Hacettepe Journal of Mathematics and Statistics
Volume 47 (1) (2018), 107 – 127

A new operational approach for solving weakly singular integro–differential equations

Khadijeh Sadri*† and Zainab Ayati‡

Abstract

Based on Jacobi polynomials, an operational method is proposed to solve weakly singular integro–differential equations. These equations appear in various fields of science such as physics and engineering, the motion of a plate in a viscous fluid under the action of external forces, problems of heat transfer, and surface waves. To solve the weakly singular integro–differential equations, a fast algorithm is used for simplifying the problem under study. The Laplace transform and Jacobi collocation methods are merged, and thus, a novel approach is presented. Some theorems are given and established to theoretically support the computational simplifications which reduce costs. In order to show the efficiency and accuracy of the proposed method some numerical results are provided. It is found that the proposed method has lesser computational size compared to other common methods, such as Adomian decomposition, Taylor expansion, and Bernstein operational methods. It is further found that the absolute errors are almost constant in the studied interval.

Keywords: Singular integro–differential equations, Shifted Jacobi polynomials, Operational matrices of integration and product.

2000 AMS Classification: 45D05, 45E10, 45F15, 47B36, 65D15

Received : 27.11.2016 *Accepted* : 20.01.2017 *Doi* : 10.15672 /HJMS.2017.519

*Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran, Email: kh.sadri@uma.ac.ir

†Corresponding Author.

‡Department of Engineerig Sciences, Faculty of Technology and Engineering East of Guilan, P.C. 44891–63157, Rudsar–Vajargah, Iran, Email: zainab.ayati@guilan.ac.ir

1. Introduction

Weakly singular integro–differential equation is found in the mathematical modeling of several models in physics and engineering, typically in phenomena in which the time evolution of the dynamical variable x is governed by diffusive processes depending on x itself. Well–known examples are the motion of a plate in a viscous fluid under the action of external forces [1], problems of heat transfer [2], surface of waves [3, 4], problem of vapor bubble growth in a superheated liquid [5], heat conduction problems [6], elasticity and fracture mechanics [7], potential problems, the Dirichlet problems, radiative equilibrium [8] and so on. The variety of applications necessitates simple and low cost by developing new methodologies to solve these equations. Weakly singular integro–differential equations have been solved using different techniques. For example, the methods including the spline collocation method [9,10], the discrete collocation method [11–13], the discrete Galerkin method [14,15], the Legendre multiwavelets method [16], the piecewise polynomial collocation method with graded meshes [17], Homotopy perturbation method [18] to determine the approximate solutions.
Some problems of mathematical physics are described in terms of second order linear and nonlinear singular Volterra integro-differential equations of the following form [19–22]:

$$\sum_{i=0}^{2} a_i(x)y^{(i)}(x) = f(x) + \int_0^x (x-t)^{-\sigma} F(t, y(t), y'(t), ..., y^{(n)}(t))dt, \tag{1.1}$$

where $k(x,t) = (x-t)^{-\sigma}$, $0 < \sigma < 1$, $0 < x < 1$, is singular kernel of the integro–differential equation which makes solving this class of equations difficult. The difficulty of the solution of these equations demands a simple approach to approximate and solve the various types of singular integro–differential equations. To achieve this objective, in this study, an operational Jacobi collocation method and Laplace transform are merged to establish a new methodology. Using Laplace transform simplifies the singular kernel of integro–differential equations. By applying the resultant matrix relations, weakly singular integro–differential equations are equated to a system of either linear or nonlinear algebraic equations.
The contents of this paper are organized as follows: Section 1 is the introduction; the shifted Jacobi polynomials and their operational matrices of integration and product are introduced in Section 2; an error and convergence analysis is presented in Section 3; in Section 4, the method of solution and necessary relations are presented; the Jacobi operational matrices are applied to solve several singular integro–differential equations in Section 5; and finally, the conclusion is presented in Section 6.

2. Shifted Jacobi polynomials and their operational matrices

The shifted Jacobi polynomials in x, on the interval $[0,1]$, can be determined with the following recurrence formula:

$$\begin{aligned} P_{i+1}^{(\alpha,\beta)}(x) &= A(\alpha,\beta,i)\ P_i^{(\alpha,\beta)}(x) + B(\alpha,\beta,i)\ (2\ x-1)\ P_i^{(\alpha,\beta)}(x) \\ &\quad - D(\alpha,\beta,i)\ P_{i-1}^{(\alpha,\beta)}(x), \qquad i = 1,2,..., \quad \alpha,\beta > -1, \end{aligned} \tag{2.1}$$

where

$$P_0^{(\alpha,\beta)}(x) = 1, \qquad P_1^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+2)(2\ x-1)}{2} + \frac{\alpha-\beta}{2},$$

and

$$A(\alpha,\beta,i)=\frac{(2i+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)},$$
$$B(\alpha,\beta,i)=\frac{(2i+\alpha+\beta+2)(2i+\alpha+\beta+1)}{2(i+1)(i+\alpha+\beta+1)},$$
$$D(\alpha,\beta,i)=\frac{(i+\alpha)(i+\beta)(2i+\alpha+\beta+2)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)}.$$

The orthogonality condition and shifted weight function, respectively, are:

$$\int_0^1 P_i^{\,(\alpha,\beta)}(x)\ P_j^{\,(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx=\theta_i\ \delta_{ij},$$

and

$$w^{(\alpha,\beta)}(x)=(1-x)^{\alpha}\ x^{\beta},\qquad x\in[0,1],$$

where δ_{ij} is the well–known Dirac function and

$$\theta_i=\frac{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{(2i+\alpha+\beta+1)\ i!\ \Gamma(i+\alpha+\beta+1)}.$$

Also, the analytic form of shifted Jacobi polynomials, on the interval $[0,1]$, is as follows:

$$P_i^{(\alpha,\beta)}(x)=\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)\ x^k}{\Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (i-k)!\ k!},\ i=0,1,\ldots\ . \tag{2.2}$$

A function $u(x)$, square integrable in the interval $[0,1]$, can be expressed in terms of shifted Jacobi polynomials as:

$$u(x)=\sum_{j=0}^{\infty}C_j\ P_j^{(\alpha,\beta)}(x), \tag{2.3}$$

where the coefficients C_j are given by:

$$C_j=\frac{1}{\theta_j}\int_0^1 u(x)\ P_j^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx,\qquad j=0,1,\ldots\ .$$

In practice, only the first $(N+1)$ terms shifted Jacobi polynomials are considered. Therefore, one has:

$$u_N(x)=\sum_{j=0}^{N}C_j\ P_j^{(\alpha,\beta)}(x)=\Phi^T(x)\ C=C^T\ \Phi(x), \tag{2.4}$$

where the vectors C and $\Phi(x)$ are given by:

$$C=[C_0,C_1,\ldots,C_N]^T,\qquad\qquad \Phi(x)=[P_0^{(\alpha,\beta)}(x),P_1^{(\alpha,\beta)}(x),\ldots,P_N^{(\alpha,\beta)}(x)]^T. \tag{2.5}$$

Similarly, a function $g(x,t)$ defined for $x,t\in[0,1]$ can be expanded in terms of double shifted Jacobi polynomials as:

$$g_N(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}g_{ij}\ P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(t)=\Phi^T(x)\ G\ \Phi(t),$$

where G is the known matrix, and its entries are given by:

$$g_{ij}=\frac{1}{\theta_i\ \theta_j}\int_0^1\int_0^1 g(x,t)\ P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(t)\ w^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(t)\ dx\ dt, \tag{2.6}$$
$$i,j=0,1,\ldots,N.$$

Some other properties of the shifted Jacobi polynomials are presented as follows:

$$(1)\ P_i^{\ (\alpha,\beta)}(0) = (-1)^i \begin{pmatrix} i+\alpha \\ i \end{pmatrix},$$

$$(2)\ \frac{d^i}{dx^i}\ P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+\beta+i+1)}{\Gamma(n+\alpha+\beta+1)}\ P_{n-i}^{(\alpha+i,\beta+i)}(x).$$

Some useful lemmas and theorems, which were introduced by authors in [19], are presented as follows.

