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A graph associated to a fixed automorphism of a finite group

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Abstract

Let G be a finite group and $Aut(G)$ be the group of automorphisms of G. We associate a graph to a group G and fixed automorphism α of G denoted by Γ_G^α as follows. The vertex set of Γ_G^α is $G \setminus Z^\alpha(G)$ and two vertices $x, g \in G \setminus Z^\alpha(G)$ are adjacent if $[g, x]_\alpha \neq 1$ or $[x, g]_\alpha \neq 1$, where $[g, x]_\alpha = g^{-1}x^{-1}gx^\alpha$ and $Z^\alpha(G) = \{x \in G \mid [g, x]_\alpha = 1$ for all $g \in G\}$. In this paper, we state some basic properties of the graph, like connectivity, diameter, girth and Hamiltonian. Moreover, planarity and 1-planarity are also investigated here.

Keywords: Automorphism group, diameter , independent set, dominating set, planer, outer planar.

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1. Introduction

There are many papers on assigning a graph to a group and investigating the algebraic properties of the group through the associated graph. For instance, commuting graph [9], non-commuting graph [1], non-cyclic graph [3], non-normal graph [5], prime graph [12], power graph [7] and so on. In this paper, we are going to assign a new graph to a finite group G and an automorphism α in the automorphism group $Aut(G)$. Let G be a finite group and α is an arbitrary but fixed element in $Aut(G)$. We define a graph denoted by Γ_G^α as an undirected simple graph with vertex set consisting all elements of $G \setminus Z^\alpha(G)$ and two distinct vertices $g, x \in G \setminus Z^\alpha(G)$ are adjacent whenever $[g, x]_\alpha \neq 1$ or $[x, g]_\alpha \neq 1$, where $[g, x]_\alpha = g^{-1}x^{-1}gx^\alpha$ and $Z^\alpha(G)$ is the set of all element x in G such that $[g, x]_\alpha = 1$ for every $g \in G$. It is clear that if α is an identity automorphisms then $[x, g]_\alpha = [g, x]_\alpha$ and Γ_G^α is the know non-commuting graph. Moreover, if G is an

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abelian group, then Γ_G^α is a complete graph. Thus throughout the paper, we may always assume that α is non-identity automorphism and G is a finite non-abelian group. Since we removed $Z^\alpha(G)$ from G, so it would imply that Γ_G^α has no isolated vertex.
In section 2, we investigate about the degree of vertices, diameter, girth and suitable condition for the graph to be Hamiltonian. Section 3 is devoted to a determination of some numerical invariants of the graph. Planarity, outer planarity and 1-planarity of the graph are also considered in section 4.
In the rest of this section, we remind some necessary definitions in graph theory. We remind that the girth of a graph is the length of a shortest cycle contained in the graph. The distance between two vertices in a graph is the number of edges in a shortest path connecting them. The diameter of a graph is the greatest distance between any pair of vertices. In graph theory, a dominating set for a graph X with vertex set V and edge set E is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(X)$ is the number of vertices in a smallest dominating set for X. An independent set or stable set is a set of vertices in a graph such that no two of which are adjacent. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges crosses each other. Moreover, a 1-planar graph is a graph that can be drawn in the Euclidean plane in such a way that each edge has at most one crossing point, where it crosses a single additional edge. A Hamiltonian path is a path in a graph that visits each vertex exactly once a Hamiltonian cycle is a Hamiltonian path that is a cycle. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

2. Diameter and girth

Let G be a non-abelian finite group, and α an arbitrary but fixed element in $Aut(G)$. We denote α-conjugacy class containing x by x_α^G and it is clear that $x_\alpha^G = \{g^{-1}xg^\alpha \ : \ g \in G\}$. If $|x_\alpha^G| = 1$, then α is nothing but conjugation by x. Moreover a subgroup $C_G^\alpha(x) = \{g \in G \mid [x,g]_\alpha = 1\}$ which satisfies $|x_\alpha^G| = [G \ : \ C_G^\alpha(x)]$. We have α-center of the group G as $Z^\alpha(G) = \bigcap_{x\in G} C_G^\alpha(x) = \{g \in G \mid [x,g]_\alpha = 1 \text{ for all } x \in G\}$. It is easy to show that $C_G^\alpha(x)$ is subgroup of G and also $Z^\alpha(G)$ is normal subgroup of G. One can see that $Z^\alpha(G) = Z(G) \bigcap \text{Fix}(\alpha)$ where $\text{Fix}(\alpha) = \{x \in G \ : \ x^\alpha = x\}$. It is interesting to see that the number of generalized conjugacy classes is the number of ordinary conjugacy classes which are invariant under α and it is also equal to the number of irreducible characters which are invariant under α (see [4] for more details). By the above notations we may define subsets $R^\alpha(x) = \{g \in G \mid [g,x]_\alpha = 1\}$, $T^\alpha(G) = \bigcap_{x\in G} R^\alpha(x) = \{g \in G \mid [g,x]_\alpha = 1 \text{ for all } x \in G\}$ of group G, $C_G^\alpha(S) = \{g \in G \mid [s,g]_\alpha = 1 \text{ for all } s \in S\}$ and $R_G^\alpha(S) = \{g \in G \mid [g,s]_\alpha = 1 \text{ for all } s \in S\}$, where S is a subset of group G. First, let us remind the definition of graph as the following.

