^{-1} f_i \rangle \psi^* \delta_i + \sum_{i \in I} \langle f, \psi^* \delta_i \rangle \psi^* \delta_i \\
&= S_F^{-1} f + \psi^* \psi f = \frac{1}{A} f + \psi^* \psi f.
\end{aligned}
$$

It implies that $\psi^* \psi = 0$, i.e $\psi = 0$ and so $\{g_i\}_{i \in I}$ is the canonical dual of F. $\square$

The following example shows that the condition tightness cannot be removed in the above proposition.

2.2. Example. Let $\{e_i\}_{i \in I}$ be an orthonormal basis of Hilbert space $\mathcal{H}$, and also let $\{f_i\}_{i \in I} = \{e_1, e_1, e_2, e_2, e_3, e_4, ...\}$. Then $\{f_i\}_{i \in I}$ is a frame for $\mathcal{H}$ with the optimal bounds 1 and 2, respectively. Its canonical dual is

$$\{S_F^{-1} f_i\}_{i \in I} = \{1/2e_1, 1/2e_1, 1/2e_2, 1/2e_2, e_3, e_4, ...\},$$

with the optimal bounds $1/2$ and 1, respectively. Now, it is not difficult to check that $\{g_i\}_{i \in I} = \{1/2e_1, 1/2e_1, 0, e_2, e_3, ...\}$ is also an alternate dual frame of $\{f_i\}_{i \in I}$ with the same optimal bounds.

Now, we are going to distinct a relation between the bounds of a frame and its alternate dual, to this end, first we recall the following definition.

2.3. Definition. Suppose that X and Y are Banach spaces and $T \in B(X, Y)$, the minimum modulus of T is defined as the following

$$\gamma(T) = \inf\{\|Tf\| : \quad f \in X, \quad \|f\| = 1\}.$$

It is well known that $\gamma(T) > 0$ if and only if T is injective and has closed range, see for instance [16].

2.4. Theorem. *Let $F = \{f_i\}_{i \in I}$ be a frame with the optimal bounds A and B. If $G = \{g_i\}_{i \in I}$ is a dual of F with the optimal bounds C and D then*

$$(2.1) \qquad \frac{1}{B} + (\gamma(\psi))^2 \le C \le \left(\|\psi\| + \frac{1}{\sqrt{B}} \right)^2,$$

$$(2.2) \qquad \frac{1}{A} \le D \le \frac{1}{A} + \|\psi\|^2,$$

where $\psi \in B(\mathcal{H}, l^2)$ is associated with G given by Proposition 1.1.

Proof. First, we show the lower bound in 2.1,

$$
\begin{aligned}
C &= \inf_{\|f\|=1} \langle S_G f, f \rangle = \inf_{\|f\|=1} \langle S_F^{-1} f, f \rangle + \|\psi f\|^2 \\
&\ge \inf_{\|f\|=1} \langle S_F^{-1} f, f \rangle + \inf_{\|f\|=1} \|\psi f\|^2 \\
&= \frac{1}{B} + (\gamma(\psi))^2.
\end{aligned}
$$

Also, for each $f \in \mathcal{H}$,

$$
\begin{aligned}
\sum_{i \in I} |\langle f, g_i \rangle|^2 &= \sum_{i \in I} |\langle f, S_F^{-1} f_i + \psi^* \delta_i \rangle|^2 \\
&\leq \sum_{i \in I} |\langle f, S_F^{-1} f_i \rangle|^2 + \sum_{i \in I} |\langle f, \psi^* \delta_i \rangle|^2 + 2 \sum_{i \in I} |\langle f, S_F^{-1} f_i \rangle| |\langle f, \psi^* \delta_i \rangle| \\
&\leq \sum_{i \in I} |\langle f, S_F^{-1} f_i \rangle|^2 + \|\psi f\|^2 + 2\|\psi f\| \left(\sum_{i \in I} |\langle f, S_F^{-1} f_i \rangle|^2 \right)^{1/2}.
\end{aligned}
$$