2.1. Lemma. *The shifted Jacobi polynomial $P_i^{(\alpha,\beta)}(x)$ can be obtained in the form of*

$$P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} \gamma_k^{(i)}\ x^k,$$

where $\gamma_k^{(i)}$ are

$$\gamma_k^{(i)} = (-1)^{i-k}\ \begin{pmatrix} i+k+\alpha+\beta \\ k \end{pmatrix} \begin{pmatrix} i+\alpha \\ i-k \end{pmatrix}.$$

Proof. See [23]. □

2.2. Lemma. *If $p \geqslant 0$, then*

$$\int_0^1 x^p\ P_n^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx =$$
$$\sum_{l=0}^{n} \frac{(-1)^{n-l}\Gamma(n+\beta+1)\ \Gamma(n+l+\alpha+\beta+1)\ \Gamma(p+l+\beta+1)\ \Gamma(\alpha+1)}{\Gamma(l+\beta+1)\ \Gamma(n+\alpha+\beta+1)\ \Gamma(p+l+\alpha+\beta+2)\ (n-l)!\ l!}.$$

Proof. See [23]. □

2.3. Lemma. *If $P_j^{(\alpha,\beta)}(x)$ and $P_k^{(\alpha,\beta)}(x)$ are respectively $j-th$ and $k-th$ shifted Jacobi polynomials, the product of $P_j^{(\alpha,\beta)}(x)$ and $P_k^{(\alpha,\beta)}(x)$ can be written as:*

$$Q_{j+k}^{(\alpha,\beta)}(x) = \sum_{r=0}^{j+k} \lambda_r^{(j,k)}\ x^r,$$

where coefficients $\lambda_r^{(j,k)}$ are determined as follows:
If $j \geqslant k$:

$r = 0, 1, ..., j+k,$
$if\ \ r > j\ \ then$
$\quad \lambda_r^{(j,k)} = \sum_{l=r-j}^{k} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$else$
$\quad r_1 = min\{r, k\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$end.$

If** $j < k$ **:

$r = 0, 1, ..., j + k,$
$if\ \ r \leqslant j\ \ then$
$\quad r_1 = min\{r, j\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$else$
$\quad r_2 = min\{r, k\},$
$\quad \lambda_r^{(j,k)} = \sum_{l=r-j}^{r_2} \gamma_{r-l}^{(j)}\ \gamma_l^{(k)},$
$end.$

and the coefficients $\gamma_l^{(k)}$ *and* $\gamma_{r-l}^{(j)}$ *are introduced by Lemma 2.1.*

Proof. See [23]. □

2.4. Lemma. *If* $P_i^{(\alpha,\beta)}(x)$, $P_j^{(\alpha,\beta)}(x)$, *and* $P_k^{(\alpha,\beta)}(x)$ *are respectively* $i-$, $j-$ *and* $k-th$ *shifted Jacobi polynomials, then*

$$\begin{aligned} q_{ijk} &= \int_0^1 P_i^{(\alpha,\beta)}(x)\ P_j^{(\alpha,\beta)}(x)\ P_k^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx \\ &= \sum_{n=0}^{j+k} \sum_{l=0}^{i} \left\{ \frac{(-1)^{i-l} \lambda_n^{(j,k)}\ \Gamma(i+\beta+1)\ \Gamma(i+l+\alpha+\beta+1)}{\Gamma(l+\beta+1)\ \Gamma(i+\alpha+\beta+1)} \right. \\ &\times \left. \frac{\Gamma(n+l+\beta+1)\ \Gamma(\alpha+1)}{\Gamma(n+l+\alpha+\beta+2)\ (i-l)!\ l!} \right\}, \end{aligned}$$

where $\lambda_n^{(j,k)}$ *has been introduced by Lemma 2.3.*

Proof. See [23]. □

Next theorem presents a general formula for the operational matrix of integration.

2.5. Theorem. *Suppose vector* $\Phi(x)$ *is the shifted Jacobi vector in Eq. (2.5). Then, the operational matrix of integration,* $\mathbf{P}$, *is defined as follows:*

$$\int_0^x \Phi(t)\ dt \simeq \mathbf{P}\ \Phi(x), \tag{2.7}$$

where $\mathbf{P}$ *is* $(N+1) \times (N+1)$ *operational matrix of integration as:*

$$\mathbf{P} = \begin{bmatrix} \pi(0,0) & \pi(0,1) & \dots & \pi(0,N) \\ \pi(1,0) & \pi(1,1) & \dots & \pi(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \pi(N,0) & \pi(N,1) & \dots & \pi(N,N) \end{bmatrix},$$

and its entries are defined as:

$$\pi(i,j) = \sum_{k=0}^{i} \omega_{ijk}, \qquad i, j = 0, 1, 2, ..., N, \tag{2.8}$$

where ω_{ijk} *are given by:*

$$\begin{aligned} \omega_{ijk} &= \frac{(-1)^{i-k}\ \Gamma(j+\beta+1)\ \Gamma(i+\beta+1)\ \Gamma(i+k+\alpha+\beta+1)\ \Gamma(\alpha+1)}{\theta_j\ \Gamma(j+\alpha+\beta+1)\ \Gamma(k+\beta+1)\ \Gamma(i+\alpha+\beta+1)\ (k+1)!\ (i-k)!} \\ &\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\ \Gamma(j+l+\alpha+\beta+1)\ \Gamma(l+k+\beta+2)}{\Gamma(l+\beta+1)\ \Gamma(l+k+\alpha+\beta+3)\ l!(j-l)!}, \\ & i, j = 0, 1, ..., N, \quad k = 0, 1, ..., i. \end{aligned}$$

Proof. See [23]. □

In performing arithmetic and other operations on the Jacobi bases, we frequently encounter the product of $\Phi(x)$ and $\Phi^T(x)$, called the operational matrix of product. In this section, some necessary matrix relations will be derived.
First, a general formula is presented for finding the $(N+1)\times(N+1)$ operational matrix of product $\tilde{U}$ whenever

$$\Phi(x)\ \Phi^T(x)\ U \simeq \tilde{U}\ \Phi(x), \tag{2.9}$$

where U is a $(n+1)$ given vector.
The following theorem determines the entries of the matrix $\tilde{U}$.

2.6. Theorem. *The entries of matrix $\tilde{U}$ in Eq. (2.9) are computed as*

$$\tilde{U}_{jk} = \frac{1}{\theta_k} \sum_{i=0}^{N} U_i\ q_{ijk}, \qquad j,k = 0,1,...,N,$$

where q_{ijk} are computed using Lemma 2.4 whereas U_i are the components of vector U in Eq. (2.9).

Proof. See [23]. □

Next theorem presents the general formula for approximating the nonlinear term $v^r(x)\ u^s(x)$ which may appear in nonlinear equations.

2.7. Theorem. *If*

$$u(x) \simeq U^T\ \Phi(x) = \Phi(x)^T\ U, \qquad v(x) \simeq V^T\ \Phi(x) = \Phi^T(x)\ V,$$
$$\Phi(x)\ \Phi^T(x)\ U \simeq \tilde{U}\ \Phi(x), \qquad \Phi(x)\ \Phi^T(x)\ V \simeq \tilde{V}\ \Phi(x),$$

where U and V are the $(N+1)$ vectors and $\tilde{U}$ and $\tilde{V}$ are the $(N+1)\times(N+1)$ operational matrices of product, the following proposition is held:

$$v^r(x)\ u^s(x) \simeq V^T\ (\tilde{V})^{r-1}\ \tilde{B}_{s-1}\ \Phi(x), \quad B_{s-1} = (\tilde{U}^T)^{s-1}\ U, \quad r,s = 1,2,\dots\ .$$

Proof. See [23]. □

3. Solution method

In this section, the necessary matrix relations will be obtained and presented to solve some weakly singular integro–differential equations.

3.1. Theorem. *Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by:*

$$\mathcal{L}[f_1(x)] = F_1(s), \qquad \mathcal{L}[f_2(x)] = F_2(s).$$

The Laplace convolution product of these functions is defined by:

$$\mathcal{L}\left[\int_0^x f_1(x-t) f_2(t)\ dt\right] = F_1(s)F_2(s).$$

3.2. Theorem. *Suppose $F(s)$ is the Laplace transform of $f(x)$, which has a Maclaurin power series expansion in the following form:*

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}. \tag{3.1}$$

By taking the Laplace transform we obtain:

$$F(s) = \sum_{i=0}^{\infty} \frac{a_i}{s^{i+1}}. \tag{3.2}$$

Conversely, Eq. (3.1) is derived from a given expansion of Eq. (3.2).

