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Ideal Rothberger spaces

Aslı Güldürdek*

Abstract

In this paper we introduce Rothberger with respect to an ideal and examine some basic properties. We also investigate its relation to weak Rothberger and almost Rothberger properties.

Keywords: Ideal Topological Space; Rothberger Space; Almost Rothberger; Weakly Rothberger.

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1. Introduction and Preliminaries

The concept of ideal captured the attention of general topologists starting with the works of Kuratowski [3], and Vaidyanathaswamy [10]. After that the notion of ideal topological space and applications have been deeply examined. An ideal $\mathbf{I}$ on a set X is a nonempty collection of subsets of X, which is closed under the subset and finite union operations. We denote a topological space (X, τ) with an ideal $\mathbf{I}$ defined on X by $(X, \tau, \mathbf{I})$. If (X, τ) is a topological space then it is clear that the collections $\mathcal{N}(\tau)$ of nowhere dense subsets, and $\mathcal{M}(\tau)$ of first category(meager) subsets are both ideals on X. An ideal $\mathbf{I}$ on (X, τ) is said to be τ-codense if $\mathbf{I} \cap \tau = \{\emptyset\}$. Note that $\mathcal{N}(\tau)$ is a τ-codense ideal.

On the other hand the properties arising from coverings are of great interest in topology. One of these covering properties is Rothberger property. A space X is Rothberger ([5], [6]) if for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} U_n$. In [7], Scheepers generalized this concept by calling a space X almost Rothberger, if for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} \overline{U}_n$. Besides, in [1] Daniels defines weakly Rothberger property. According to this definition, a space X is weakly Rothberger if for every sequence

*Department of Mathematics, American University of the Middle East, KUWAIT
Email: asli.guldurdek@aum.edu.kw

$\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$. It is easy to conclude that every Rothberger space is Lindelöf, and any separable space is weakly Rothberger. The relationship among these properties is summarized by the following in [8]

$$\textit{Rothberger} \Longrightarrow \textit{almost Rothberger} \Longrightarrow \textit{weakly Rothberger.}$$

Also Song [8] presents counterexamples which show that converses do not hold.

In this work we first introduce the ideal Rothberger space definition, and investigate some basic properties. After that we give characterizations of these weaker forms of Rothberger property by using ideals. Next we deal with the preservation of these by subspaces and functions of ideal topological spaces. Finally, we examine the Rothberger property on the new topology introduced via the ideal. Throughout this work, (X, τ), $\mathcal{P}(X)$, $\overline{A}$, and A^o will denote the topological space, power set, closure, and interior, respectively.

2. Main Results

Let us begin with the definition of ideal Rothberger space.

2.1. Definition. Let $(X, \tau, \mathbf{I})$ be an ideal topological space. (X, τ) is said to be $\mathbf{I}$-Rothberger or Rothberger with respect to $\mathbf{I}$, if for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X \setminus \bigcup_{n\in\mathbb{N}} U_n \in \mathbf{I}$.

2.2. Lemma. *If (X, τ) is a Rothberger space, then (X, τ) is $\mathbf{I}$-Rothberger for every ideal $\mathbf{I}$ on X.*

Proof. Since being Rothberger implies the existence of a sequence $\{U_n \mid n \in \mathbb{N}\}$ of open sets for every sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X, such that $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} U_n$, it is clear that this sequence satisfies:

$X \setminus \bigcup_{n\in\mathbb{N}} U_n = \emptyset \in \mathbf{I}$ for any ideal $\mathbf{I}$. □

The following example shows that being $\mathbf{I}$-Rothberger space does not imply being Rothberger space, in general.

2.3. Example. [9] Let X be the set of real numbers $\mathbb{R}$, and let us define the topology τ on X as follows: We declare each rational number open, and the basic open neighborhoods of an irrational number x, are of form: $U_n = \{x_i\} \cup \{x\}$ where (x_i) is a sequence of rational numbers converging to x. Since the set of irrationals $\mathbb{R} \setminus \mathbb{Q}$ is an uncountable dense subset, X is not Lindelöf, hence not Rothberger. On the other hand the set of rationals $\mathbb{Q}$ is a countable dense subset which means X is separable. So, X is weakly Rothberger. Note that $\mathcal{N}(\tau) = \mathcal{P}(\mathbb{R} \setminus \mathbb{Q})$ the power set of irrationals, and (X, τ) is $\mathcal{N}(\tau)$-Rothberger.