Hence

$$
C = \inf_{\|f\| \leq 1} \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq \left(\|\psi\| + \frac{1}{\sqrt{B}} \right)^2.
$$

Moreover, for every dual frame $G = \{g_i\}_{i \in I}$ of F by Lemma 5.3.6 of [11] we have

$$
\sum_{i \in I} |\langle f, S_F^{-1} f_i \rangle|^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2,
$$

so applying the fact that the canonical dual $\{S_F^{-1} f_i\}_{i \in I}$ has optimal bounds $\dfrac{1}{B}$ and $\dfrac{1}{A}$, we obtain the lower bound in (2.2). Finally, for computing of the upper bound of D, by using of Proposition 1.1 there exists a bounded operator ψ such that $T_F \psi = 0$ and $g_i = S_F^{-1} f_i + \psi^* \delta_i$ for each $i \in I$. Therefore

$$
\begin{aligned}
\langle S_G f, f \rangle &= \left\langle \sum_{i \in I} \langle f, S_F^{-1} f_i + \psi^* \delta_i \rangle (S_F^{-1} f_i + \psi^* \delta_i), f \right\rangle \\
&= \left\langle S_F^{-1} \sum_{i \in I} \langle S_F^{-1} f, f_i \rangle f_i, f \right\rangle + \langle \psi^* \psi f, f \rangle \\
&= \langle S_F^{-1} f, f \rangle + \|\psi f\|^2,
\end{aligned}
$$

for each $f \in \mathcal{H}$. Now, we obtain

$$
\begin{aligned}
D &= \|S_G\| \\
&= \sup_{\|f\| \leq 1} |\langle S_G f, f \rangle| \\
&= \sup_{\|f\| \leq 1} | \langle S_F^{-1} f, f \rangle + \|\psi f\|^2 | \\
&\leq \frac{1}{A} + \|\psi\|^2.
\end{aligned}
$$

$\square$

2.5. Remark.
(i) However the operator ψ in Theorem 2.4 may not be injective in general, there are dual frames for which the corresponding operator ψ is injective and has closed range. For example, consider $\mathcal{H} = \mathbb{R}^3$ with the standard orthonormal basis $\{e_i\}_{i=1}^3$, also let $F = \{e_1, e_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3\}$. Then F is a frame of $\mathcal{H}$, with the canonical dual

$$
S_F^{-1} F = \{ \frac{1}{2} e_1, \frac{1}{2} e_1, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{2}} e_2, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3, \frac{1}{\sqrt{3}} e_3 \}.
$$

On the other hand, the sequense $G = \{e_1, 0, \sqrt{2} e_2, 0, \sqrt{3} e_3, 0, 0\}$ is a dual frame of F, and we have

$$
G - S_F^{-1} F = \{ \frac{1}{2} e_1, -\frac{1}{2} e_1, \frac{1}{\sqrt{2}} e_2, -\frac{1}{\sqrt{2}} e_2, \frac{2}{\sqrt{3}} e_3, -\frac{1}{\sqrt{3}} e_3, -\frac{1}{\sqrt{3}} e_3 \}.
$$

Since $G - S_F^{-1}F$ is a frame for $\mathcal{H}$, so $\psi = T_G^* - T_F^* S_F^{-1}$ is an injective operator and has closed range, i.e., $\gamma(\psi) > 0$.

(ii) There are some frames in Hilbert spaces for which there exist dual frames with arbitrary large optimal bounds. For example, let $\{e_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. Then $F = \{e_1, 0, e_2, 0, e_3, 0, ...\}$ is a Parseval frame for $\mathcal{H}$ and the sequence $G = \{e_1, ne_1, e_2, ne_2, e_3, ne_3, ...\}$ is a dual of F, for each $n \in \mathbb{N}$. Moreover,

$$\sum_{i \in I} |\langle f, g_i \rangle|^2 = (1 + n^2)\|f\|^2 \to \infty, \quad (n \to \infty).$$

The following proposition, for a frame $F = \{f_i\}_{i \in I}$, characterizes all operators $U \in B(\mathcal{H})$, where $\{Uf_i\}_{i \in I}$ is a Parseval frame. As a particular case, when $U = S_F^{-1/2}$ we obtain the well-known Parseval frame $\{S_F^{-1/2} f_i\}_{i \in I}$. The proof is obtained by a straightforward calculation.