3.3. Theorem. *The following relation is established for $0 < \sigma < 1$:*

$$\int_0^x \frac{t^k}{(x-t)^\sigma} dt = \frac{\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)} x^{k-\sigma+1}, \qquad k = 0, 1, 2, \dots . \tag{3.3}$$

Proof. Taking Laplace transform from the left hand side of Eq. (3.3) leads to:

$$\begin{aligned} \mathcal{L}\left[\int_0^x \frac{t^k}{(x-t)^\sigma} dt\right] &= \mathcal{L}[x^k]\ \mathcal{L}\left[\frac{1}{x^\sigma}\right] \\ &= \frac{\Gamma(k+1)\Gamma(1-\sigma)}{s^{k-\sigma+2}}, \qquad k = 0, 1, 2, \dots . \end{aligned} \tag{3.4}$$

Now, by applying the inverse Laplace operator in Eq. (3.4), the following result is obtained:

$$\begin{aligned} \int_0^x \frac{t^k}{(x-t)^\sigma} dt &= \mathcal{L}^{-1}\left[\frac{\Gamma(k+1)\Gamma(1-\sigma)}{s^{k-\sigma+2}}\right] \\ &= \frac{\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)} x^{k-\sigma+1}, \qquad k = 0, 1, 2, \dots . \end{aligned}$$

□

To approximate the integral part in Eq. (1.1), the following theorem is stated:

3.4. Theorem. *Suppose $y(x) \in C[0,1]$, $0 < \sigma < 1$, and $y(x) \simeq \Phi^T(x)\ C$, where vectors $\Phi(x)$ and C are defined by Eq. (6), then*

$$\int_0^x \frac{y(t)}{(x-t)^\sigma} dt \simeq \mathbf{B}^{(\sigma)}\ \Phi(x)\ C,$$

where $\mathbf{B}^{(\sigma)}$ is $(N+1) \times (N+1)$ matrix and its entries are determined as follows:

$$\mathbf{B}_{ij}^{(\sigma)} = \sum_{k=0}^{i-1} \frac{\gamma_k^{(i-1)}\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)} a_{j-1}^{(k-\sigma+1)}, \qquad i, j = 1, 2, ..., N+1,$$

where $\gamma_k^{(l)}$ and $a_j^{(n)}$ are computed using Lemmas 1 and 2, respectively.

Proof. By the definition of vector $\Phi(x)$ and Lemma 2.1, one has:

$$\begin{aligned} \Phi^T(t) &= [P_0^{(\alpha,\beta)}(t), P_1^{(\alpha,\beta)}(t), ..., P_N^{(\alpha,\beta)}(t)] \\ &= \left[\sum_{k=0}^{0} \gamma_k^{(0)} t^k, \sum_{k=0}^{1} \gamma_k^{(1)} t^k, ..., \sum_{k=0}^{N} \gamma_k^{(N)} t^k\right]. \end{aligned}$$

Using Theorem 3.3 leads to:

$$\int_0^x \frac{\Phi^T(t)}{(x-t)^\sigma}dt = \Bigg[\sum_{k=0}^{0}\gamma_k^{(0)}\int_0^x \frac{t^k}{(x-t)^\sigma}dt, \sum_{k=0}^{1}\gamma_k^{(1)}\int_0^x \frac{t^k}{(x-t)^\sigma}dt,$$
$$..., \sum_{k=0}^{N}\gamma_k^{(N)}\int_0^x \frac{t^k}{(x-t)^\sigma}dt\Bigg]$$

$$= \Bigg[\sum_{k=0}^{0}\frac{\gamma_k^{(0)}\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)}x^{k-\sigma+1}, \tag{3.5}$$
$$\sum_{k=0}^{1}\frac{\gamma_k^{(1)}\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)}x^{k-\sigma+1},$$
$$..., \sum_{k=0}^{N}\frac{\gamma_k^{(N)}\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)}x^{k-\sigma+1}\Bigg].$$

Now, we approximate $x^{k-\sigma+1}$ in terms of shifted Jacobi polynomials as follows:

$$x^{k-\sigma+1} = \sum_{j=0}^{N} a_j^{(k-\sigma+1)} P_j^{(\alpha,\beta)}(x),$$

and

$$a_j^{(k-\sigma+1)} = \frac{1}{\theta_j}\int_0^1 x^{k-\sigma+1}P_j^{(\alpha,\beta)}(x)\ w^{(\alpha,\beta)}(x)\ dx, \qquad j = 0, 1, ..., N,$$

where $a_j^{(k-\sigma+1)}$ are computed using Lemma 2.2. Thus:

$$\sum_{k=0}^{l}\frac{\gamma_k^{(l)}\Gamma(k+1)\Gamma(1-\sigma)}{\Gamma(k-\sigma+2)}x^{k-\sigma+1} = \sum_{j=0}^{N}\left\{\sum_{k=0}^{l}\frac{\gamma_k^{(l)}\Gamma(k+1)\Gamma(1-\sigma)a_j^{(k-\sigma+1)}}{\Gamma(k-\sigma+2)}\right\}P_j^{(\alpha,\beta)}(x)$$
$$= \sum_{j=0}^{N} b(l,j)\ P_j^{(\alpha,\beta)}(x), \qquad l = 0, 1, ..., N.$$

Eq. (3.5) is then obtained as follows:

$$\int_0^x \frac{\Phi^T(t)}{(x-t)^\sigma}dt \simeq \begin{bmatrix} b(0,0) & b(0,1) & \dots & b(0,N) \\ b(1,0) & b(1,1) & \dots & b(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ b(N,0) & b(N,1) & \dots & b(N,N) \end{bmatrix}\begin{bmatrix} P_0^{(\alpha,\beta)}(x) \\ P_1^{(\alpha,\beta)}(x) \\ \vdots \\ P_N^{(\alpha,\beta)}(x) \end{bmatrix}$$
$$= \mathbf{B}^{(\sigma)}\ \Phi(x).$$

The desired result is achieved. □

3.5. Theorem. *Suppose $f(x,t) \in C([0,1]\times[0,1])$, $0 < \sigma < 1$, and $y(x) \in C[0,1]$. If:*

$$y(x) \simeq \Phi^T(x)\ C, \qquad f(x,t) \simeq \Phi^T(x)\ F\ \Phi(t), \tag{3.6}$$

where F is a known $(N+1)\times(N+1)$ matrix and obtained from Eq. (2.6), we have:

$$\int_0^x \frac{f(x,t)}{(x-t)^\sigma}y(t)\ dt \simeq \Phi^T(x)\ F\ \tilde{C}\ \mathbf{B}^{(\sigma)}\ \Phi(x),$$

where $\tilde{C}$ is the operational matrix of product corresponding to vector C.

Proof. Using approximations in Eq. (3.6), the following integral converts to:

$$\begin{aligned}\int_0^x \frac{f(x,t)}{(x-t)^\sigma} y(t)\ dt &\simeq \int_0^x \frac{\Phi^T(x)\ F\ \Phi(t)\ \Phi^T(t)\ C}{(x-t)^\sigma} dt \\ &= \Phi^T(x)\ F \int_0^x \frac{\Phi(t)\ \Phi^T(t)\ C}{(x-t)^\sigma} dt \\ &= \Phi^T(x)\ F\ \tilde{C} \int_0^x \frac{\Phi(t)}{(x-t)^\sigma} dt \\ &= \Phi^T(x)\ F\ \tilde{C}\ \mathbf{B}^{(\sigma)}\ \Phi(x).\end{aligned}$$

□

In nonlinear integral equations, the integral parts can be approximated as follows:

3.6. Theorem. *If $y(x), u(x) \in C[0,1]$ and $y(x) \simeq \Phi^T(x)\ C_1$ and $u(x) \simeq \Phi^T(x)\ C_2$, we have:*

$$\int_0^x \frac{y^2(t)}{(x-t)^\sigma} dt \simeq C_1^T\ \tilde{C}_1\ \mathbf{B}^{(\sigma)}\ \Phi(x),$$
$$\int_0^x \frac{y(t)\ u(t)}{(x-t)^\sigma} dt \simeq C_1^T\ \tilde{C}_2\ \mathbf{B}^{(\sigma)}\ \Phi(x).$$

Proof. By applying Theorem 2.7 we have:

$$\begin{aligned}\int_0^x \frac{y^2(t)}{(x-t)^\sigma} dt &\simeq \int_0^x \frac{C_1^T\ \Phi(t)\ \Phi^T(t)\ C_1}{(x-t)^\sigma} dt \\ &= C_1^T \int_0^x \frac{\Phi(t)\ \Phi^T(t)\ C_1}{(x-t)^\sigma} dt \\ &\simeq C_1^T\ \tilde{C}_1 \int_0^x \frac{\Phi(t)}{(x-t)^\sigma} dt \\ &\simeq C_1^T\ \tilde{C}_1\ \mathbf{B}^{(\sigma)}\ \Phi(x).\end{aligned}$$

Similarly, the second part is established. □

By using the introduced matrices and approximations, the terms of the equations under study are approximated and substituted into related equations. Hereby, the solving of main problem is changed to solving systems of linear or nonlinear algebraic equations. The resultant algebraic equations are collocated at $(N+1)$ roots of the $(N+1)$–th shifted Jacobi polynomial on the interval $[0,1]$. These equations generate $(N+1)$ linear or nonlinear algebraic equations. The resultant nonlinear systems can be solved using Newton's iterative method. By solving this algebraic system, the unknown vector C can be determined, and an approximate solution is acquired from Eq. (2.4) for various values of α and β parameters.