If $\mathbf{I} = \{\emptyset\}$ then we have: (X, τ) is Rothberger if and only if (X, τ) is $\{\emptyset\}$-Rothberger.

2.4. Theorem. *For a topological space (X, τ) the following are equivalent:*
1. (X, τ) *is weakly Rothberger,*
2. (X, τ) *is* $\mathcal{N}(\tau)$*-Rothberger.*

Proof. Suppose (X,τ) is weakly Rothberger and let $\{\mathcal{U}_n \mid n\in\mathbb{N}\}$ be a sequence of open covers of X. By assumption, there exists a sequence $\{U_n \mid n\in\mathbb{N}\}$ of open sets so that, $U_n\in\mathcal{U}_n$ for every $n\in\mathbb{N}$ and $X=\overline{\bigcup_{n\in\mathbb{N}} U_n}$. Then $X\setminus\bigcup_{n\in\mathbb{N}} U_n$ is a closed subset of X, and $(X\setminus\bigcup_{n\in\mathbb{N}} U_n)^o=\emptyset$. So, $X\setminus\bigcup_{n\in\mathbb{N}} U_n\in\mathcal{N}(\tau)$, implies (X,τ) being $\mathcal{N}(\tau)$-Rothberger.

On the other hand, suppose (X,τ) is $\mathcal{N}(\tau)$-Rothberger, and let $\{\mathcal{U}_n \mid n\in\mathbb{N}\}$ be a sequence of open covers of X. By definition, there exists a sequence $\{U_n \mid n\in\mathbb{N}\}$ of open sets so that, $U_n\in\mathcal{U}_n$ for every $n\in\mathbb{N}$ and $X\setminus\bigcup_{n\in\mathbb{N}} U_n$ is a nowhere dense subset of X. Then:
$(X\setminus\bigcup_{n\in\mathbb{N}} U_n)^o=\emptyset$. That is, $X\setminus\overline{\bigcup_{n\in\mathbb{N}} U_n}=\emptyset$. □

2.5. Theorem. (X,τ) *is weakly Rothberger if and only if it is* **I**-*Rothberger with respect to some codense ideal* **I**.

Proof. If (X,τ) is weakly Rothberger, then by previous theorem, it is (X,τ) is $\mathcal{N}(\tau)$-Rothberger and $\mathcal{N}(\tau)$ is a τ-codense ideal.

Next, we suppose that (X,τ) is **I**-Rothberger with respect to some codense ideal **I**. Let $\{\mathcal{U}_n \mid n\in\mathbb{N}\}$ be a sequence of open covers of X. By our assumption, there exists a sequence $\{U_n \mid n\in\mathbb{N}\}$ of open sets so that, $U_n\in\mathcal{U}_n$ for every $n\in\mathbb{N}$ and $X\setminus\bigcup_{n\in\mathbb{N}} U_n\in\mathbf{I}$. So, $X\setminus\bigcup_{n\in\mathbb{N}} U_n$ has an empty interior. Then, $X=\overline{\bigcup_{n\in\mathbb{N}} U_n}$. □

2.6. Theorem. *If* (X,τ) *is almost Rothberger, then it is* $\mathcal{M}(\tau)$-*Rothberger.*

Proof. Let $\{\mathcal{U}_n \mid n\in\mathbb{N}\}$ be a sequence of open covers of X. By hypothesis, there exists a sequence $\{U_n \mid n\in\mathbb{N}\}$ of open sets so that, $U_n\in\mathcal{U}_n$ for every $n\in\mathbb{N}$ and $X\setminus\bigcup_{n\in\mathbb{N}}\overline{U_n}=\emptyset$. Since $X\setminus\bigcup_{n\in\mathbb{N}} U_n\subset\bigcup_{n\in\mathbb{N}}(\overline{U_n}\setminus U_n)$ and $\overline{U_n}\setminus U_n\in\mathcal{N}(\tau)$ for every $n\in\mathbb{N}$, we conclude $X\setminus\bigcup_{n\in\mathbb{N}} U_n\in\mathcal{M}(\tau)$. □

