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Volterra type operator on the convex functions

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Abstract

In this paper we study the Volterra type operator I_g on convex functions. Furthermore, some new properties for convex, starlike and spirallike functions of complex order are discussed.

Keywords: Integral operators; convex functions of complex order, starlike functions of complex order; spirallike functions of type λ with complex order; Schwarzian derivative, Schwarzian norm, composition operator; Volterra type operator.

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1. Introduction

The convolution or Hadamard product of two power series functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.2}$$

which are analytic in $\mathbb{D}$. If $p(z) \in \mathcal{P}$ satisfies $\Re\{p(z)\} > 0$ $(z \in \mathbb{D})$, then we say that $p(z)$ is the Carathéodory function, (see [2]).

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If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}),$$

for some α $(0 \leqslant \alpha < 1)$, then $f(z)$ is said to be starlike of order α in $\mathbb{D}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{D}$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{D}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}),$$

for some α $(0 \leqslant \alpha < 1)$.

As usual, in the present investigation, we write

$$\mathcal{S}^* = \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} = \mathcal{K}(0).$$

Moreover, for some non-zero complex number b, we consider the subclasses $\mathcal{S}_b^*$ and $\mathcal{K}_b$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_b^* = \left\{ f(z) \in \mathcal{A} : \Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (z \in \mathbb{D})\right\}.$$

and

$$\mathcal{K}_b = \left\{ f(z) \in \mathcal{A} : \Re\left\{1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad (z \in \mathbb{D})\right\}.$$

Then we can see that

$$\mathcal{S}_{1-\alpha}^* = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha).$$

Let f be a function analytic and locally univalent in the unit disk $\mathbb{D}$. Let

$$S_f = (f''/f')' - 1/2(f''/f')^2$$

denote its Schwarzian derivative, and let

$$\|S_f(z)\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f|$$

denote its Schwarzian norm.

Recall first, that if f maps the disk conformally onto a convex region, then the function

$$g(z) = 1 + \frac{zf''(z)}{f'(z)}$$

has positive real part in $\mathbb{D}$. (see for instance [2]). Since $g(0) = 1$, this say that g is subordinate to the half-plan mapping $\mathcal{L}(z) = (1+z)/(1-z)$, so that $g(z) = \mathcal{L}(\varphi(z))$ for some Schwarz functions φ. In other word,

$$\frac{zf''(z)}{f'(z)} = \frac{1+\varphi(z)}{1-\varphi(z)} - 1 = \frac{2\varphi(z)}{1-\varphi(z)},$$

where φ is analytic and has the property $|\varphi(z)| \leqslant |z|$ in $\mathbb{D}$. With the notation $\psi(z) = \varphi(z)/z$ this gives the representation

$$\frac{f''(z)}{f'(z)} = \frac{2\psi(z)}{1 - z\psi(z)} \tag{1.3}$$

for the pre-Schwarzian, where ψ is analytic and satisfies $|\psi(z)| \leqslant 1$ in $\mathbb{D}$. Straight forward calculation now gives the Schwarzian of f in the form

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{2\psi'(z)}{(1 - z\psi(z))^2}.$$

But $|\psi'(z)| \leqslant \left(1-|\psi(z)|^2\right)/(1-|z|^2)$ by the invariant form of the Schwarz lemma, so we conclude that

$$|S_f(z)| \leqslant 2\frac{1-|\psi(z)|^2}{(1-|z|^2)\,(1-|z\psi(z)|)^2} \leqslant \frac{2}{(1-|z|^2)^2}. \tag{1.4}$$

In other words, the inequality (1.4) says that the Schwarzian norm $\|S_f(z)\|$ of convex mapping is no large than 2. The bound is best possible since the parallel strip mapping

$$L(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

has Schwarzian $S_L(z) = 2(1-z^2)^{-2}$. Nehari [6] also stated that $|S_f(z)| < 2$ if the convex mapping f is bounded.
For $g \in \mathcal{H}(\mathbb{D})$, the integral operator I_g

$$I_g h(z) = \int_0^z h'(\xi)g(\xi)\mathrm{d}\xi, \qquad (h \in \mathcal{H}(\mathbb{D}))$$

was introduced in [12] and is called the Volterra type operator.
In this paper we introduce some new subclasses of $\mathcal{H}(\mathbb{D})$ as follow

$$P(\beta, b) := \left\{p(z) \in \mathcal{P} : \Re\left\{\frac{1}{b}\left(\frac{zp'(z)}{p(z)}\right)\right\} \geq \beta\right\}$$

and

$$P'(\beta, b) := \left\{p(z) \in \mathcal{P} : \Re\left\{\frac{1}{b}\left(\frac{zp'(z)}{p(z)}\right)\right\} \leq \beta\right\}$$

for some real number β and non-zero complex number b.