4. Error and convergence analysis

In this section, the error and convergence analysis of the approximate solution of Eq. (1.1) by using the Jacobi collocation method is considered. Before stating the our results, first some hypotheses, that will be used in the subsequent discussion, must be set. These hypotheses are listed as follows for the case $a_0(x) \neq 0$.

$H1$) There exists a non–negative constant L_k such that

$$\| y^{(k)}(x) - y_N^{(k)}(x) \|_{L^2(\Omega)} \leqslant L_k \| y(x) - y_N(x) \|_{L^2(\Omega)},$$

for $x \in [0,1]$ and $k = 0,1,2$ $(\Omega = [0,1])$.
$H2$) There exists an integer such as μ, $0 \leqslant \mu \leqslant 2$, such that F in Eq. (1.1) satisfies uniform Lipschitz condition,

$$\|F(x, y(x), y', ..., y^{(n)}(x)) - F(x, y_N(x), y'_N, ..., y_N^{(n)}(x)) \|_{L^2(\Omega)} \leqslant L_{F_\mu} \| y^{(\mu)}(x) - y_N^{(\mu)}(x) \|_{L^2(\Omega)},$$

for all $x \in [0,1]$ where $L_{F_\mu} \geqslant 0$.
$H3$) There exists a positive constant $\tilde{K}$ such that

$$\| k(x,t) \|_{L^2(\Omega)} \leqslant \tilde{K},$$

for $(x,t) \in [0,1] \times [0,1]$.
$H4$) There exists a non–negative constant $\tilde{F}$ such that

$$\| F(x, y(x), y', ..., y^{(n)}(x)) \|_{L^2(\Omega)} \leqslant \tilde{F},$$

for all $x \in [0,1]$.
In here, we rewrite the Eq. (1.1) as follows,

$$\begin{aligned} y(x) = g(x) + a(x) \int_0^x k(x,t) F(t, y(t), y', ..., y^{(n)}(t))\ dt \\ + b_1(x) y'(x) + b_2(x) y''(x), \end{aligned} \tag{4.1}$$

where,

$$g(x) = \frac{f(x)}{a_0(x)}, \quad a(x) = \frac{1}{a_0(x)}, \quad b_1(x) = -\frac{a_1(x)}{a_0(x)},$$
$$b_2(x) = -\frac{a_2(x)}{a_0(x)}, \quad k(x,t) = (x-t)^{-\sigma}.$$

Also, substituting the appropriate solution y_N and the approximation of $k(x,t)$ given by Eq. (2.4) and theorems 3.4–3.6 into Eq. (4.1) leads to,

$$\begin{aligned} y_N(x) = g(x) + a(x) \int_0^x k_N(x,t) F(t, y_N(t), y'_N, ..., y_N^{(n)}(t))\ dt \\ + b_1(x) y'_N(x) + b_2(x) y''_N(x) + H_N(x), \end{aligned} \tag{4.2}$$

where $H_N(x)$ is perturbation term. Obviously, $H_N(x) \longrightarrow 0$ as $N \longrightarrow \infty$.
To prove the convergence of the method, we the following lemma that will be applied in proof of the main theorem.

4.1. Lemma. *(Schwarz's Inequality) For any two functions $f, g \in L^2(\Omega)(\Omega = [0,1])$, we have,*

$$\| (f,g) \| \leqslant \| f \| \| g \|,$$

where $(.,.)$ indicates the inner product on $L^2(\Omega)$.

4.2. Lemma. *If $f \geqslant 0$ be a measurable function and $A \subseteq B$, then*

$$\int_A f\ d\mu \leqslant \int_B f\ d\mu.$$

Now, the following convergence theorem is stated.

4.3. Theorem. *Suppose that the hypotheses $H1 - H4$ hold. Then, for the $N-$order approximation of the solution to Eq. (4.1), denoted by $y_N(x)$ corresponding to the approximation of the kernel, we have,*

$$\| e_N(x) \|_{L^2(\Omega)} \leqslant \frac{\| H_N \|_{L^2(\Omega)} + \| a \|_{L^2(\Omega)} \tilde{F} \| e(K_N) \|_{L^2(\Omega)}}{1 - \| a \|_{L^2(\Omega)} \tilde{K}\ L_{F_\mu} - \sum_{k=1}^{2} L_k \ \ \| b_k \|_{L^2(\Omega)}},$$

where $e_N(x) = y(x) - y_N(x)$, $e(K_N) = k(x,t) - k_N(x,t)$, *and* $\Omega = [0,1]$.

Proof. Substracting Eq. (4.2) from Eq. (4.1) implies,

$$\begin{aligned} e_N(x) = {} & -H_N(x) + a(x)\int_0^x k(x,t)F(t, y(t), y', ..., y^{(n)}(t))\ dt \\ & - a(x)\int_0^x k_N(x,t)F(t, y_N(t), y'_N, ..., y_N^{(n)}(t))\ dt \\ & + b_1(x)e'_N(x) + b_2(x)e''_N(x). \end{aligned} \tag{4.3}$$

By adding and substracting appropriate terms in Eq. (4.3), it can be rewritten as,

$$\begin{aligned} e_N(x) = {} & -H_N(x) + a(x)\int_0^x k(x,t)\bigg(F(t, y(t), y', ..., y^{(n)}(t)) \\ & - F(t, y_N(t), y'_N, ..., y_N^{(n)}(t))\bigg)\ dt \\ & + a(x)\int_0^x \bigg(k(x,t) - k_N(x,t)\bigg)F(t, y_N(t), y'_N, ..., y_N^{(n)}(t))\ dt \\ & + b_1(x)e'_N(x) + b_2(x)e''_N(x). \end{aligned}$$

According $H1 - H4$, Lemmas 4.1 and 4.2, and since $[0,x] \subseteq [0,1]$, we obtain,

$$\begin{aligned} \| e_N(x) \|_{L^2(\Omega)} \leqslant {} & \| H_N(x) \|_{L^2(\Omega)} + \| a \|_{L^2(\Omega)} \tilde{K} L_{F_\mu} \| e_N(x) \|_{L^2(\Omega)} \\ & + \| a \|_{L^2(\Omega)} \tilde{F} \| e(K_N) \|_{L^2(\Omega)} + \| b_1 \|_{L^2(\Omega)} L_1 \| e_N(x) \|_{L^2(\Omega)} \\ & + \| b_2 \|_{L^2(\Omega)} L_2 \| e_N(x) \|_{L^2(\Omega)} . \end{aligned}$$

Therefore,

$$\begin{aligned} \| e_N(x) \|_{L^2(\Omega)} & \bigg(1- \| b_1 \|_{L^2(\Omega)} L_1 - \| b_2 \|_{L^2(\Omega)} L_2 - \| a \|_{L^2(\Omega)} \tilde{K} L_{F_\mu}\bigg) \\ & \leqslant \| H_N(x) \|_{L^2(\Omega)} + \| a \|_{L^2(\Omega)} \tilde{F} \| e(K_N) \|_{L^2(\Omega)}, \end{aligned}$$

or

$$\| e_N(x) \|_{L^2(\Omega)} \leqslant \frac{\| H_N \|_{L^2(\Omega)} + \| a \|_{L^2(\Omega)} \tilde{F} \| e(K_N) \|_{L^2(\Omega)}}{1- \| a \|_{L^2(\Omega)} \tilde{K}\ L_{F_\mu} - \sum_{k=1}^2 L_k\ \ \| b_k \|_{L^2(\Omega)}}.$$

By assumption $1- \| a \|_{L^2(\Omega)} \tilde{K}\ L_{F_\mu} - \sum_{k=1}^2 L_k\ \ \| b_k \|_{L^2(\Omega)} > 0$, the desired result is aquired and easily follows from that the right hand side tends to zero when $N \longrightarrow \infty$. So, $e_N(x) = y - y_N \longrightarrow 0$. □

5. Illustrative examples

In this section, six examples are given to demonstrate the efficiency and accuracy of the proposed method. The maximum absolute errors are reported for different values of α and β parameters. Also, the absolute errors are calculated at some arbitrary selected points. All the results are calculated using the symbolic calculus software Maple 13.