2.7. Lemma. *Let* **I** *and* **J** *be given two ideals on* X, *with* $\mathbf{I}\subset\mathbf{J}$. *If* (X,τ) *is* **I**-*Rothberger, then it is* **J**-*Rothberger.*

Proof. Clear. □

2.8. Remark. One can deduce that any separable space which is not almost Rothberger proves that: being $\mathcal{M}(\tau)$-Rothberger does not imply being almost Rothberger. Indeed any separable space is weakly Rothberger and by *theorem* 2.4, $\mathcal{N}(\tau)$-Rothberger and the lemma above, $\mathcal{M}(\tau)$-Rothberger.

3. Subspaces and Functions

Let $(X,\tau,\mathbf{I})$ be a given ideal topological space, and let A be a nonempty subset of X. Then $\mathbf{I_A}=\{I\cap A \mid I\in\mathbf{I}\}$ is an ideal on A, and $(A,\tau_A,\mathbf{I_A})$ is an ideal topological space[2].

The following theorem states that, being an **I**-Rothberger space is hereditary on closed subspaces, where on contrary, being Rothberger space is not.

3.1. Theorem. *Let (X, τ) be* **I**-*Rothberger, and A be a closed subset. Then (A, τ_A) is* $\mathbf{I_A}$-*Rothberger.*

Proof. Let $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ be a sequence of open covers of A by open sets from τ_A. So, for every $n \in \mathbb{N}$, we can think of $\mathcal{U}_n$ as $\mathcal{U}_n = \{U \cap A \mid U \in \mathcal{V}_n \subset \tau\}$. Then $\{\mathcal{V}'_n \mid n \in \mathbb{N}\}$ is a sequence of open covers of X, where $\mathcal{V}'_n = \mathcal{V}_n \cup \{X \setminus A\}$. By assumption, there exists a sequence $\{V_n \mid n \in \mathbb{N}\}$ of open sets so that, $V_n \in \mathcal{V}'_n$ for every $n \in \mathbb{N}$ and $X \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathbf{I}$.

<u>**Case1**</u>: $V_n = X \setminus A$, for every $n \in \mathbb{N}$.
In this case, $X \setminus \bigcup_{n\in\mathbb{N}} V_n = X \setminus (X \setminus A) = A \in \mathbf{I}$, and $A \in \mathbf{I_A}$. So for any sequence $\{U^n \cap A \mid n \in \mathbb{N}\}$ such that $U^n \cap A \in \mathcal{U}_n$ for every $n \in \mathbb{N}$, we conclude that: $A \setminus \bigcup_{n\in\mathbb{N}} (U^n \cap A) \subset A$.

<u>**Case2**</u>: $V_n = U^n$ for every $n \in \mathbb{N}$.
We have $X \setminus \bigcup_{n\in\mathbb{N}} V_n = X \setminus \bigcup_{n\in\mathbb{N}} U^n \in \mathbf{I}$. Then
$A \setminus \bigcup_{n\in\mathbb{N}} (U^n \cap A) = A \cap (X \setminus \bigcup_{n\in\mathbb{N}} U^n) \in \mathbf{I_A}$.

<u>**Case3**</u>: For at least one $k \in \mathbb{N}$, suppose $V_k = X \setminus A$, and for at least one $j \in \mathbb{N}$, suppose $V_j = U^j$. Then, $X \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathbf{I}$. Now we define a sequence $\{V'_n \mid n \in \mathbb{N}\}$ of open subsets of A as follows:

$$V'_m = \begin{cases} U^m = V_m, & \text{if } V_m \neq X \setminus A, \\ U_m \in \mathcal{V}_m (for\ any\ U_m \in \mathcal{V}_m), & \text{if } V_m = X \setminus A. \end{cases}$$

<u>Claim</u>: $A \setminus \bigcup_{n\in\mathbb{N}} (V'_n \cap A) \in \mathbf{I_A}$.