Examples:

$$p(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots \in P(-1/2, 1),$$

$$p(z) = \frac{1}{1+z} = 1 - z + z^2 - z^3 \pm \cdots \in P(-1/2, 1),$$

$$p(z) = 1 + z \in P'(1/2, 1), \qquad p(z) = 1 - z \in P'(1/2, 1).$$

Moreover, for some non-zero complex numbers b and real λ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$ we define the subclass $\hat{\mathcal{S}}_\alpha(b)$ of $\mathcal{A}$ as follow:

$$\hat{\mathcal{S}}_\alpha(b) := \left\{f(z) \in \mathcal{A} : \Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right)\right\} > 0, \quad (z \in \mathbb{D})\right\}.$$

If a function $f(z)$ belong to the class $\hat{\mathcal{S}}_\alpha(b)$, we say that $f(z)$ is spirallike of type λ with the complex order b, $b \neq 0$.

In this paper, we get some properties for functions in $\mathcal{S}^*_b$, $\mathcal{K}_b$ and $\hat{\mathcal{S}}_\alpha(b)$. Also we study the Volterra type operator I_g on K and $\mathcal{K}_b$. Furthermore, we get necessary and sufficient condition such that $I_g h(\mathbb{D})$ is bounded, moreover we obtain sufficient condition such that $|S_f(z)| < 2$.

2. Preliminaries

In this section we prove some properties of functions in $\mathcal{S}_b^*$, $\mathcal{K}_b$ and $\hat{\mathcal{S}}_\alpha(b)$ as the form of the lemmas. Also, Here we quote some auxiliary results which will be used in the proofs of the main results in this paper.

2.1. Lemma. (1) *Let $b \in \mathbb{C}$, $b \neq 0$ and $\beta \geqslant 0$. If $g \in P(\beta, b)$ then I_g is an operator on $\mathcal{K}_b$.*
(2) *Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$ and $0 \leqslant \alpha + \beta < 1$. If $g \in P(\beta, 1)$, then I_g is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}(\alpha+\beta)$.*

Proof. Let $h \in \mathcal{K}_b$, then

$$\begin{aligned}
\Re\left\{1+\frac{1}{b}\left(\frac{z\left(I_g h\right)''(z)}{\left(I_g h\right)'(z)}\right)\right\} &= \Re\left\{1+\frac{1}{b}\left(\frac{zh''(z)g(z)+zh'(z)g'(z)}{h'(z)g(z)}\right)\right\} \\
&= \Re\left\{1+\frac{1}{b}\left(\frac{zh''(z)}{h'(z)}+\frac{zg'(z)}{g(z)}\right)\right\} \qquad (2.1)\\
&= \Re\left\{1+\frac{1}{b}\left(\frac{zh''(z)}{h'(z)}\right)\right\}+\Re\left\{\frac{1}{b}\left(\frac{zg'(z)}{g(z)}\right)\right\}.
\end{aligned}$$

By hypothesis of this Lemma and (2.1), we have

$$\Re\left\{1+\frac{1}{b}\left(\frac{z\left(I_g h\right)''(z)}{\left(I_g h\right)'(z)}\right)\right\} > 0,$$

therefore $I_g h \in \mathcal{K}_b$ for each $h \in \mathcal{K}_b$.
The proof of (2) is similar to the proof of (1).

□

2.2. Lemma. *The function f is convex of complex order b ($b \neq 0$) in $\mathbb{D}$ if and only if*

$$f' * \frac{z\left(\dfrac{1+x}{2b}\right)+1-z}{(1-z)^2} \neq 0, \qquad (z \in \mathbb{D},\ |x|=1).$$

Proof. The function f is convex of complex order b if and only if

$$\Re\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad (z \in \mathbb{D}). \tag{2.2}$$

It is easy to see that (2.2) is equivalent to

$$1+\frac{1}{b}\left(\frac{(zf'(z))'}{f'(z)}-1\right) \neq \frac{x-1}{x+1}, \qquad (z \in \mathbb{D},\ |x|=1,\ x \neq -1).$$