2.6. Proposition. *Let $\{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$ with the operator frame S, and $U \in B(\mathcal{H})$. Then $\{Uf_i\}_{i \in I}$ is a Parseval frame if and only if $US_F U^* = I_{\mathcal{H}}$.*

In the end of this section we are going to construct a family of Parseval frames from an alternate dual. In fact let $\{g_i\}_{i \in I}$ be a dual frame of $\{f_i\}_{i \in I}$. Take

$$\psi = T_G^* - T_F^* S_F^{-1}.$$

Then

$$S_G = S_F^{-1} + \psi^*\psi.$$

Hence $\{(S_F^{-1} + \psi^*\psi)^{-1/2} g_i\}_{i \in I}$ is a Parseval frame. By replacing ψ with ψS_F, we can obtain the following family of Parseval frames as,

$$\left\{ (S_F^{-1} + S_F^n \psi^* \psi S_F^n)^{-1/2} (S_F^{-1} f_i + S_F^n \psi^* \delta_i) \right\}_{i \in I}, \quad (n \in \mathbb{N}).$$

3. Fusion frames

In this section, we first briefly recall the basic definitions and notations of fusion frames. Then we give some results on dual fusion frames. Throughout this section, π_V denotes the orthogonal projection from $\mathcal{H}$ onto a closed subspace V.

3.1. Definition. Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i \in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. Then $\{(W_i, \omega_i)\}_{i \in I}$ is called a *fusion frame* for $\mathcal{H}$ if there exist the constants $0 < A \leq B < \infty$ such that

(3.1) $$A\|f\|^2 \leq \sum_{i \in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}).$$

The constants A and B are called *fusion frame bounds*. If we only have the upper bound in (3.1) we call $\{(W_i, \omega_i)\}_{i \in I}$, a *Bessel fusion sequence*. A fusion frame is called *tight*, if $A = B$, and *Parseval* if $A = B = 1$. Also if $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i, \omega_i)\}_{i \in I}$ is called ω-*uniform*. A fusion frame $\{(W_i, \omega_i)\}_{i \in I}$ is said to be an *orthonormal fusion basis* if $\mathcal{H} = \bigoplus_{i \in I} W_i$. Recall that for each sequence $\{W_i\}_{i \in I}$ of closed subspaces in $\mathcal{H}$, the space

$$\sum_{i \in I} \oplus W_i = \left\{ \{f_i\}_{i \in I} : f_i \in W_i, \sum_{i \in I} \|f_i\|^2 < \infty \right\},$$

with the inner product $\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle$ is a Hilbert space. For a Bessel fusion sequence $\{(W_i, \omega_i)\}_{i \in I}$ of $\mathcal{H}$, the *synthesis operator* $T_W : \sum_{i \in I} \oplus W_i \to \mathcal{H}$ is

defined by

$$T_W(\{f_i\}_{i\in I}) = \sum_{i\in I} \omega_i f_i, \qquad \left(\{f_i\} \in \sum_{i\in I} \oplus W_i\right).$$

Its adjoint operator $T_W^* : \mathcal{H} \to \sum_{i\in I} \oplus W_i$, which is called the *analysis operator*, is obtained by $T_W^* f = \{\omega_i \pi_{W_i} f\}_{i\in I}$. If $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame, the *fusion frame operator* $S_W : \mathcal{H} \to \mathcal{H}$ defined by $S_W f = T_W T_W^* f = \sum_{i\in I} \omega_i^2 \pi_{W_i} f$ is a bounded, invertible as well as positive. A connection between local and global properties is given in the next result, see [7].