5.1. Example. Consider the following singular integro–differential equation:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x) + \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt, \tag{5.1}$$

$$y(0) = -1, \qquad y'(0) = 1, \tag{5.2}$$

Table 1. Comparison of absolute errors errors of Example 5.1 for various values of α and β

(α,β)	Error_{Abs}	(α,β)	Error_{Abs}
$(0,0)$	3.4084×10^{-6}	$(1,1)$	5.3472×10^{-6}
$(2,2)$	9.1709×10^{-6}	$(\frac{1}{2},\frac{1}{2})$	3.7623×10^{-6}
$(-\frac{1}{2},\frac{1}{2})$	4.5621×10^{-6}	$(\frac{1}{2},-\frac{1}{2})$	2.1224×10^{-5}
$(-\frac{1}{3},-\frac{1}{3})$	7.6308×10^{-6}	$(\frac{1}{2},1)$	5.8128×10^{-6}
$(1,-\frac{1}{2})$	4.5273×10^{-5}	$(\frac{1}{3},\frac{3}{2})$	9.0113×10^{-6}

Table 2. Maximum absolute errors of Example 5.1 for values $N=10,15,20$, and $\alpha=\beta=0$

N	10	15	20
$Error_{Abs}$	3.4084×10^{-6}	9.6297×10^{-7}	2.2418×10^{-7}

where

$$p(x)=\frac{1-x}{x}-\left[\frac{1}{5}(1-x)+\frac{2}{3}\right]x^{\frac{3}{2}},\quad q(x)=\frac{1}{x}-\frac{x^{\frac{3}{2}}}{5}+2\sqrt{x}.$$

The exact solution of this problem is $y(x)=x-1$. First, let us consider the following approximation:

$$y''(x)\simeq\Phi^T(x)\,C. \tag{5.3}$$

The iterative integrating of Eq. (5.3) and using conditions (5.2) lead to the following approximations for $y'(x)$ and $y(x)$.

$$\begin{aligned} y'(x)&\simeq\Phi^T(x)\,\mathbf{P}^T\,C+\Phi^T(x)\,Y_1,\\ y(x)&\simeq\Phi^T(x)\,(\mathbf{P}^T)^2\,C+\Phi^T(x)\,\mathbf{P}^T\,Y_1+\Phi^T(x)\,Y_2, \end{aligned} \tag{5.4}$$

where

$$y'(0)\simeq\Phi^T(x)\,Y_1,\qquad y(0)\simeq\Phi^T(x)\,Y_2,$$

and $\mathbf{P}$ is the operational matrix of integration. By using the presented algorithm in Section 3 and above approximations, Eq. (5.1) is converted to the following linear algebraic equation:

$$\begin{aligned} &\Phi^T(x)\,C+p(x)\Phi^T(x)\{\mathbf{P}^T\,C+Y_1\}+q(x)\Phi^T(x)\{(\mathbf{P}^T)^2\,C+\mathbf{P}^T\,Y_1+Y_2\}\\ &\qquad-\mathbf{B}^{(\frac{1}{2})}\,\Phi^T(x)\{(\mathbf{P}^T)^2\,C+\mathbf{P}^T\,Y_1+Y_2\}-f(x)\approx0. \end{aligned} \tag{5.5}$$

The setting of $N=10$ and by using the roots of $P_{11}^{(\alpha,\beta)}(x)$, Eq. (5.5) is collocated at 11 points for different values of α and β parameters. Hereby, the solution of Eq. (5.1) is reduced to solving a system of linear algebraic equations and unknown coefficients can be obtained for some values of α and β parameters. Table 1 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 2 shows the values of the maximum absolute errors for $\alpha=\beta=0$ and $N=10,15,20$; the errors decrease as N increases. A comparison between the exact and an approximate solution obtained by the proposed method is shown in part (a) of Figure 1 for values of $\alpha=1/3$, $\beta=3/2$, and $N=10$. Also, part (b) of Figure 1 shows the absolute error function for $\alpha=1/3$, $\beta=3/2$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

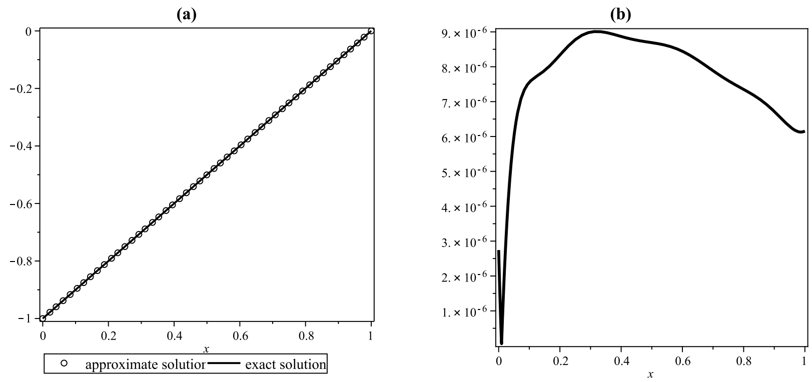

Figure 1. (a) Comparison of exact and approximate solutions, (b) absolute error function for $\alpha = \frac{1}{3}$, $\beta = \frac{3}{2}$, and $N = 10$ for Example 5.1

5.2. Example. Now, consider Eq. (5.1) with the following functional coefficients and initial conditions:

$$p(x) = \frac{1}{x} - \frac{13}{15}x^{\frac{3}{2}}, \qquad q(x) = -\frac{1}{x^2} + \frac{11}{5}\sqrt{x},$$

$$y(0) = y' = 0. \tag{5.6}$$

The exact solution of this problem is $y(x) = x^2$. By using the approximations (5.3) and (5.4) and conditions (5.6), Eq. (5.1) is converted to the following linear algebraic equation:

$$\begin{aligned} &\Phi^T(x)\ C + p(x)\Phi^T(x)\{\mathbf{P}^T\ C + Y_1\} + q(x)\Phi^T(x)\{(\mathbf{P}^T)^2\ C + \mathbf{P}^T\ Y_1 + Y_2\} \\ &\qquad - \mathbf{B}^{(\frac{1}{2})}\ \Phi^T(x)\{(\mathbf{P}^T)^2\ C + \mathbf{P}^T\ Y_1 + Y_2\} - f(x) \approx 0. \end{aligned} \tag{5.7}$$

The setting of $N = 10$ and by using the roots of $P_{11}^{(\alpha,\beta)}(x)$, Eq. (5.7) is collocated at 11 points for different values of α and β parameters. Hereby, the solution of Eq. (5.1) is reduced to solving a system of linear algebraic equations and unknown coefficients can be obtained for some values of α and β parameters. Table 3 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 4 displays different values of the exact and approximate solutions at points $x_i = 0.2i, (i = 1, 2, ..., 5)$, for $\alpha = \beta = 1$. As it is seen from Table 4, the results of the solutions obtained by the Jacobi polynomials method are almost the same as the results of the exact solutions. A comparison between the exact and an approximate solution obtained by the proposed method is shown in part (a) of Figure 2 for values of $\alpha = -1/2$, $\beta = 1/2$, and $N = 10$. Also, part (b) of Figure 2 shows the absolute error function for $\alpha = -1/2$ and $\beta = 1/2$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

5.3. Example. Consider the following singular integro–differential equation [24]:

$$y''(x) + y(x) + \int_0^x x\ y(t)\ dt + \int_0^x \frac{y(t)}{(x-t)^{\frac{1}{2}}} dt = 2 + x^2 + \frac{16}{15}x^{\frac{5}{2}} + \frac{x^4}{3}, \tag{5.8}$$

$$y(0) = y'(0) = 0. \tag{5.9}$$

Table 3. Comparison of absolute errors errors of Example 5.2 for various values of α and β

(α,β)	Error_{Abs}	(α,β)	Error_{Abs}
$(0,0)$	2.7527×10^{-8}	$(1,1)$	9.2896×10^{-8}
$(2,2)$	1.5469×10^{-7}	$(\frac{1}{2},\frac{1}{2})$	5.9518×10^{-8}
$(-\frac{1}{2},\frac{1}{2})$	7.3368×10^{-8}	$(\frac{1}{2},-\frac{1}{2})$	4.7167×10^{-9}
$(-\frac{1}{5},-\frac{1}{5})$	1.6105×10^{-8}	$(\frac{1}{3},3)$	3.1881×10^{-7}
$(2,-\frac{2}{3})$	8.6151×10^{-9}	$(-\frac{3}{4},-\frac{1}{3})$	9.4886×10^{-9}

Table 4. Values of exact and approximate solutions at arbitrary points for Example 5.2 for values $N=10$ and $\alpha=\beta=1$

x_i	exact values	approximate values	Error_{Abs}
0.2	0.04	0.039999981620	1.8380×10^{-8}
0.4	0.16	0.159999963454	3.6546×10^{-8}
0.6	0.36	0.359999944899	5.5101×10^{-8}
0.8	0.64	0.639999926340	7.3660×10^{-8}
1.0	1.00	0.999999907104	9.2896×10^{-8}