Which simplifies to

$$(1+x)\left(zf'(z)\right)' + (2b-x-1)f'(z) \neq 0,$$

we have

$$(1+x)\left(zf'(z)\right)' = f'(z) * \frac{1+x}{(1-z)^2}$$

and

$$(2b-x-1)f'(z) = f'(z) * \frac{2b-x-1}{1-z}.$$

So that

$$\begin{aligned}
&(1+x)\left(zf'(z)\right)' + (2b-x-1)f'(z) = \\
&f'(z) * \frac{1+x}{(1-z)^2} + f'(z) * \frac{2b-x-1}{1-z} = \\
&f'(z) * \frac{(1+x)+(1-z)(2b-x-1)}{(1-z)^2} = \\
&f'(z) * \frac{(1+x-2b)z+2b}{(1-z)^2} \neq 0.
\end{aligned}$$

Since $b \neq 0$ therefore we get

$$f' * \frac{z\left(\dfrac{1+x-2b}{2b}\right)+1}{(1-z)^2} \neq 0, \qquad (z \in \mathbb{D},\ |x|=1,\ x \neq -1).$$

The case $x = 1$ in the convolution condition is equivalent to stating $f' \neq 0$ for each $z \in \mathbb{D}$, which is a necessary condition for univalence, and the proof of this lemma is complete. □

Note: If we put $b = 1-\alpha$ in the above lemma, we get the Theorem 1 in [11].

2.3. Lemma. *Let $b \in \mathbb{C}$, $b \neq 0$, $\lambda \in (-\pi/2, \pi/2)$ and $\beta = e^{-i\lambda}\cos\lambda$. Then $f \in \hat{\mathcal{S}}_\alpha(b)$ if and only if there is $g \in \mathcal{S}_b^*$ such that*

$$f(z) = z\left(\frac{g(z)}{z}\right)^{\beta}. \tag{2.3}$$

The branch of the power function is chosen such that $\left(\dfrac{g(z)}{z}\right)^{\beta}\Big|_{z=0} = 1.$

Proof. First assume $f \in \hat{\mathcal{S}}_\alpha(b)$. Clearly (2.3) is equivalent to

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{\cos\lambda}}, \quad (z \in \mathbb{D}).$$

we choose the branch of the power function such that $\left(\dfrac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{\cos\lambda}}\Big|_{z=0} = 1$. A simple computation yields the relation

$$1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right) = (1 + i\tan\lambda)\left(1 + \frac{zf'(z)}{bf(z)}\right) - \frac{1+i\tan\lambda}{b} - i\tan\lambda.$$

Therefore

$$\Re\left\{1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right)\right\} = \frac{1}{\cos\lambda}\Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right)\right)\right\}.$$

Since $f \in \hat{\mathcal{S}}_\alpha(b)$, consequently g is starlike of complex order b.

Conversely, if $g \in \mathcal{S}_b^*$, then in view of the above relation and the fact that $\lambda \in (-\pi/2, \pi/2)$, one deduces that

$$\Re\left\{e^{i\lambda}\left(1 + \frac{1}{b}\left(\frac{zg'(z)}{g(z)} - 1\right)\right)\right\} > 0 \quad (z \in \mathbb{D}).$$

Thus, f is spirallike of type λ with complex order b. This completes the proof. □

2.4. Lemma. *Let $b \in \mathbb{C}$ and $b \neq 0$. Then we have the following equality:*

$$\mathcal{S}_b^* = \left\{zh'(z) : h \in \mathcal{K}_b\right\}.$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_b^*$. It's obvious that

$$f(z) = z\left(1 + \sum_{n=2}^{\infty} a_n z^{n-1}\right).$$

We put

$$h'(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1},$$

therefore

$$h(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n.$$

Then $f(z) = zh'(z)$. Applying that $h(z) \in \mathcal{K}_b$ if and only if $zh'(z) \in \mathcal{S}_b^*$, we deduce, $h(z) \in \mathcal{K}_b$ and the proof of this lemma is complete. □

2.5. Lemma. *For the function $f(z) \in \mathcal{A}$, it follows that*

$$f(z) \in \mathcal{S}_b^* \Longleftrightarrow z\left(\frac{f(z)}{z}\right)^{1/b} \in \mathcal{S}^*.$$

Proof. Let $f(z)$ be a starlike of complex order b. By using of Lemma 2.4, there is $h \in \mathcal{K}_b$ such that $f(z) = zh'(z)$. Since $h \in \mathcal{K}_b$, then by using Theorem 1.2 in [3] we have $z\left(h'(z)\right)^{\frac{1}{b}} \in \mathcal{S}^*$. □

(Another proof for this theorem: we set $F(z) = z\left(\frac{f(z)}{z}\right)^{1/b}$. Therefore

$$\Re\left\{\frac{zF'(z)}{F(z)}\right\} = \Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0,$$

and then $f \in \mathcal{S}_b^*$.)