3.2. Theorem. *For each $i \in I$, let W_i be a closed subspace of $\mathcal{H}$ and $\omega_i > 0$. Also let $\{f_{i,j}\}_{j\in J_i}$ be a frame for W_i with frame bounds α_i and β_i such that*

(3.2) $$0 < \alpha = inf_{i\in I}\alpha_i \le \beta = sup_{i\in I}\beta_i < \infty.$$

Then the following conditions are equivalent.
(i) $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame of $\mathcal{H}$ with bounds C and D.
(ii) $\{\omega_i f_{i,j}\}_{i\in I, j\in J_i}$ is a frame of $\mathcal{H}$ with bounds αC and βD, .

For every fusion frame as $\{(W_i, \omega_i)\}_{i\in I}$, there exist frames $\{f_{i,j}\}_{j\in J_i}$ for W_i, such that satisfy (3.2), [7]. These frames are called the *local frames* of $\{(W_i, \omega_i)\}_{i\in I}$.

A Bessel fusion sequence $\{(V_i, \nu_i)\}_{i\in I}$ is called a *dual* of $\{(W_i, \omega_i)\}_{i\in I}$ if [13]

(3.3) $$f = \sum_{i\in I} \omega_i \nu_i \pi_{V_i} S_W^{-1} \pi_{W_i} f, \qquad (f \in \mathcal{H}).$$

Every fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$ has a dual as $\{(S_W^{-1} W_i, \omega_i)\}_{i\in I}$, so called the *canonical dual*. Also, it is not difficult to see that a Bessel fusion sequence $\{(V_i, \upsilon_i)\}_{i\in I}$ is a dual of fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ if and only if $T_V \phi_{vw} T_W^* = I_{\mathcal{H}}$, where the bounded operator $\phi_{vw} : \sum_{i\in I} \bigoplus W_i \to \sum_{i\in I} \bigoplus V_i$ is given by

(3.4) $$\phi_{vw}(\{f_i\}_{i\in I}) = \{\pi_{V_i} S_W^{-1} f_i\}_{i\in I}.$$

In [15], it is proved that for a fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$ with the frame operator S_W, a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i\in I}$ that $V_i = S_W^{-1} W_i \oplus U_i$, is dual of W in which U_i is a closed subspace of $\mathcal{H}$ for all $i \in I$. Although, there is no complete characterization of dual fusion frames.

3.1. Opposite relations between duals of frames and fusion frames.
In the sequel, we present some discussions on duals of fusion frames. In particular, we show that unlike ordinary frames, there is no one-to-one correspondence between duals of a fusion frame and the bounded left inverses of its analysis operator. In the following proposition, we investigate a relation between dual fusion frames and some their local frames.

3.3. Proposition. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\{e_j\}_{j\in J}$ be an orthonormal basis for $\mathcal{H}$. Then a Bessel sequence $V = \{(V_i, \upsilon_i)\}_{i\in I}$ is a dual of W if and only if $G = \{\upsilon_i \pi_{V_i} e_j\}_{i\in I, j\in J}$ is a dual frame of $F = \{\omega_i \pi_{W_i} S_W^{-1} e_j\}_{i\in I, j\in J}$. In particular, W is also a dual of V if $S_W = S_V$.*

Proof. The result is obtained from the following equalities

$$\sum_{i\in I, j\in J} \langle f, \omega_i \pi_{W_i} S_W^{-1} e_j \rangle \upsilon_i \pi_{V_i} e_j = \sum_{i\in I, j\in J} \omega_i \upsilon_i \pi_{V_i} \langle S_W^{-1} \pi_{W_i} f, e_j \rangle e_j$$

$$= \sum_{i\in I} \omega_i \upsilon_i \pi_{V_i} S_W^{-1} \pi_{W_i} f.$$

Moreover, since F is also a dual of G, we have

$$
\begin{aligned}
f &= \sum_{i \in I, j \in J} \langle f, v_i \pi_{V_i} e_j \rangle \omega_i \pi_{W_i} S_W^{-1} e_j \\
&= \sum_{i \in I, j \in J} \omega_i v_i \pi_{W_i} S_W^{-1} \langle \pi_{V_i} f, e_j \rangle e_j \\
&= \sum_{i \in I} \omega_i v_i \pi_{W_i} S_V^{-1} \pi_{V_i} f = T_W \varphi_{w,v} T_V^* f,
\end{aligned}
$$

for each $f \in \mathcal{H}$. This means that W is also a dual of V. $\qquad \square$

It is worthwhile to mention that two fusion frames are not dual of each other in general. As an easy consequence of Proposition 3.3, the result holds if fusion frames are Parseval.