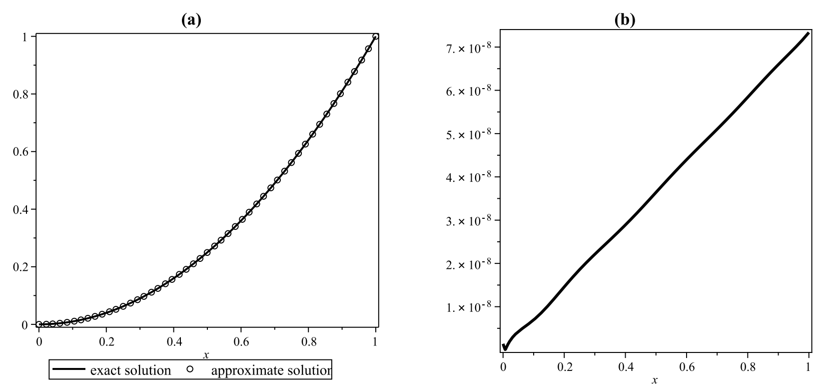

Figure 2. (a) Comparison of exact and approximate solutions, (b) absolute error function for $\alpha=-\frac{1}{2}$, $\beta=\frac{1}{2}$, and $N=10$ for Example 5.2

The exact solution is $y(x)=x^2$. By using the approximations (5.3) and (5.4) and conditions (5.9), Eq. (5.8) is converted to the following linear algebraic equation:

$$\begin{aligned}\Phi^T(x)\,C+\Phi^T(x)\{(\mathbf{P}^T)^2\,C+\mathbf{P}^T\,Y_1+Y_2\}+\Phi^T(x)\,K\,\tilde{V}\,\mathbf{P}\,\Phi(x)\\ +\mathbf{B}^{(\frac{1}{2})}\,\Phi^T(x)\{(\mathbf{P}^T)^2\,C+\mathbf{P}^T\,Y_1+Y_2\}-2-x^2-\frac{16}{15}x^{\frac{5}{2}}-\frac{x^4}{3}\approx 0,\end{aligned}\tag{5.10}$$

where K is the known matrix and is computed as $x\simeq\Phi^T(x)\,K\,\Phi(t)$ and $\tilde{V}$ is the operational matrix of product corresponding to the vector $V=(\mathbf{P}^T)^2\,C+\mathbf{P}^T\,Y_1+Y_2$. The setting of $N=7$ and by using the roots of $P_8^{(\alpha,\beta)}(x)$, Eq. (5.10) is collocated at 8 points for different values of α and β parameters. Hereby, the solution of Eq. (5.8) is

Table 5. Comparison of absolute errors errors of Example 5.3 for various values of α and β

(α,β)	Error$_{Abs}$	(α,β)	Error$_{Abs}$
$(0,0)$	4.3601×10^{-8}	$(1,1)$	3.8198×10^{-7}
$(2,2)$	6.4917×10^{-7}	$(\frac{1}{2},\frac{1}{2})$	2.1682×10^{-7}
$(-\frac{1}{2},\frac{1}{2})$	3.0737×10^{-7}	$(\frac{1}{2},-\frac{1}{2})$	9.3028×10^{-8}
$(-\frac{1}{3},-\frac{1}{2})$	1.1227×10^{-7}	$(\frac{2}{3},\frac{1}{5})$	9.5589×10^{-8}
$(2,\frac{3}{2})$	4.6612×10^{-7}	$(-\frac{3}{4},\frac{1}{3})$	2.3351×10^{-7}

Table 6. Absolute errors for various N and $\alpha=\beta=0$ for Example 5.3

x_i	error for $N=7$	error for $N=10$	error for $N=20$	error for $N=10$ in Ref. [24]	error for $N=20$ in Ref. [24]
0.0	6.5118×10^{-9}	7.0858×10^{-10}	6.5192×10^{-12}	0.0000	0.0000
0.2	7.0196×10^{-8}	1.8858×10^{-9}	1.5886×10^{-11}	0.1568×10^{-11}	0.0130
0.4	3.0091×10^{-8}	2.2747×10^{-9}	3.1050×10^{-11}	0.2064×10^{-11}	0.1598
0.6	3.0138×10^{-8}	3.9583×10^{-9}	4.6083×10^{-11}	0.2388×10^{-11}	0.3603
0.8	3.9085×10^{-8}	4.5432×10^{-9}	5.0761×10^{-11}	0.3244×10^{-11}	0.6611
1.0	3.2557×10^{-8}	5.2826×10^{-9}	2.9208×10^{-11}	0.6227×10^{-10}	0.8770

reduced to solving a system of linear algebraic equations and unknown coefficients can be obtained for some values of α and β parameters. Table 5 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 6 displays values of absolute errors at points $x_i = 0.2i, (i = 0, 1, ..., 5)$, for $\alpha = \beta = 0$ and $N = 7, 10, 20$. In Ref. [24], the errors of the approximate solutions obtained from the Bernstein polynomials method at $x_i = 0.2i$, $(i = 0, 1, ..., 5)$, and for $N = 10, 20$. The resultan errors are listed in the last columns in Table 6. As it is seen from Table 6, the results obtained by the Jacobi polynomials method have more errors than the results obtained by the Bernstein polynomials method for $N = 10$, but for $N = 20$, the resultant solutions of Jacobi operational method are more precise than those obtained by the Bernstein operational method in [24]. Also, a comparison between the exact and an approximate solution obtained by the proposed method is shown in part (a) of Figure 3 for values of $\alpha = \beta = 2$ and $N = 7$. Also, part (b) of Figure 3 shows the absolute error function for $\alpha = \beta = 2$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

5.4. Example. Consider the following second order singular integro–differential equation [24]:

$$y''(x) + y(x) + \frac{1}{\sqrt{\pi}}\int_0^x \frac{y''(t)}{(x-t)^{\frac{1}{2}}}dt = 3 + x + x^2 + \frac{4\sqrt{x}}{\sqrt{\pi}}, \tag{5.11}$$

$$y(0) = y'(0) = 1. \tag{5.12}$$

The exact solution is $y(x) = 1 + x + x^2$. By using the approximations (5.3) and (5.4) and conditions (5.12), Eq. (5.11) is converted to the following linear algebraic equation:

$$\begin{aligned}\Phi^T(x)\ C + \Phi^T(x)\{(\mathbf{P}^T)^2\ C + \mathbf{P}^T\ Y_1 + Y_2\} + \frac{1}{\sqrt{\pi}}\mathbf{B}^{(\frac{1}{2})}\ \Phi^T(x)\ C \\ - 3 - x - x^2 - \frac{4\sqrt{x}}{\sqrt{\pi}} \approx 0.\end{aligned} \tag{5.13}$$

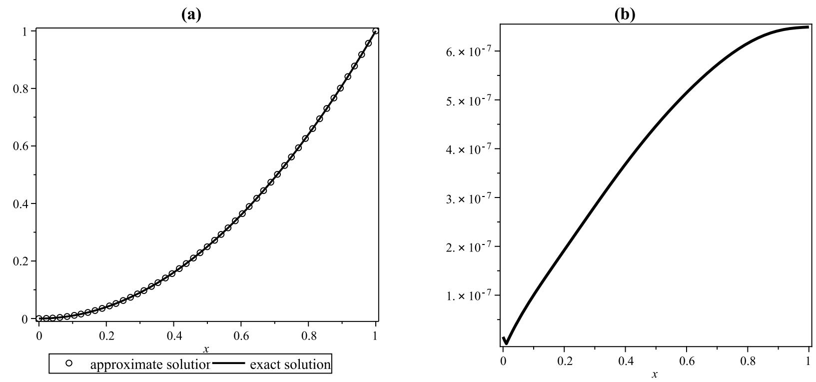

Figure 3. (*a*) Comparison of exact and approximate solutions, (*b*) absolute error function for $\alpha = \beta = 2$ and $N = 7$ for Example 5.3

Table 7. Comparison of absolute errors errors of Example 5.4 for various values of α and β

(α, β)	Error_{Abs}	(α, β)	Error_{Abs}
$(0, 0)$	7.7990×10^{-5}	$(1, 1)$	1.1873×10^{-4}
$(\frac{1}{2}, 1)$	1.2379×10^{-4}	$(\frac{1}{2}, \frac{1}{2})$	1.0623×10^{-4}
$(-\frac{1}{2}, \frac{1}{2})$	1.1699×10^{-4}	$(\frac{1}{2}, -\frac{1}{2})$	3.8610×10^{-5}
$(\frac{1}{3}, -\frac{1}{3})$	4.8554×10^{-5}	$(-\frac{1}{3}, -\frac{1}{5})$	5.8353×10^{-5}
$(1, -\frac{1}{2})$	7.5052×10^{-5}	$(\frac{3}{2}, 1)$	1.1487×10^{-4}
$(1, 3)$	1.3811×10^{-4}	$(-\frac{1}{4}, 3)$	1.5198×10^{-4}

