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Quality and pricing decisions for substitutable items under imperfect production process over a random planning horizon

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Abstract

The paper determines the optimum qualities and prices of two substitute products for a manufacturer cum retailer in an imperfect production process over a random planning horizon for maximum profit. In this Economic Production Lot-size (EPL) process, items are produced simultaneously, defective production commences during the out-of-control state after the passage of some time from the commencement of production and the defective units are partially reworked. The items are substitutable to each other depending on their prices and qualities jointly or either of these two. Unit production cost depends directly on raw-material, labour and quality improvement costs and inversely on the production rate. A part of it is spent against environment protection. Here learning effect is introduced in the set-up and maintenance costs. For the whole process, the planning horizon is random with normal distribution, which is treated as a chance constraint. The models are formulated as profit maximization problems subject to a chance constraint and solved using Genetic Algorithm with Variable Populations (GAVP). The models are demonstrated numerically and the near-optimum results are presented graphically.

Keywords: Imperfect production, Out-of-control sate, Price and quality dependent substitution, Quality improvement cost, Environment Protection cost, Random horizon, Genetic Algorithm.

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1. Introduction

It is a fact that the demand of an item is influenced by the selling price of that item i.e. whenever the selling price of an item increases, the demand of that decreases and vice-verse. Mondal et al. [27] investigated the inventory system of ameliorating items for price dependent demand. Liang and Zhou [20] solved two warehouse inventory models for deteriorating items with price dependent demand. Maiti et al.[22] introduced the concept of advanced payment for determining the optimal ordering policy under stochastic lead-time and price dependent demand.

Now-a-days, due to strong competitive market, retailers prefer to do the business / production of several items with the hope that due to dull market, if one item does not fetch profit, the other one will save the situation. There are several investigations for substitutable items in the newsboy setting. Abdel-Maleka and Montanari [1] analysed a multi-product newsboy problem with a budget constraint. Das and Maiti [10] studied a single period newsboy type inventory problem for two substitutable deteriorating items with resource constraint involving a wholesaler and several retailers. Stavrulaki [39] modelled the joint effect of demand (stock-dependent) stimulation and product substitution on inventory decisions by considering a single period and stochastic demand. Gurler and Yilmaz [13] assumed substitution of a product when the other one is out of stock and presented a two level supply chain newsboy problem with two substitutable products. Kim and Bell [18] investigated the impact of the symmetrical and asymmetrical demand substitution on optimal prices, production levels and revenue and the impact of changes in the production cost on the optimal solutions. Recently Zhao et al.[45] developed a two-stage supply chain where two different manufacturers compete to sell substitutable products through a common retailer and analysed the problem using game theory. Here the consumer demand function is defined as a linear form of the two products' retail prices-downward slopping in its own price and increasing with respect to the competitor's price. In marketing substitutable items, the demand of an item is sometimes affected by the other, depending upon the other item's inventory level (Maity & Maiti, [24]). Ahiska and kurtul [3] presented a one-way product substitution strategy for a stochastic manufacturing/ re-manufacturing system and illustrated using real life data.

Rosenblatt and Lee [35] studied the effects of an imperfect production process on the optimal production run time by assuming that time to out-of-control state is exponentially distributed. Hu et al.[14] obtained optimal production run length for imperfect production processes in fuzzy-random environment allowing back-orders. Sana [37] presented an EPL model with random imperfect production process and defective units were repaired immediately when they were produced. Sarkar et al. [38] obtained the optimal reliability for an EPL model connecting process reliability with imperfect production system. Not all of the defective items are repairable, a portion of them are scrap and discarded beforehand. Recently Chen [6] investigated a problem of production preventive maintenance, inspection and inventory for an imperfect production process. Pal et al. [30] presented an EPQ model with imperfect production and stochastic demand. Very recently, Paul et al. [32] outlined joint replenishment policy for imperfect items allowing price discount. Rad et al. [33] also obtained optimized inventory and sales decisions for a two stage-supply chain with imperfect production process allowing back-orders.

Classical inventory models are usually developed over the infinite planning horizon. According to Chung and Kim [9], the assumption of infinite planning horizon is not realistic due to several reasons such as variation of inventory costs, change in product specifications and designs, technological development, etc. Moreover, for seasonal products like fruits, vegetables, warm garments, fashionable goods, etc., business period is not infinite, rather fluctuates with each season. Hence the planning horizon for these products varies over the years depending upon the environmental effects. Therefore, it is better to estimate this type of products with finite time horizon as in nature. Moon and Yun [29] and Guria et al. [12] developed an EOQ model in random planning horizon.