2.1. Definition. Let G be a non-abelian finite group, and α an arbitrary but fixed element in $Aut(G)$. We define a graph, denoted by Γ_G^α, such that vertex set of Γ_G^α is $G \setminus Z^\alpha(G)$ and two different vertices $x, g \in G \setminus Z^\alpha(G)$ are adjacent if $[g,x]_\alpha \neq 1$ or $[x,g]_\alpha \neq 1$.

In the following lemma, we give the degree of any vertex in Γ_G^α.

2.2. Lemma. *The degree of every vertex of the graph* Γ_G^α *is*

(i) *If* $x = x^\alpha$ *then* $\deg(x) = |G| - |C_G(x) \bigcap C_G^\alpha(x)|$,
(ii) *If* $x \neq x^\alpha$ *then* $\deg(x) = |G| - |Z^\alpha(G)| - |R^\alpha(x) \bigcap C_G^\alpha(x)| - 1$.

Proof. For the degree of any vertex x of the graph Γ_G^α we have two cases: (i) $x = x^\alpha$ and (ii) $x \neq x^\alpha$. In case (i),

$$\deg(x) = |G| - |Z^\alpha(G)| - |R^\alpha(x) \bigcap C_G^\alpha(x)| + |Z^\alpha(G) \bigcap R^\alpha(x)|.$$

Furtheremore, if $x = x^\alpha$, then $R^\alpha(x) = C_G(x)$ and therefore $\deg(x) = |G| - |C_G(x) \bigcap C_G^\alpha(x)|$. In case (ii) the degree of the vertex x is

$$|G| - |Z^\alpha(G)| - |R^\alpha(x) \bigcap C_G^\alpha(x)| + |Z^\alpha(G) \bigcap R^\alpha(x)| - 1.$$

In this case $R^\alpha(x) \bigcap Z^\alpha(G) = \emptyset$ which implies that $\deg(x) = |G| - |Z^\alpha(G)| - |R^\alpha(x) \bigcap C_G^\alpha(x)| - 1$. □

2.3. Lemma. *Let G be a non-abelian finite group. Then the Γ_G^α is Hamiltonian graph.*

Proof. Let G be a non-abelian finite group. We know that if $x = x^\alpha$ then $\deg(x) = |G| - |C_G(x) \bigcap C_G^\alpha(x)|$. Hence

$$\begin{aligned} |V(\Gamma_G^\alpha)| &= |G| - |Z^\alpha(G)| < |G| \\ &= 2|G| - |G| < 2|G| - 2|C_G(x) \bigcap C_G^\alpha(x)| \\ &= 2(|G| - |C_G(x) \bigcap C_G^\alpha(x)|) = 2\deg(x). \end{aligned}$$

Therefore $\deg(x) > \dfrac{|V(\Gamma_G^\alpha)|}{2}$.

If $x \neq x^\alpha$ then $\deg(x) = |G| - |Z^\alpha(G)| - |R^\alpha(x) \bigcap C_G^\alpha(x)| - 1$. We have

$$\begin{aligned} |V(\Gamma_G^\alpha)| &= |G| - |Z^\alpha(G)| < |G| = 2|G| - |G| \\ &\leqslant 2|G| - 2(|Z^\alpha(G)| + |R^\alpha(x) \bigcap C_G^\alpha(x)| + 1). \end{aligned}$$

Thus $\deg(x) > \dfrac{|V(\Gamma_G^\alpha)|}{2}$ and by Dirac's Theorem [6], the graph Γ_G^α is Hamiltonian. □

By the above lemma, we can see that Γ_G^α is always connected and also is not a star graph. Now in the following lemma, we determine of the diameter of Γ_G^α.