2.6. Lemma. *Let $b \in \mathbb{C}$ and $b \neq 0$, also let $\lambda \in (-\pi/2, \pi/2)$ and $\beta = e^{-i\lambda}\cos\lambda$. Then $f \in \hat{\mathcal{S}}_\lambda(b)$ if and only if there is $h \in \mathcal{K}_b$ such that*

$$f(z) = z\left(h'(z)\right).$$

Proof. By using Lemmas 2.3 and 2.4, the proof of this lemma is obvious. □

2.7. Lemma. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a starlike function of complex order b $(b \neq 0)$. Then $|a_2| \leqslant 2|b|$. This bound is sharp. Equality is attained for $f_b(z) = \frac{z}{(1-z)^{2b}}$.*

Proof. Let $f(z) \in \mathcal{S}_b^*$, by using Lemma 2.5, we have $g(z) = z\left(\frac{f(z)}{z}\right)^{1/b} \in \mathcal{S}^*$. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, therefore $b_2 = \frac{1}{b} a_2$. So, by using Bieberbach theorem we have $|a_2| = |b||b_2| \leqslant 2|b|$. Since

$$\frac{z}{(1-z)^{2b}} = z + \frac{1}{2b-1}\sum_{n=2}^{\infty} \frac{\prod_{j=1}^{n}\left(j + 2(b-1)\right)}{(n-1)!} z^n,$$

Then it is obvious that equality is attained for f_b. □

2.8. Lemma. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a spirallike function of type λ with complex order b $(b \neq 0)$. Then $|a_2| \leqslant 2|b|\cos\lambda$.*

Proof. Let $f(z) \in \hat{\mathcal{S}}_\lambda(b)$, by using Lemma 2.3, there is $g(z) = z + b_2z^2 + b_3z^3 + \cdots \in \mathcal{S}_b^*$ such that $f(z) = z\left(\frac{g(z)}{z}\right)^\beta$, $(\beta = e^{-i\lambda}\cos\lambda)$. Then $a_2 = b_2\beta$, so $|a_2| = |b_2|\cos\lambda$. By using Lemma 2.6 we have $|a_2| \leqslant 2|b|\cos\lambda$ and the proof of this Lemma is complete. □

Note: since $\hat{\mathcal{S}}_\lambda(1) = \hat{\mathcal{S}}_\lambda$. then we get the Corollary 2.4.12 in [4] as a result of the above lemma.

3. Volterra type operator on $\mathcal{K}$ and $\mathcal{K}(\alpha)$

3.1. Theorem. *Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$, $\alpha + \beta \geqslant 0$ and let $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. Then the image $(I_gh)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)\left|2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}\right| < 1.$$

Proof. By using of the Lemma 2.1 part (2), we have $I_gh \in \mathcal{K}$. In (1.3), we put $f = I_gh$. Then there is an analytic function ψ such that

$$\frac{(I_gh)''(z)}{(I_gh)'(z)} = \frac{2\psi(z)}{1 - z\psi(z)},$$

therefore

$$\psi(z) = \frac{(I_gh)''(z)/(I_gh)'(z)}{2 + z(I_gh)''(z)/(I_gh)'(z)} = \frac{g'(z)/g(z) + h''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)},$$

we have

$$\begin{aligned}\frac{1-|z|}{|1 - z\psi(z)|} &= \frac{1-|z|}{\left|1 - \frac{zg'(z)/g(z) + zh''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)}\right|} \\ &= \frac{1-|z|}{\frac{2}{2+zg'(z)/g(z)+zh''(z)/h'(z)}} \qquad (3.1) \\ &= \frac{1}{2}(1-|z|)\left|2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}\right|.\end{aligned}$$

By Theorem 2 in [1] we get the image $(I_gh)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z|\to 1}\frac{1-|z|}{|1 - z\psi(z)|} < \frac{1}{2}.$$

By (3.1), the proof is complete.