The setting of $N = 10$ and by using the roots of $P_{11}^{(\alpha,\beta)}(x)$, Eq. (5.13) is collocated at 11 points for different values of α and β parameters. Hereby, the solution of Eq. (5.11) is reduced to solving a system of linear algebraic equations and unknown coefficients can be obtained for some values of α and β parameters. Table 7 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 8 displays values of absolute errors at points $x_i = 0.2i, (i = 0, 1, ..., 5)$, for $\alpha = -1/4$, $\beta = 3$, and $N = 4$. In Refs. [24, 25], the errors of the approximate solutions obtained by the Bernstein polynomials and Taylor expansion methods at $x_i = 0.2i$, $(i = 0, 1, ..., 5)$, and for $N = 4$. The resultan errors are listed in the third and forth columns in Table 8. As it is seen from Table 8, the resultant errors obtained by the Jacobi polynomials method are almost constant on the interval [0, 1], while the errors of the Bernstein and Taylor expansion methods increase in the end of the studied interval. A comparison between the exact and an approximate solution obtained by the proposed method is shown in part (*a*) of Figure 4 for values of $\alpha = 3/2$, $\beta = 1$, and $N = 10$. Also, part (*b*) of Figure 4 shows the absolute error function for $\alpha = 3/2$, $\beta = 2$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

5.5. Example. Consider the following first order singular integro–differential equation:

$$y'(x) = x\, y(x) + (2 - x)\exp(2x) - \frac{4}{3}x^{\frac{3}{4}} + \int_0^x \frac{\exp(-2t)y(t)}{(x-t)^{\frac{1}{4}}}dt, \tag{5.14}$$

Table 8. Absolute errors for various $N = 4$, $\alpha = -\frac{1}{4}$, and $\beta = 3$ for Example 5.4

x_i	error for $N = 4$	error for $N = 4$ in Ref. [24]	error for $N = 4$ in Ref. [25]
0.0	3.5094×10^{-5}	0.0000	0.0000
0.2	2.1208×10^{-4}	0.1452×10^{-4}	0.1372×10^{-3}
0.4	4.0433×10^{-4}	0.3665×10^{-4}	0.1500×10^{-2}
0.6	5.5568×10^{-4}	0.3186×10^{-4}	0.6300×10^{-2}
0.8	6.7594×10^{-4}	0.3220×10^{-3}	0.1720×10^{-1}
1.0	7.7072×10^{-4}	0.2100×10^{-2}	0.3690×10^{-1}

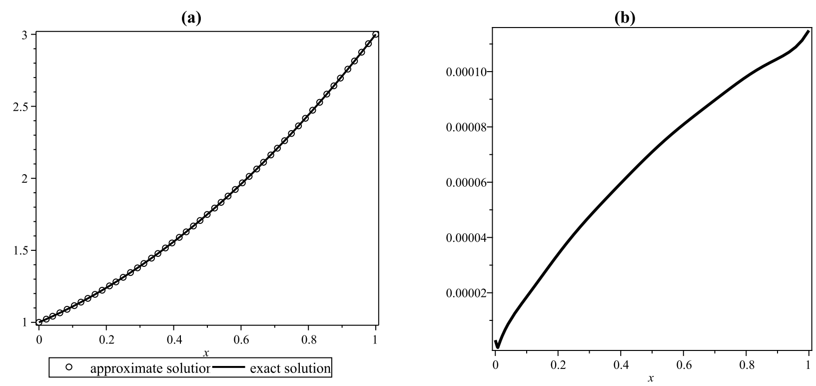

Figure 4. (a) Comparison of exact and approximate solutions, (b) absolute error function for $\alpha = \frac{3}{2}, \beta = 1$, and $N = 10$ for Example 5.4

$$(5.15) \quad y(0) = 1.$$

The exact solution is $y(x) = \exp(2x)$. According to condition (5.15), first consider the following approximations:

$$(5.16) \quad y'(x) \simeq \Phi^T(x)\ C, \qquad y(x) \simeq \Phi^T(x)\ \mathbf{P}^T\ C + \Phi^T(x)\ Y,$$

By using the approximations (5.16) and Theorem 3.5, Eq. (5.14) is converted to the following linear algebraic equation:

$$(5.17) \quad \Phi^T(x)\ C - x\ \Phi^T(x)\{\mathbf{P}^T\ C + Y\} - F\ \tilde{V}\ \mathbf{B}^{(\frac{1}{4})}\ \Phi(x) - (2-x)\exp(2x) + \frac{4}{3}x^{\frac{3}{4}} \approx 0,$$

where

$$\exp(2t) \simeq \Phi^T(t)\ F, \qquad V = \mathbf{P}^T\ C + Y, \qquad \Phi(x)\ \Phi^T(x)\ V \simeq \tilde{V}\ \Phi(x).$$

The setting of $N = 10$ and by using the roots of $P_{11}^{(\alpha,\beta)}(x)$, Eq. (5.17) is collocated at 11 points for different values of α and β parameters. Table 9 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 10 displays values of absolute errors at points $x_i = 0.2i, (i = 0, 1, ..., 5)$, for $\alpha = -1/2$, $\beta = 1$, and $N = 10$. A comparison between the exact and an approximate solution obtained by the proposed method is shown in part (a) of Figure 5 for values of $\alpha = 1/2$, $\beta = -1/2$, and $N = 10$. Also, part (b) of Figure 5 shows the absolute error function

Table 9. Comparison of absolute errors errors of Example 5.5 for various values of α and β

(α,β)	Error_{Abs}	(α,β)	Error_{Abs}
$(0,0)$	4.3543×10^{-5}	$(1,1)$	7.5706×10^{-5}
$(2,2)$	7.0969×10^{-5}	$(1,-\frac{1}{2})$	6.6520×10^{-4}
$(-\frac{1}{2},\frac{1}{2})$	6.6944×10^{-5}	$(\frac{1}{2},-\frac{1}{2})$	2.5105×10^{-4}
$(3,3)$	1.3462×10^{-2}	$(-\frac{1}{2},1)$	8.7132×10^{-5}

Table 10. Values of exact and approximate solutions at points x_i for $N=10$, $\alpha=-\frac{1}{2}$, and $\beta=1$ for Example 5.5

x_i	exact values	approximate values	error $_{Abs}$
0.0	1.00000000	0.99996967	3.0327×10^{-5}
0.2	1.49182470	1.49178150	4.3201×10^{-5}
0.4	2.22554093	2.22549277	4.8155×10^{-5}
0.6	3.32011692	3.32005794	5.8988×10^{-5}
0.8	4.95303242	4.95295924	7.3182×10^{-5}
1.0	7.38905610	7.38896381	9.2287×10^{-5}

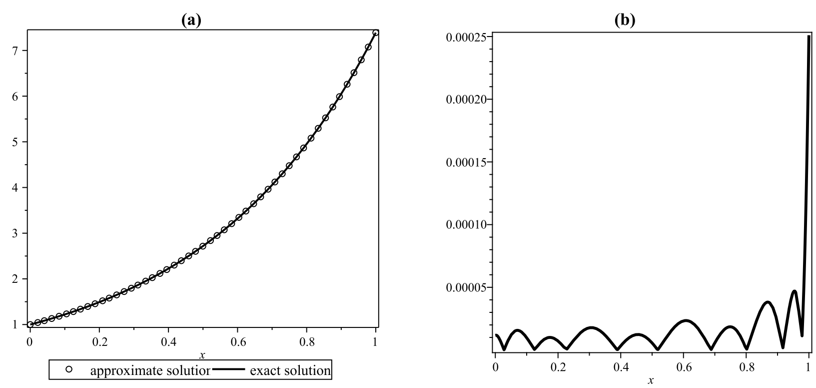

Figure 5. (a) Comparison of exact and approximate solutions, (b) absolute error function for $\alpha=\frac{1}{2},\beta=-\frac{1}{2}$, and $N=10$ for Example 5.5

for $\alpha=1/2$ and $\beta=-1/2$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

5.6. Example. In the current example, we consider the following nonlinear singular integro–differential equation:

$$y''(x)=\int_0^x \frac{(y(t))^2}{(x-t)^{\frac{1}{2}}}dt+f(x), \tag{5.18}$$

Table 11. Comparison of absolute errors errors of Example 5.6 for various values of α and β

(α,β)	Error_{Abs}	(α,β)	Error_{Abs}
$(0,0)$	6.9820×10^{-5}	$(1,1)$	1.2470×10^{-4}
$(2,2)$	1.4318×10^{-4}	$(\frac{1}{2},\frac{1}{2})$	1.0456×10^{-4}
$(-\frac{1}{2},\frac{1}{2})$	1.1503×10^{-4}	$(\frac{1}{2},-\frac{1}{2})$	1.0350×10^{-5}
$(-\frac{1}{3},0)$	7.2500×10^{-5}	$(-\frac{1}{2},\frac{1}{4})$	9.6780×10^{-5}