2.4. Lemma. *For any non-abelian group G,* $\mathrm{diam}(\Gamma_G^\alpha) = 2$.

Proof. First assume that there exists a vertex x in $Z(G) \setminus \mathrm{Fix}(\alpha)$ and y is an arbitrary vertex in $V(\Gamma_G^\alpha)$. Then $[y, x]_\alpha = y^{-1}x^{-1}yx^\alpha = x^{-1}x^\alpha \neq 1$, it implies that x is adjacent to y hence $d(x, y) = 1$. Secondly, assume that there exists x is an arbitrary vertex in $\mathrm{Fix}(\alpha) \setminus Z(G)$ and y is an arbitrary vertex in $V(\Gamma_G^\alpha)$. Suppose that $y \neq y^\alpha$, then we have two cases, (i) x and y commute and (ii) x does not commute with y. If case (i) occurs then $[x, y]_\alpha = [x, y]y^{-1}y^\alpha \neq 1$ thus x is adjacent to y. If case (ii) occurs then $[y, x]_\alpha = y^{-1}x^{-1}yx^\alpha = [y, x] \neq 1$. It implies that x is adjacent to y. Therefore $d(x, y) = 1$. Assume that $y = y^\alpha$, since α is not identity automorphisms in $Aut(G)$, so there exists $z \in G$ such that $z \neq z^\alpha$. Hence z is a vertex in $V(\Gamma_G^\alpha)$. By the similar method, we can see that z is adjacent to x and y. It implies that $d(x, y) = 2$. Finally assume that x and y are two non-adjacent vertices in $V(\Gamma_G^\alpha)$ such that $x, y \notin Z(G)$ and $x, y \notin \mathrm{Fix}(\alpha)$. We have $[x, xy] = [x, y] \neq 1$ hence $xy \notin Z(G)$. Thus xy is a vertex in Γ_G^α. We know that $[x, xy]_\alpha = [x, y]_\alpha [x, x]_\alpha^{y^\alpha}$. Since x is not adjacent to y so $[x, y]_\alpha = 1$ and $[y, x]_\alpha = 1$, hence $[x, xy]_\alpha = [x, x]_\alpha^{y^\alpha}$. Assume that $[x, x]_\alpha^{y^\alpha} = 1$, then $[x, x]_\alpha = 1$ it is a contradiction. Therefore x is adjacent to xy. By the similar method we can show that y is adjacent to xy. It implies that $d(x, y) = 2$. The proof is now completed. □

2.5. Lemma. *Let G be a non-abelian finite group. The girth of the graph Γ_G^α is at most 4.*

Proof. We have to show that Γ_G^α indeed has a cycle of length at most four. Firstly suppose that there exists a vertex x in $Z(G) \setminus \mathrm{Fix}(\alpha)$. Let y and z be two arbitrary vertices in $V(\Gamma_G^\alpha)$. If y is adjacent to z, then three elements x, y, z induce a cycle of length 3. If y is not adjacent to z, then we can see that y and z are adjacent to yz. Hence elements x, y, yz induce a cycle of length 3. Secondly, assume that x is an arbitrary vertex in $\mathrm{Fix}(\alpha) \setminus Z(G)$ and y, z are two vertex in $V(\Gamma_G^\alpha)$ such that $y \neq y^\alpha$ and $z \neq z^\alpha$. Then by Lemma 2.4. we have x is adjacent to y and z. If y is adjacent to z, then elements x, y, z induce a cycle of length 3. If y is not adjacent to z, then we can see that y and z are adjacent to yz. Hence elements x, y, z, yz induce a cycle of length 4. Finally, $Z^\alpha(G) = Z(G) = \mathrm{Fix}(\alpha)$. Since G is non-abelian group, there exists $x \in V(\Gamma_G^\alpha)$ such that $x^2 \neq 1$, so x^{-1} is vertex in Γ_G^α and therefore x is adjacent to x^{-1}. For y in $V(\Gamma_G^\alpha)$ that y is adjacent to x, it is easy to show that y is adjacent to x^{-1}. Thus elements x, y, x^{-1} induce a cycle of length 3. Therefore the girth of Γ_G^α is at most 4. □

3. Dominating set and independent set

Suppose that $[G : Z^\alpha(G)] = r$. We know that G can be partitioned into disjoint cosets of $Z^\alpha(G)$. Now, assume that $G \setminus Z^\alpha(G) = \bigcup_{i=1}^{r-1} g_i Z^\alpha(G)$.