Now, we are going to discuss on some differences between fusion frames and ordinary frames. First, let $\{(W_i, \omega_i)\}_{i \in I}$ be a fusion frame of $\mathcal{H}$ and $T \in B(\mathcal{H})$ be an invertible operator. It is well known that $\{(TW_i, \omega_i)\}_{i \in I}$ is also a fusion frame of $\mathcal{H}$, see [8]. However, in fusion frames unlike ordinary frames a surjective operator $T \in B(\mathcal{H})$ may not preserve Besselian property. The next example shows this fact.

3.4. Example. Let $\{e_i\}_{n \in I}$ be an orthonormal basis of $\mathcal{H}$ and $W_i = span\{e_i\}$, for all $i \in I$. Clearly $\{(W_i, 1)\}$ is an orthonormal fusion basis for $\mathcal{H}$. Define

$$
Te_i = \begin{cases} \dfrac{1}{m} e_1 & i = 2m - 1, \\ e_m & i = 2m. \end{cases}
$$

Then the mapping T can be extended to a bounded and surjective linear operator on $\mathcal{H}$, i.e $T \in B(\mathcal{H})$. Also

$$
TW_i = \begin{cases} span\{e_1\} & i = 2m - 1, \\ span\{e_m\} & i = 2m. \end{cases}
$$

Hence, we can easily see that for $f = e_1$

$$
\sum_{i=1}^n \|\pi_{TW_i} f\|^2 \to \infty, \quad (n \to \infty),
$$

i.e. $\{(TW_i, 1)\}$ is not a Bessel fusion sequence.

See [17] for more examples. Also, the construction of Parseval fusion frames, which have a key role in fusion frame theory is different from ordinary frames. More precisely, let $\{(W_i, \omega_i)\}_{i \in I}$ be a fusion frame with bounds A and B, we can see that $\{(S_W^{-1/2} W_i, \omega_i)\}_{i \in I}$ is not a Parseval fusion frame, in general. Moreover, there exists some fusion frames for $\mathcal{H}$ such that for every invertible operator $U \in B(\mathcal{H})$, $\{UW_i\}_{i \in I}$ fails to be a Parseval fusion frame, see Example 7.6 of [17].

Now, we show that Proposition 1.1 for fusion frames fails. Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a fusion frame with a dual as $\{(V_i, v_i)\}_{i \in I}$. Then there exists an operator $\Psi \in B(\mathcal{H}, \sum_{i \in I} \bigoplus W_i)$ such that $T_W \Psi = 0$. In fact there exists a bounded operator ϕ_{vw}, given by (3.4), such that $T_V \phi_{vw} T_W^* = I_{\mathcal{H}}$. Take

$$
(3.5) \qquad\qquad \Psi = \phi_{vw}^* T_V^* - T_W^* S_W^{-1}.
$$

Then it is easy to see that $\Psi \in B(\mathcal{H}, \sum_{i \in I} \bigoplus W_i)$ and $T_W \Psi = 0$. However, the converse is not true. Indeed, consider $W_1 = span\{(1, 1, 0)\}$ and $W_2 = \{0\} \times \mathbb{R}^2$, then $W = \{W_i\}_{i=1}^2$ is a 1-uniform fusion frame for $\mathbb{R}^3$. Now let $T_W \Psi = 0$, then for every $f = (a, b, c) \in \mathbb{R}^3$ we have $\Psi^* T_W^* f = 0$ so $\Psi^* \{((a+b)/2, (a+b)/2, 0), (0, b, c)\} = 0$. This implies that $\Psi^* \{f_1, f_2\} = 0$, for every $\{f_1, f_2\} \in \sum_{i=1}^2 \bigoplus W_i$. Hence the only