□

3.2. Theorem. *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$, such that $(h'g)(z) * \frac{(x-1)z}{2(1-z)^2} \neq 0$. Then the image $(I_gh)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)|2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}| < 1.$$

Proof. The proof of this Theorem is similar to the proof of Theorem 3.1. Only in this proof we using of the Lemma 2.2. By Lemma 2.2, we have $I_gh \in \mathcal{K}$, the remain proof is analogue the above proof. □

By using of the Lemma 2.1 part (2), we have the following corollary.

Corollary 3.3 Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$, $\alpha + \beta \geqslant 0$. If $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$, then

$$\left|S_{I_g h}\right| \leqslant 2.$$

By using of the Lemma 2.2, we have the following corollary.

Corollary 3.4. Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$. If $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$ then

$$\left|S_{I_g h}\right| \leqslant 2.$$

By using of the Theorem 3.1, we have the following corollary.

Corollary 3.5. Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$ and $\alpha + \beta \geqslant 0$. Let $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. If

$$\limsup_{|z|\to 1}(1-|z|)\left|2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}\right| < 1,$$

then

$$\left|S_{I_g h}\right| < 2.$$

By using of the Theorem 3.2, we have the following corollary.

Corollary 3.6. Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$, such that $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. If

$$\limsup_{|z|\to 1}(1-|z|)\left|2 + \frac{zh''(z)}{h'(z)} + \frac{zg'(z)}{g(z)}\right| < 1,$$

then

$$\left|S_{I_g h}\right| < 2.$$

4. Product of Composition operators and Volterra-type operator on $\mathcal{K}$ and $\mathcal{K}(\alpha)$

Products of composition operators and integral type operators have been recently introduced by S. Li and S. Stevic in [7, 8, 9, 10]. Here we shall be interested in studding the product of composition operators and Volterra-type integral operators, which are defined by

$$(C_\sigma I_g h)(z) = \int_0^{\sigma(z)} h'(\xi) g(\xi) \mathrm{d}\xi \qquad (z \in \mathbb{D})$$

on subclasses of $\mathcal{H}(\mathbb{D})$, where $g \in \mathcal{H}(\mathbb{D})$ and σ is an analytic self-map of the unit disk. In this section we assume that $\sigma(z)$ be the Möbius automorphism $\sigma(z) = \dfrac{z + z_0}{1 + \bar{z}_0 z}$ on $\mathbb{D}$, where, z_0 be the fixed point in $\mathbb{D}$.

4.1. Theorem. *Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$ and $0 \leqslant \alpha + \beta < 1$. If $g \in P(\beta, 1)$, then $C_\sigma I_g$ is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}$.*

Proof. By hypothesis of this theorem and by using of the Lemma 2.1 part (2), it is obvious that I_g is an operator from $\mathcal{K}(\alpha)$ to $\mathcal{K}$. Therefore $I_g h$ is a convex map. We let $f = I_g h$. By using of the Lemma 1 in [4] we have $fo\sigma$ is a convex mapping of $\mathbb{D}$. We know

$$fo\sigma(z) = f(\sigma(z)) = (I_g h)(\sigma(z)) = \int_0^{\sigma(z)} h'(\xi) g(\xi) \mathrm{d}\xi = (C_\sigma I_g h)(z)$$

and the proof is complete. $\square$

4.2. Theorem. *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$. If*

$$\left(h'g\right)(z) * \frac{(x-1)z}{2(1-z)^2} \neq 0,$$

then $C_\sigma I_g h \in \mathcal{K}$.

Proof. By using of the Lemma 2.2 and Lemma 1 in [1] the proof is obvious. □

4.3. Theorem. *Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$ and $\alpha + \beta \geqslant 0$. Let $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. Then the image $(C_\sigma I_g h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)\Big|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\Big| < \frac{1}{2},$$

where $A(z) = \dfrac{g'\left(\sigma(z)\right)}{g\left(\sigma(z)\right)} + \dfrac{h''\left(\sigma(z)\right)}{h'\left(\sigma(z)\right)}$.