3.1. Lemma. *By the above notation let $g \in G \setminus Z^\alpha(G)$, then $gZ^\alpha(G)$ is an independent set in Γ_G^α if and only if $g \in \mathrm{Fix}(\alpha)$.*

Proof. Assume that x and y two arbitrary elements in $gZ^\alpha(G)$ such that $x = gz_1$ and $y = gz_2$ where $z_1, z_2 \in Z^\alpha(G)$. Let x be not adjacent to y. We know that x is not adjacent to y if and only if $[x, y]_\alpha = 1$ and $[y, x]_\alpha = 1$. Hence $1 = [x, y]_\alpha = [gz_1, gz_2]_\alpha = [g, g]_\alpha [z_2, \alpha]$. It implies that $[g, g]_\alpha = [g, \alpha] = 1$, thus $g \in \mathrm{Fix}(\alpha)$. □

Let A be an independent set of $\{g_1, g_2, \ldots, g_{r-1}\} \bigcap \mathrm{Fix}(\alpha)$ and $|A| = k$, $B = \bigcup_{a \in A} aZ^\alpha(G)$. By the above lemma, we can see that $\alpha(\Gamma_G^\alpha) \geq |B| = k|Z^\alpha(G)|$ where $\alpha(\Gamma_G^\alpha)$ is the independence number of Γ_G^α. The following lemma states that every vertex of the coset $gZ^\alpha(G)$ is adjacent to every vertex of coset $xZ^\alpha(G)$ if and only if g is adjacent to x, where g and x are in $G \setminus Z^\alpha(G)$.

3.2. Lemma. *Let $gZ^\alpha(G)$ and $hZ^\alpha(G)$ be two arbitrary disjoint cosets of $G \setminus Z^\alpha(G)$, where $g, h \notin Z^\alpha(G)$. Then every element of $gZ^\alpha(G)$ is adjacent to every element of $hZ^\alpha(G)$ if and only if g is adjacent to h.*

Proof. Suppose that x is an arbitrary element in $gZ^\alpha(G)$ and y is an arbitrary element in $hZ^\alpha(G)$ such that $x = gz_1$ and $y = hz_2$ where $z_1, z_2 \in Z^\alpha(G)$. Let x is adjacent to y. We know that x is adjacent to y if and only if $[x, y]_\alpha \neq 1$ or $[y, x]_\alpha \neq 1$. Let $[x, y]_\alpha \neq 1$ then

$$1 \neq [x, y]_\alpha = [gz_1, hz_2]_\alpha = [g, h]_\alpha [z_2, \alpha] = [g, h]_\alpha.$$

Therefore g is adjacent to h. □

3.3. Lemma. *Let x be in $\mathrm{Fix}(\alpha)$. If $\{x\}$ is a dominating set of Γ_G^α, then $Z^\alpha(G) = 1$ and $x^2 = 1$.*

Proof. Suppose on the contrary that $Z^\alpha(G) \neq 1$. Then every element of the set $xZ^\alpha(G) - \{x\}$ is a vertex which is adjacent to x. This is a contradiction. Now, suppose that $x^2 \neq 1$, then $x^{-1} \in V(\Gamma_G^\alpha)$. We have $[x, x^{-1}]_\alpha = 1$, this is a contradiction. The proof is now complete. □

3.4. Lemma. *Let G be a finite group. Then subset S of $V(\Gamma_G^\alpha)$ is a dominating set of Γ_G^α if and only if $R^\alpha(S) \bigcap C_G^\alpha(S) \subseteq Z^\alpha(G) \cup S$.*

Proof. First assume that S is a dominating set of Γ_G^α and $g \in G$. If $g \notin Z^\alpha(G) \cup S$, then $g \notin S$. Since S is a dominating set, hence there exists $x \in S$ such that g is adjacent to x. Thus $[x,g]_\alpha \neq 1$ or $[g,x]_\alpha \neq 1$. If $[x,g]_\alpha \neq 1$ then $g \notin C_G^\alpha(S)$. Also if $[g,x]_\alpha \neq 1$, then $g \notin R^\alpha(S)$. Therefore $R^\alpha(S) \bigcap C_G^\alpha(S) \subseteq Z^\alpha(G) \cup S$. Conversely, suppose that $R^\alpha(S) \bigcap C_G^\alpha(S) \subseteq Z^\alpha(G) \cup S$. If there exists $g \in G$ such that $g \notin Z^\alpha(G) \cup S$, then either $g \notin C_G^\alpha(S)$ or $g \in C_G^\alpha(S)$ but $g \notin R^\alpha(S)$. In each case, there exists $x \in S$ such that $[x,g]_\alpha \neq 1$ or $[g,x]_\alpha \neq 1$. It implies that g is adjacent to x. □