operator $\Psi \in B(\mathcal{H}, \sum_{i=1}^{2} \bigoplus W_i)$ such that $T_W \Psi = 0$ is $\Psi = 0$. While, there exist more than one dual for W, such as $V = \{S_W^{-1} W_1 \oplus span\{(0, 1/2, 1)\}, S_W^{-1} W_2\}$ and $Z = \{S_W^{-1} W_1, S_W^{-1} W_2 \oplus span\{(1, 0, 3/2)\}\}$, which are 1-uniform alternate duals of W. Hence, unlike discrete frames, the corresponding assertion in Proposition 1.1 for fusion frames fails. This also shows that there is no one-to-one correspondence between duals of a fusion frame and bounded left inverses of T_W^*. However, we show that for the mapping $\Psi \in B(H, \sum_{i \in I} \bigoplus W_i)$ such that $T_W \Psi = 0$ we can construct a dual for $F = \{\omega_i f_{i,j}\}_{i \in I, j \in J_i}$, where $\{f_{i,j}\}_{j \in J_i}$ is a local frame for W_i for each $i \in I$. Define $\psi_F : \mathcal{H} \to l^2$ by $\psi_F f = \{c_{i,j}\}_{i \in I, j \in J_i}$ where $\{\Psi f\}_{i \in I} = \{\sum_{j \in J_i} c_{i,j} f_{i,j}\}_{i \in I}$ and $\{c_{i,j}\}_{j \in J_i}$ is a unique sequence whose elements represent the components of Ψf in W_i with minimal l^2-norm, see Lemma 5.3.6 of [11]. The mapping ψ_F is well defined since $\{c_{i,j}\}_{j \in J_i}$ is unique. Thus

$$T_F \psi_F f = \sum_{i \in I, j \in J_i} \omega_i c_{i,j} f_{i,j} = \sum_{i \in I} \omega_i (\Psi f)_i = T_W \Psi f = 0.$$

Therefore, we obtain a dual frame $\{g_{i,j}\}_{i \in I, j \in J_i}$ for $\{\omega_i f_{i,j}\}_{i \in I, j \in J_i}$ by Proposition 1.1. Hence, characterization of all dual fusion frames is an intricate problem.

3.2. Characterization of duals of some fusion frames. In the following, we try to characterize duals of some fusion frames in Hilbert spaces. First, we need to the following lemma.

3.5. Lemma. [9] *Let W_1 and W_2 be closed subspaces of $\mathcal{H}$, and $\omega_1, \omega_2 > 0$. The following conditions are equivalent;*
(i) *$W = \{(W_i, \omega_i)\}_{i=1}^{2}$ is a Parseval fusion frame for $\mathcal{H}$.*
(ii) *Either $W_1 = W_2 = \mathcal{H}$ and $\omega_1^2 + \omega_2^2 = 1$ or $W_1 \perp W_2$ and $\omega_1 = \omega_2 = 1$*

3.6. Theorem. *Let $W = \{(W_i, \omega_i)\}_{i=1}^{2}$ be a Parseval fusion frame and $V = \{(V_i, v_i)\}_{i=1}^{2}$ be a dual of W. Then one of the following conditions are hold:*
(i) *$V_1 = V_2 = \mathcal{H}$ and $\omega_1 v_1 + \omega_2 v_2 = 1$.*
(ii) *$V_1 \perp V_2$ and $\omega_1 v_1 = \omega_2 v_2 = 1$.*
(iii) *$V_1 \supseteq W_1$, $V_2 \supseteq W_2$ and $v_1 = v_2 = 1$.*