Table 12. Values of exact and approximate solutions at points x_i for $N = 15$, $\alpha = -\frac{1}{3}$, and $\beta = 0$ for Example 5.6

x_i	exact values	approximate values	error $_{Abs}$
0.0	1.00000000	1.00000075	7.4543×10^{-7}
0.2	1.22140276	1.22139335	9.4061×10^{-6}
0.4	1.49182470	1.49180515	1.9544×10^{-5}
0.6	1.82211880	1.82208723	3.1575×10^{-5}
0.8	2.22554093	2.22549307	4.7858×10^{-5}
1.0	2.71828183	2.71820933	7.2495×10^{-5}

where $f(x) = \exp(x) - \frac{\sqrt{2\pi}}{2}\exp(2x)erf(\sqrt{2x})$ and $erf(x)$ is the error function and is defined as $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt$, with initial conditions,

$$y(0) = y'(0) = 1. \tag{5.19}$$

The exact solution is $y(x) = \exp(x)$. By using the approximations (5.3), (5.4), conditions (5.19), and Theorem 3.6, Eq. (5.18) is converted to the following linear algebraic equation:

$$\Phi^T(x)\ C - V^T\ \tilde{V}\ \mathbf{B}^{(\frac{1}{2})}\ \Phi(x) - f(x) \approx 0, \tag{5.20}$$

where $V = (\mathbf{P}^T)^2\ C + (\mathbf{P}^T\ Y_1 + Y_2)$ and $(y(x))^2 \simeq V^T\ \tilde{V}\ \Phi(x)$. The setting of $N = 15$ and by using the roots of $P_{16}^{(\alpha,\beta)}(x)$, Eq. (5.20) is collocated at 16 points for different values of α and β parameters. Table 11 represents the maximum absolute (Abs) error of the approximate solutions for different values of α and β. Table 12 displays values of absolute errors at points $x_i = 0.2i, (i = 0, 1, ..., 5)$, for $\alpha = -1/3$, $\beta = 0$, and $N = 15$. A comparison between the exact and an approximate solution obtained by the proposed method is shown in part (a) of Figure 6 for values of $\alpha = -1/2$, $\beta = 1/4$, and $N = 15$. Also, part (b) of Figure 6 shows the absolute error function for $\alpha = -1/2$ and $\beta = 1/4$. It can be seen that the solution obtained from the present method is in good agreement with the exact solution.

6. Conclusion

In this paper, a simple approach has been developed to solve the weakly singular integro–differential equations by using Laplace transform and the Jacobi collocation method. Also, algorithms have been adapted to solve nonlinear integral equations. The proposed method compared to other methods, such as the Bernstein operational method and the Taylor expansion method, has lesser computational cost. The Jacobi polynomials are easily determined with the recurrence formula, Eq. (2.1). The resultant errors of the proposed method are almost constant over the interval $[0, 1]$; while the errors of the Bernstein and Taylor expansion methods increase at the end of the studied interval.

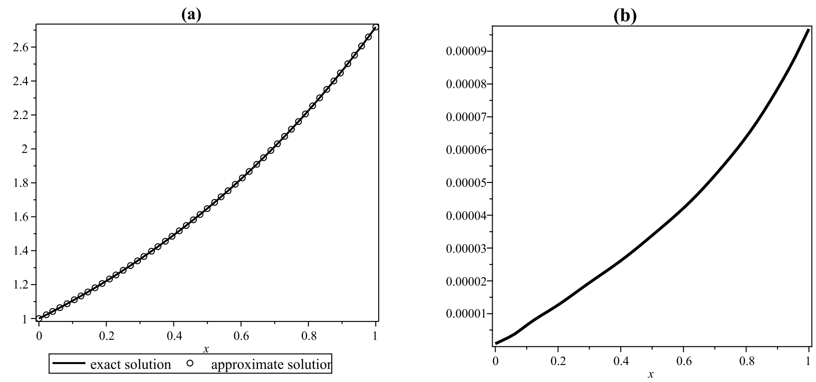

Figure 6. (a) Comparison of exact and approximate solutions, (b) absolute error function for $\alpha = -\frac{1}{2}, \beta = \frac{1}{4}$, and $N = 15$ for Example 5.6

Six examples are solved with the proposed method. The results demonstrate the better efficiency and accuracy of the Jacobi operational method compared to the methods used by others for the same order approximate solutions. In addition, the obtained results here verify that the Jacobi operational method is efficient to solve the linear and nonlinear systems and the approximate solutions are in agreement with the exact solutions. The proposed Laplace–Jacobi operational method is easy and efficient to obtain numerical solutions of singular integro–differential equations. In short, the Jacobi collocation method can be a powerful tool for investigating approximate and even analytic solutions for linear and nonlinear functional equations.

References

[1] Lamb, H., *Hydrodynamics*, 6th ed., Cambridge Univ. Press, London/New York, 1932.

[2] Carslaw, H.S. and Jaeger, J.C. *Conduction of Heat in Solids*, 2nd ed., Oxford Univ. (Clarendon) Press, London/New York, 1953.

[3] Ursell, F. J. Fluid Mech. 19, 305, 1964.

[4] Prosperett, A. Phys. Fluids. 19, 195, 1976.

[5] Prosperett, A. and Plesse, M.S. J. Fluid Mech. 85, 349, 1978.

[6] Tang, B.Q. and Li, X.F. *Solution of a class of Volterra integral equations with singular and weakly singular kernels*, Appl. Math. Comput. **199**, 406–413, 2008.

[7] Zozulya, V.V. and Gonzalez–Chi, P.I. *Weakly singular, singular and hypersingular integrals in 3–D elasticity and fracture mechanics*, J. Chin. Inst. Eng. **22**, 763–775, 1999.

[8] Kythe, P.K. and Puri, P. *Computational Methods for Linear Integral Equations*, Birkhäuser, Boston, 2002.

[9] Pedas, A. and Tamme, E. *Spline collocation method for integro–differential equations with weakly singular kernels*, J. Comput. Appl. Math. **197**, 253–269, 2006.

[10] Cao, Y., Huang, M., Liu, L. and Xu, Y. *Hybrid collocation methods for Fredholm integral equations with weakly singular kernels*, Appl. Numer. Math. **57**, 549–561, 2007.

[11] Pedas, A. and Tamme, E. *A discrete collocation method for Fredholm integro–differential equations with weakly singular kernels*, Appl. Numer. Math. **61**, 738–751, 2011.

[12] Pedas, A. and Tamme, E. *Product integration for weakly singular integro–differential equations*, Math. Model. Anal. **16**, 153–172, 2011.

[13] Kangro, R. and Tamme, E. *On fully discrete collocation methods for solving weakly singular integro–differential equations*, Math. Model. Anal. **15**, 69–8, 2010.

[14] Pedas, A. and Tamme, E. *Discrete Galerkin method for Fredholm integro–differential equations with weakly singular kernels*, J. Comput. Appl. Math. **213**, 111–126, 2008.

[15] Pedas, A. and Tamme, E. *Fully discrete Galerkin method for Fredholm integro–differential equations with weakly singular kernels*, Comput. Methods Appl. Math. **8**, 294–30, 2008.

[16] Lakestani, M., Nemati Saray, B. and Dehghan, M. *Numerical solution for the weakly singular Fredholm integro–differential equations using Legendre multiwavelets*, J. Comput. Appl. Math. **235**, 3291–3303, 2011.

[17] Parts, I., Pedas, A. and Tamme, E. *Piecewise polynomial collocation for Fredholm integro–differential equations with weakly singular kernels*, SIAM J. Numer. Anal. **43**, 1897–1911, 2005.

[18] Chen, Z. and Cheng, X. *An efficient algorithm for solving Fredholm integro–differential equations with weakly singular kernels*, J. Comput. Appl. Math. **257**, 57–64, 2014.

[19] Agarwal, R.P. *Boundary value problems for higher order integro–differential equations*, Nonlinear Anal. **7**, 259–270, 1983.

[20] Morchalo, J. *On two point boundary value problem for integro–differential equation of second order*, Fasc. Math. **9**, 51–56, 1975.

[21] Wazwaz, A.M. *A reliable algorithm for solving boundary value problems for higher–order integro–differential equations*, Appl. Math. Comput. **118**, 327–342, 2001.

[22] Babolian, E. and Shamloo, A.S. *Numerical solution of Volterra integral and integro–differential equations of convolution type by using operational matrices of piecewise constant orthogonal functions*, J. Comput. Appl. Math. **214**, 495–508, 2008.

[23] Borhanifar, A. and Sadri, K. *A new operational approach for numerical solution of generalized functional integro–differential equations*, J. Comput. Appl. Math. **279**, 80–9 2015.

[24] Rasit Isik, O., Sezer, M. and Guney, Z. *Bernstein series solution of a class of linear integro–differential equations with weakly singular kernel*, Appl. Math. Comput. **217**, 7009–7020, 2011.

[25] Huang, Y. and Li, X.F. *Approximate solution of a class of linear integro–differential equations by Taylor expansion method*, Int. J. Comput. Math. 1–12, 2009.