3.5. Lemma. *Let G be a non-abelian group such that $G =< X >$. Then $X \setminus Z^\alpha(G)$ is a dominating set of Γ_G^α.*

Proof. Suppose that G is a non-abelian group such that $G =< X >$, then $Y = X \setminus Z^\alpha(G) \neq \emptyset$. We show that $R^\alpha(Y) \bigcap C_G^\alpha(Y) \subseteq Z^\alpha(G) \bigcup Y$. Let $g \in R^\alpha(Y) \bigcap C_G^\alpha(Y)$, then $g \in C_G^\alpha(Y) = C_G^\alpha(X \setminus Z^\alpha(G))$. Hence $g \in C_G^\alpha(< X > - Z^\alpha(G)) = C_G^\alpha(G - Z^\alpha(G))$. Clearly $g \in C_G^\alpha(Z^\alpha(G))$ and therefore $g \in Z^\alpha(G) \subseteq Z^\alpha(G) \bigcup Y$. Now, Lemma 3.4 implies that $X \setminus Z^\alpha(G)$ is a dominating set. □

3.6. Lemma. *Let G be a non-abelian simple group and $\gamma(\Gamma_G^\alpha)$ be the dominating number of the graph Γ_G^α. Then $\gamma(\Gamma_G^\alpha) \leqslant 2$.*

Proof. Assume that G is a non-abelian simple group, hence we can see that $G =< g, x >$. Since G is a non-abelian simple group then $Z^\alpha(G) = 1$, we set $X = \{g, x\}$. It is clear that $Z^\alpha(G) \bigcap X = \emptyset$, thus by Lemma 3.5, it is clear X is a dominating set of Γ_G^α. □

4. Planarity

This section is devoted to a determination of planarity, outer planarity and 1−planar graph of the graph Γ_G^α. We will show that, with exception of a few possible cases, Γ_G^α is not planar.

4.1. Lemma. *If $|Z^\alpha(G)| \geq 3$, then Γ_G^α is not a planar graph.*

Proof. Since G is not abelian, so $\dfrac{G}{Z(G)}$ is not cyclic and so $[G : Z(G)] \geq 4$. Similarly $[G : Z^\alpha(G)] \geq 4$, because $Z^\alpha(G) \subseteq Z(G)$. Thus $G \setminus Z^\alpha(G)$ is the union of at least three distinct cosets of $Z^\alpha(G)$. Since Γ_G^α is connected, so there exists at least an edge between two cosets of $Z^\alpha(G)$. Now, by Lemma 3.2 and the assumption that $|Z^\alpha(G)| \geq 3$, we have a subgraph $K_{3,3}$. Therefore Γ_G^α is not planar. □

4.2. Corollary. *If $|Z^\alpha(G)| \neq 1$, then Γ_G^α is not an outer planar graph.*

Proof. Let $|Z^\alpha(G)| \geq 2$, then by Lemma 4.1, we can see that Γ_G^α have a subgraph $K_{2,2}$. Hence Γ_G^α is not an outer planar graph. □

We know that if Γ_G^α is a planar graph, then there exists a vertex x such that $\deg(x) \leqslant 5$. Moreover if Γ_G^α is a 1−planar graph, then there exists a vertex y such that $\deg(y) \leqslant 7$ (see [8]). By this fact, we are going to state the following result.

4.3. Lemma. *If $|G| \neq 6, 8, 10$ then Γ_G^α is not planar graph.*

Proof. Suppose that Γ_G^α is a planar graph, then there exists a vertex x in a $V(\Gamma_G^\alpha)$ such that $\deg(x) \leqslant 5$. First assume that $x = x^\alpha$, from Lemma 2.2 we see that $\deg(x) = |G| - |C_G(x) \bigcap C_G^\alpha(x)|$. We have $|C_G(x) \bigcap C_G^\alpha(x)| \leqslant |C_G(x)|$. Thus $|G| - |C_G(x)| \leqslant$

$|G| - |C_G(x) \bigcap C_G^{\alpha}(x)| \leqslant 5$. Since x is a vertex in Γ_G^{α}, so $x \notin Z(G)$ and consequently $C_G(x) \subsetneqq G$. Therefore

$$|G| - \frac{|G|}{2} \leqslant |G| - |C_G(x)| \leqslant |G| - |C_G(x) \bigcap C_G^{\alpha}(x)| \leqslant 5.$$

It implies that $|G| \leqslant 10$.