Proof. Since W is a Parseval fusion frame, therefore by Lemma 3.5 either we have $W_1 = W_2 = \mathcal{H}$ and $\omega_1^2 + \omega_2^2 = 1$ or we have $W_1 \perp W_2$ and $\omega_1 = \omega_2 = 1$. Now if $W_1 = W_2 = \mathcal{H}$, and $\omega_1^2 + \omega_2^2 = 1$, then easily we can see that $(\{V_i, \sqrt{\omega_i v_i}\}_{i=1}^{2})$ is a Parseval fusion frame and so we have either (i) or (ii). On the other hand, let $W_1 \perp W_2$, and $\omega_1 = \omega_2 = 1$. In this case, for every $f \in W_1$ we have $v_1 \pi_{V_1} f = f$ and so $W_1 \subseteq V_1$ and $v_1 = 1$, similarly $W_2 \subseteq V_2$ and $v_2 = 1$, i.e., (iii) holds. $\square$

3.7. Theorem. *Suppose that $W = \{(W_i, \omega_i)\}_{i=1}^{\infty}$ is a fusion frame for $\mathcal{H}$ such that $W_1 \subseteq W_2 \subseteq W_3 \subseteq ...$, then a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i=1}^{\infty}$ is a dual of W if and only if $V_i \supseteq S_W^{-1} W_i$, for all i.*

Proof. Let $V = \{(V_i, \omega_i)\}_{i=1}^{\infty}$ be a dual of W. If $S_W^{-1} W_1$ is not a subspace of V_1, then there exists $f_0 \in W_1$ such that $S_W^{-1} f_0 \in S_W^{-1} W_1 \setminus V_1$ and so $\|\pi_{V_1} S_W^{-1} f_0\| < \|S_W^{-1} f_0\|$. Also

$$\sum_{i=1}^{\infty} \omega_i^2 \pi_{V_i} S_W^{-1} \pi_{W_i} f_0 = f_0 = \sum_{i=1}^{\infty} \omega_i^2 S_W^{-1} \pi_{W_i} f_0.$$

Therefore,

$$\sum_{i=1}^{\infty} \omega_i^2 \|\pi_{V_i} S_W^{-1} f_0\|^2 = \sum_{i=1}^{\infty} \omega_i^2 \|S_W^{-1} f_0\|^2,$$

which is a contradiction. Now let $S_W^{-1}W_1 \subseteq V_1,\ \dots\ ,\ S_W^{-1}W_{j-1} \subseteq V_{j-1}$ we show that $S_W^{-1}W_j \subseteq V_j$. Otherwise, as the above there exists an element $f_0 \in W_j$ such that $S_W^{-1}f_0 \in S_W^{-1}W_j \setminus V_j$. So we can write

$$\sum_{i=j}^{\infty} \omega_i^2 \pi_{V_i} S_W^{-1} f_0 = \sum_{i=j}^{\infty} \omega_i^2 S_W^{-1} f_0.$$

Hence $\sum_{i=1}^{\infty} \omega_i^2 \|\pi_{V_i} S_W^{-1} f_0\|^2 = \sum_{i=1}^{\infty} \omega_i^2 \|S_W^{-1} f_0\|^2$ which is a contradiction, therefore $S_W^{-1} W_i \subseteq V_i$ for all i.

$\square$

It is worth noticing that, if $W = \{(W_i, \omega_i)\}_{i=1}^{n}$ is a fusion frame for $\mathcal{H}$ such that $W_1 \supseteq W_2 \supseteq \dots \supseteq W_n$, then similar to the proof of Theorem 3.7 we can see that a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i=1}^{n}$ is a dual of W if and only if $V_i \supseteq S_W^{-1} W_i$, for all $1 \leq i \leq n$. Also, this condition that the weights of dual V is equal to weights of fusion frame W is necessary. For example, consider $\mathcal{H} = \mathbb{C}^3$, $W_i = \mathbb{C}^3$ and $\omega_i = 1/\sqrt{3}$, for each $1 \leq i \leq 3$. Clearly $W = \{(W_i, \omega_i)\}_{i=1}^{3}$ is a Parseval fusion frame with a dual as $V = \{(V_i, \nu_i)\}_{i=1}^{3}$, where $V_1 = \mathbb{C}^3$ and $V_2 = V_3 = \{0\}$ and $\upsilon_i = \sqrt{3}$ for each $1 \leq i \leq 3$.

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56

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