Proof. By using of the Theorem 4.1, we have $(C_\sigma I_g h) \in \mathcal{K}$. In the proof of the Theorem 3.1 we saw that

$$\psi(z) = \frac{g'(z)/g(z) + h''(z)/h'(z)}{2 + zg'(z)/g(z) + zh''(z)/h'(z)}.$$

By using of the Lemma 1 in [1], there is an analytic function λ such that

$$\frac{(C_\sigma I_g h)''}{(C_\sigma I_g h)'} = \frac{2\lambda(z)}{1 - z\lambda(z)},$$

where

$$\lambda(z) = \frac{\psi\left(\sigma(z)\right) - \bar{z}_0}{1 - z_0\psi\left(\sigma(z)\right)}.$$

We have

$$z\lambda(z) = \frac{\dfrac{zA(z)}{2+\sigma(z)A(z)} - \bar{z}_0 z}{1 - \dfrac{z_0 A(z)}{2+\sigma(z)A(z)}} = \frac{(z - \bar{z}_0 z\sigma(z))\,A(z) - 2\bar{z}_0 z}{(\sigma(z)-z_0)\,A(z)+2},$$

hence

$$\begin{aligned}\frac{1-|z|}{|1-z\lambda(z)|} &= \frac{1-|z|}{\Big|1 - \dfrac{(z-\bar{z}_0 z\sigma(z))\,A(z) - 2\bar{z}_0 z}{(\sigma(z)-z_0)\,A(z)+2}\Big|} \\ &= (1-|z|)\Big|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\Big|., \end{aligned} \tag{4.1}$$

By using of the Theorem 2 in [1]] we get the image $(C_\sigma I_g h)(\mathbb{D})$ is bounded if and only if

$$\limsup_{|z|\to 1}\frac{1-|z|}{|1-z\lambda(z)|} < \frac{1}{2}.$$

By (4.1), the proof is complete. □

4.4. Theorem. *Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$, such that $(h'g)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. Then the image $(C_\sigma I_g h)(\mathbb{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1}(1-|z|)\Big|\frac{(\sigma(z)-z_0)\,A(z)+2}{(\sigma(z)-z_0+\bar{z}_0 z\sigma(z)-z)\,A(z)+2\bar{z}_0 z+2}\Big| < \frac{1}{2},$$

where $A(z) = \dfrac{g'\left(\sigma(z)\right)}{g\left(\sigma(z)\right)} + \dfrac{h''\left(\sigma(z)\right)}{h'\left(\sigma(z)\right)}$.

Proof. The proof of this theorem is similar to the proof of Theorem 4.3. Only, in this proof we using of the Theorem 4.2. By using of the Theorem 4.2, we have $C_\sigma I_g h \in \mathcal{K}$, the remain proof of this theorem is analogue the above proof. $\square$

By using of the Theorem 4.1, we have the following corollary:

Corollary 4.5. Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$, $\alpha + \beta \geqslant 0$. If $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$, then

$$\left|S_{C_\sigma I_g h}\right| \leqslant 2.$$

By using of the Theorem 4.2, we have the following corollary:

Corollary 4.6. Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$. If

$$\left(h' g\right)(z) * \frac{(x-1)z}{2(1-z)^2} \neq 0,$$

then

$$\left|S_{C_\sigma I_g h}\right| \leqslant 2.$$

By using of the Theorem 4.3, we have the following corollary.

Corollary 4.7. Let $\beta \in \mathbb{R}$, $0 \leqslant \alpha < 1$ and $\alpha + \beta \geqslant 0$. Let $g \in P(\beta, 1)$ and $h \in \mathcal{K}(\alpha)$. If

$$\limsup_{|z| \to 1} (1 - |z|) \left| \frac{(\sigma(z) - z_0) A(z) + 2}{(\sigma(z) - z_0 + \bar{z}_0 z \sigma(z) - z) A(z) + 2\bar{z}_0 z + 2} \right| < \frac{1}{2},$$

then

$$\left|S_{C_\sigma I_g h}\right| < 2.$$

Where $A(z) = \dfrac{g'(\sigma(z))}{g(\sigma(z))} + \dfrac{h''(\sigma(z))}{h'(\sigma(z))}$.

By using of the Theorem 4.3, we have the following corollary.

Corollary 4.8. Let $g \in \mathcal{P}$ and $h \in \mathcal{A}$, such that $\left(h' g\right)(z) * \dfrac{(x-1)z}{2(1-z)^2} \neq 0$. If

$$\limsup_{|z| \to 1} (1 - |z|) \left| \frac{(\sigma(z) - z_0) A(z) + 2}{(\sigma(z) - z_0 + \bar{z}_0 z \sigma(z) - z) A(z) + 2\bar{z}_0 z + 2} \right| < \frac{1}{2},$$

then

$$\left|S_{C_\sigma I_g h}\right| < 2.$$

Where $A(z) = \dfrac{g'(\sigma(z))}{g(\sigma(z))} + \dfrac{h''(\sigma(z))}{h'(\sigma(z))}$.

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