Second assume that $x \neq x^{\alpha}$ and $x \in Z(G)$, then $\deg(x) = |G| - |Z^{\alpha}(G)| - |R^{\alpha}(x) \bigcap C_G^{\alpha}(x)| - 1$. Since $x \in Z(G)$ so $R^{\alpha}(x) = \emptyset$, hence $\deg(x) = |G| - |Z^{\alpha}(G)| - 1$. We have $|G| - \frac{|G|}{2} \leqslant |G| - |Z^{\alpha}(G)| \leqslant 6$. Thus $|G| \leqslant 12$. Finally suppose that $x \neq x^{\alpha}$ and $x \notin Z(G)$ then $Z^{\alpha}(G) \leq Z(G) \subsetneqq C_G(x) \subsetneqq G$. Hence

$$|G| - \frac{|G|}{4} - |C_G^{\alpha}(x)| \leqslant |G| - |Z^{\alpha}(G)| - |R^{\alpha}(x) \bigcap C_G^{\alpha}(x)| \leqslant 6.$$

It implies that $|G| \leqslant 24$. We know that the groups of orders 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23 are abelian groups. So associated graphs of the groups of orders 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23 are not planar graph. Moreover, if $|E(\Gamma_G^{\alpha})| \leqslant 3|V(\Gamma_G^{\alpha})| - 6$, where $|E(\Gamma_G^{\alpha})|$ is a number of edges in graph Γ_G^{α}, then Γ_G^{α} is not planar (see [6, Corollary 9.5.2]. Hence by using the group theory package GAP [11], it is easy to see that graphs of the associated to groups of orders 12, 14, 16, 18, 20, 21, 22, 24 are not planar. It implies that if $|G| \neq 6, 8, 10$ then Γ_G^{α} is not planar graph. □

By the above lemma, we can state that following corollary.

4.4. Corollary. *If* $|G| \neq 6, 8, 10, 12, 14, 16$ *then* Γ_G^{α} *is not* $1-$*planar graph.*

Proof. If Γ_G^{α} is a $1-$planar graph, then there exists a vertex x such that $\deg(x) \leqslant 7$. Then we can see that if $x = x^{\alpha}$ then $|G| \leqslant 14$. If $x \neq x^{\alpha}$ and $x \in Z(G)$ then $|G| \leqslant 16$, and if $x \neq x^{\alpha}$ and $x \notin Z(G)$ then $|G| \leqslant 32$. Moreover, the groups of orders 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29, 31 are abelian groups. So if Γ_G^{α} is a $1-$planar graph then $|G|$ is equal to 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 24, 26, 27, 28, 30, 32. Since every $1-$planar graph with n vertices has at most $4n - 8$ edges, a straightforward computation shows that the associated graph of groups of orders 18, 20, 22, 24, 26, 27, 28, 30, 32 are not $1-$planar. The proof is now complete. □

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References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, **298** (2006), 468-492.

[2] A. Abdollahi, *Engel graph associated with a group*, J. Algebra, **318** (2007), 680-691.

[3] A. Abdollahi and A. Mohammadi Hassanabadi, *Non-cyclic graph of a group*, Comm. in Algebra, **35** (2007), 2057-2081.

[4] R. Barzegar and A. Erfanian, *Nilpotency and solubility of groups relative to an automorphism*, Caspian Journal of Mathematical Sciences, **4**(2) (2015), 271-283.

[5] A. Erfanian, M. Farrokhi D.G. and B. Tolue, *Non-normal graphs of finite groups*, J. Algebra Appl., **12** (2013).

[6] J. A. Bondy and J. S. R. Murty, *Graph Theory with Applications*, Elsevier, (1977).

[7] P. J. Cameron and S. Ghosh, *The power graph of a fnite group*, Discrete Math, **311** (2011), 1220-1222.

[8] I. Fabrici and T. Madaras, *The structure of 1-planar graphs*, Discrete Mathematics, **307** (2007), 854 - 865.

[9] A. Iranmanesh and A. Jafarzadeh, *On the commuting graph associated with the symmetric and alternating groups*, J. Algebra Appl., **7** (2008), 129-146.
[10] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag: New York-Heidelberg Berlin (1982).
[11] The GAP Group, *GAP-Groups, Algorithms and Programming*, Version 4.6.4, 2013 (http://www.gap-system.org/).
[12] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, **69** (2) (1981), 487-513.