Proof of claim: Let $x \in X$ be a point in $A \setminus \bigcup_{n\in\mathbb{N}} (V'_n \cap A)$. Then $x \in A$, $x \notin U^n$, and $x \notin U_m$ for every n, m. So, $x \in A \setminus \bigcup_{n\in\mathbb{N}} V_n = X \setminus \bigcup_{n\in\mathbb{N}} V_n$. That is, $A \setminus \bigcup_{n\in\mathbb{N}} (V'_n \cap A) \subset A \setminus \bigcup_{n\in\mathbb{N}} V_n$. Since $A \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathbf{I}$ we conclude: $A \setminus \bigcup_{n\in\mathbb{N}} (V'_n \cap A) \in \mathbf{I_A}$.

□

As a corollary, we can give an alternative proof to following already known result.

3.2. Corollary. [8] *If $A \subset X$ is a clopen(both open and closed) subset of a weakly Rothberger space (X, τ), then (A, τ_A) is also weakly Rothberger.*

Proof. By previous theorem, (A, τ_A) is $\mathcal{N}(\tau_A)$-Rothberger. Since A is open, $\mathcal{N}(\tau_A) = \{I \cap A \mid I \in \mathcal{N}(\tau)\}$ is a τ_A-codense ideal of A. Also, if $(I \cap A)$ has a nonempty interior with respect to τ_A, then $(I \cap A)^o \neq \emptyset$. So we finish this by $theorem 4$. □

When we consider the preservation under continuous functions, the following setting has been suggested in [2]:
Let $f : X \to Y$ be a function, and **I**, **J** be ideals on X, and Y respectively. Then:
$f(\mathbf{I}) = \{f(I) \mid I \in \mathbf{I}\}$ is an ideal on Y, and $f^{-1}(\mathbf{J}) = \{f^{-1}(J) \mid J \in \mathbf{J}\}$ is an ideal on Y.

3.3. Theorem. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a continuous surjection and* $\mathbf{I}$ *be an ideal on* X*. If* (X, τ_1) *is* $\mathbf{I}$*-Rothberger, then* (Y, τ_2) *is* $f(\mathbf{I})$*-Rothberger.*

Proof. Suppose (X, τ_1) is an $\mathbf{I}$-Rothberger space, and $f : X \to Y$ is a continuous surjection. Let $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ be a sequence of open covers of Y. Then $\{f^{-1}(\mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a sequence of open covers of X. By definition there exists a sequence of open sets $\{f^{-1}(U_n) \mid n \in \mathbb{N}\}$ such that $f^{-1}(U_n) \in f^{-1}(\mathcal{U}_n)$ for every $n \in \mathbb{N}$, and $X \setminus \bigcup_{n\in\mathbb{N}} f^{-1}(U_n) \in \mathbf{I}$. Then it is clear that: $f[X \setminus \bigcup_{n\in\mathbb{N}} f^{-1}(U_n)] \in f(\mathbf{I})$ □

Note that if $\mathbf{I} = \{\emptyset\}$, then we get the known result stating that being a Rothberger space is preserved under continuous surjections.

3.4. Theorem. *Let $f : X \to (Y, \mu, \mathbf{J})$ be a surjection, (Y, μ) be* $\mathbf{J}$*-Rothberger, and the topology $f^{-1}(\mu)$ on X be constructed by μ and f as follows: $U \in f^{-1}(\mu)$ if and only if $U = f^{-1}(G)$ for a set $G \in \mu$. Under these assumptions, if (Y, μ) is* $\mathbf{J}$*-Rothberger, then $(X, f^{-1}(\mu))$ is* $f^{-1}(\mathbf{J})$*-Rothberger.*

Proof. Let $\mathcal{U}_n = f^{-1}(\mathcal{V}_n) = \{f^{-1}(V) \mid V \in \mathcal{V}_n\}$ be an open cover of X for every $n \in \mathbb{N}$. Then $\{\mathcal{V}_n \mid n \in \mathbb{N}\}$ is a sequence of open covers of Y, and by definition there exists a sequence $\{V_n \mid n \in \mathbb{N}\}$ such that $V_n \in \mathcal{V}_n$ for every $n \in \mathbb{N}$, and $Y \setminus \bigcup_{n\in\mathbb{N}} V_n \in \mathbf{J}$. We conclude:
$f^{-1}[Y \setminus \bigcup_{n\in\mathbb{N}} V_n] = X \setminus \bigcup_{n\in\mathbb{N}} f^{-1}(V_n) \in f^{-1}(\mathbf{J})$. □

3.5. Lemma. [2] *If $f : X \to (Y, \mu, \mathbf{J})$ is a surjection and* $\mathbf{J}$ *is a μ-codense ideal, then* $f^{-1}(\mathbf{J})$ *is $f^{-1}(\mu)$-codense.*

3.6. Corollary. *Let $f : X \to (Y, \mu)$ be a surjection and let $f^{-1}(\mu)$ be the topology on X defined as in theorem* 3.4*. Then:*
1. *$(X, f^{-1}(\mu))$ is Rothberger, if (Y, μ) is Rothberger.*
2. *$(X, f^{-1}(\mu))$ is weakly Rothberger, if (Y, μ) is weakly Rothberger.*

Proof. 1. If we take $\mathbf{J} = \{\emptyset\}$, we are done.
2. If (Y, μ) is weakly Rothberger, then there exists a μ-codense ideal $\mathbf{J}$ so that (Y, μ) is $\mathbf{J}$-Rothberger. Then $f^{-1}(\mathbf{J})$ is $f^{-1}(\mu)$-codense and $(X, f^{-1}(\mu))$ is $f^{-1}(\mathbf{J})$-Rothberger. This finally implies that $(X, f^{-1}(\mu))$ is weakly Rothberger. □

Recall that a map $f : X \to Y$ is closed(open) if it preserves closed(open) subsets, perfect if it is continuous,closed and $f^{-1}(y) \subset X$ is compact for every $y \in Y$.

3.7. Theorem. *Let $f : (X, \tau) \to (Y, \mu)$ be a perfect open surjection, and (Y, μ) be* $\mathbf{J}$*-Rothberger. Then (X, τ) is* $f^{-1}(\mathbf{J})$*-Rothberger.*

Proof. Let $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ be a sequence of open covers of X. Then for each $y \in Y$ and $n \in \mathbb{N}$ there is a finite subcollection $\mathcal{U}'_n = \{U^n_{y_i} \mid i = 1, 2, ..., k_y\}$ of $\mathcal{U}_n$ so that $\mathcal{U}'_n$ covers $f^{-1}(y)$. Let $U^n(y) = \bigcup_{i=1}^{k_y} U^n_{y_i}$. It is easy to see that each $U^n(y)$ is an open subset of X, each $f(X \setminus U^n(y))$ is a closed subset of Y, and $y \notin f(X \setminus U^n(y))$. If we define an open neighborhood of $y \in Y$ as $V^n(y) = Y \setminus f(X \setminus U^n(y))$ then $f^{-1}(V^n(y)) \subset U^n(y)$. The collection $\mathcal{V}_n = \{V^n(y) \mid y \in Y\}$ is an open cover, and the sequence $\{\mathcal{V}_n \mid n \in \mathbb{N}\}$ is a sequence of open covers of Y. Since (Y, μ) is $\mathbf{J}$-Rothberger, there exists a sequence

$\{V^n(y_n) \mid n \in \mathbb{N}\}$ such that $Y \setminus \bigcup_{n\in\mathbb{N}} V^n(y_n) \in \mathbf{J}$. Since $f^{-1}(V^n(y_n)) \subset U^n(y_n)$ we have: $X \setminus \bigcup_{n\in\mathbb{N}} U^n(y_n) \subset f^{-1}(Y \setminus \bigcup_{n\in\mathbb{N}} V^n(y_n))$, and this implies: $X \setminus \bigcup_{n\in\mathbb{N}} U^n(y_n) \in f^{-1}(\mathbf{J})$. □

3.8. Lemma. [2] *Let (X,τ) be a topological space, $(Y,\mu,\mathbf{J})$ be an ideal topological space, and $f : X \to Y$ be an open surjection. If $\mathbf{J}$ is μ-codense, then $f^{-1}(\mathbf{J})$ is τ-codense.*

The following is also a known result, which can also be proved by *theorem*4 and the previous lemma.

3.9. Corollary. *Let $f : (X,\tau) \to (Y,\mu)$ be a perfect open surjection.*
i− If (Y,μ) is Rothberger, then (X,τ) is Rothberger.
ii− If (Y,μ) is weakly Rothberger, then (X,τ) is weakly Rothberger.

4. The Space (X,τ^*)

We start this section by recalling the well-known definitions of local function and the space (X,τ^*).

4.1. Definition. [3] Let (X,τ) be a topological space, and $\mathbf{I}$ be an ideal on X. Then the local function $A^*(\mathbf{I},\tau)$ of $A \subset X$ is defined as following:
$A^*(\mathbf{I},\tau) = \{x \in X \mid A \cap U \notin \mathbf{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$.

One can see that, $(.)^* : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfies the conditions to make $c^*(A) = A \cup A^*(\mathbf{I},\tau)$ a Kuratowski closure operator.

4.2. Definition. [4] Let (X,τ) be a topological space, and $\mathbf{I}$ be an ideal on X. Since $c^*(A) = A \cup A^*(\mathbf{I},\tau)$ is a Kuratowski closure operator, generates a topology $\tau^*(\mathbf{I},\tau)$ on X. If there is no chance of confusion this topological space is denoted as (X,τ^*).

It is easy to show that $\tau \subset \tau^*$. Also note that, if $\mathbf{I} = \{\emptyset\}$ then $\tau = \tau^*$, and if $\mathbf{I} = \mathcal{P}(X)$ then $A^*(\mathcal{P}(X),\tau) = \emptyset$ which implies $\tau^* = \mathcal{P}(X)$.

Now we will examine the Rothberger property for the space (X,τ^*).

4.3. Theorem. *Let (X,τ) be a topological space, $\mathbf{I}$ be an ideal on X, and let (X,τ^*) be a Rothberger space. Then (X,τ) is also Rothberger and hence $\mathbf{I}$-Rothberger.*

Proof. The proof is clear via the fact that $\tau \subset \tau^*$. □

On the other hand the following example shows that, being $\mathbf{I}$-Rothberger for (X,τ) does not imply being Rothberger for (X,τ^*).

4.4. Example. Let X be the set of real numbers $\mathbb{R}$, the topology τ be the usual topology of $\mathbb{R}$, and the ideal $\mathbf{I}$ be the power set $\mathcal{P}(\mathbb{R})$. It is clear that, $(\mathbb{R},\tau)$ is $\mathcal{P}(\mathbb{R})$-Rothberger. On the other hand τ^* on $\mathbb{R}$ is the discrete topology which is not even Lindelöf, so not Rothberger.

4.5. Question. What extra conditions we can add on $\mathbf{I}$, in order to have the converse of the previous theorem?

Now we will try to explain the situation for the weaker forms of Rothberger property.

4.6. Theorem. [8] *A space X is almost(weakly) Rothberger if anf only if for each sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of covers of X by regular open subsets, there exists a sequence $\{U_n \mid n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \overline{U}_n$ ($X = \overline{\bigcup_{n\in\mathbb{N}} U_n}$).*

Note that the previous theorem proves that being almost Rothberger, and weakly Rothberger properties are semiregular properties.

4.7. Lemma. [4] *Semiregular properties are shared by* (X, τ) *and* (X, τ^*) *if* $X = X^*$.

4.8. Corollary. *If* $X = X^*$*, then* (X, τ) *is almost(weakly) Rothberger if and only if* (X, τ^*) *is almost(weakly) Rothberger.*

5. Conclusion and Future Work

In this work, the Rothberger property and some of its weaker forms are examined in the case of ideal topological spaces and the notion of ideal Rothberger property is introduced. Some generalizations are made for ideal topological spaces. Besides these, a problem (4.5 Question) is suggested for a future work. If one comes up with some extra properties for the ideal **I**, to have the converse of (4.3 Theorem), this would suggest a direct transition between the Rothberger and **I**-Rothberger spaces.

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