All the production inventory models emit green house gases, specially CO_2 in the atmosphere. Several authors have outlined production inventory policies for minimum cost/maximum profit with the constraints on carbon emission. A multi-sourcing deterministic lot-sizing model with carbon constraint is investigated by Absi et al. [2]. An EOQ model with a constraint on the emission of carbon is considered by Chen et al. [7]. They concluded that without increasing cost significantly carbon emissions be reduced through operational adjustment. A multi-item production planning problem with carbon cap and trade policy, where the firm can buy or sell the right to emit carbon on a carbon trading market, was investigated by Zhang and Xu [44]. Recently, Jin et al. [15] studied the impact of three carbon emissions reduction policies including cap and trade and carbon tax regulations on a major retailer determining its supply network design and choice of transportation. Zakeri et al. [43] presented an analytical supply chain planning model that can be used to examine the supply chain performance at the tactical/operational planning level under carbon pricing and trading schemes. Swami and Shah [40] and Ghosh and Shah [11] introduced green supply chain system explore the impact of cost sharing contract on the green initiative decisions of supply chain members. In the first paper [40], they exhibited that greening effects by the manufacturer and retailer result in demand expansion at the retail end. It is also pointed out that profits and efforts are higher in the integrated channel as compared to the case of the disconnected channel. In the second paper [11] two models of cost sharing- one in which the retailer offers a cost sharing contract and the other in which retailer and manufacturer gain on the share, are considered with linearly price and product greening improvement level dependent demand.

Unit production cost has been assumed constant in some EPL models. In reality, it depends on several factors such as the raw materials, labours engaged, rate of production, product's quality and environment protection, etc. Khouja and Mehrez [16] assumed a unit production cost involving costs of raw materials, labours and wear and tear of components. After that, several authors (Mandal et al. [25]) have implemented this in their EPL models. In a production system, better machinery and control systems, expert labours, etc. are required to have the quality of product. So, unit production cost varies directly with the product's quality. Moreover, in every manufacturing process, it is fact that environment is polluted to some extent and for that, now-a-days attention is paid not to pollute the environment taking some measures for it. This involves some expenditures and hence unit production cost increases with this process [4]. So far, these considerations are ignored by the researchers.

In spite of all these developments in imperfect EPL models, there are still some lacunas in making the models more realistic. These are

- None has considered the production-marketing system for substitutable products under imperfect production process introducing learning effect in the set-up and maintenance costs.

- There is no production systems (inventory) management research and the pricing decisions with product substitution depending on the joint effect of price and quality or on the basis of either price or quality.
- Unit production cost is normally assumed to be dependent on the raw material and labour costs. But none have considered that quality improvement cost which is a function of quality of an item, is a part of unit production cost.
- Several authors [2, 40, 7, 15, 43, 11] have studied the environmental effect on the production inventory/inventory management systems, mainly considering the carbon emission or product greening improvement. But, none introduce the environment protection cost (EPC) for EPL models, which is again varies with the rate of production.
- In the literature, there is no model for substitutable products formulated over a random planning horizon.

Therefore, there is a strong motivation for further research in this area. Hence, in this investigation, we consider all the above lacunas and formulate an imperfect substitutable multi-item production-inventory model with selling price and quality dependent demand, partially reworked, disposal of not reworkable defective units incorporating environmental protection cost over a finite time horizon. In real life EPL models, a production system remains in control at the beginning and after some time, it goes to out-of-control state and then defective units are produced. The unit production cost has four components- the raw material cost to produce an unit, labour cost per unit production and quality improvement and environment protection costs. Here demands of the substitute products are defined as linear functions of the products' selling prices and qualities. The demand of a merchandise has downward slopping in its own price and increasing with respect to the competitor's price. It is reversed with respect to quality e.g. increases in its own quality and decreases for other's quality. There may be different relations amongst the coefficients of demand functions. The models are formulated as profit maximization problems in which number of cycles, selling prices, production rates and qualities are decision variables. With the different relations in demand functions, it is solved by using GAVP. The models are illustrated with numerical examples and some results are presented graphically.

The remainder of this paper is organised as follows. Section 2 describes different types of demand functions and formulates the models. Section **??** presents the solution methodology. Models are illustrated numerically in Section 4. Section 5 gives the discussion about the model's results. Section 6 outlines the practical implication and conclusion is derived in Section7.

2. Model formulation

2.1. Notations for the proposed models. *The following notations are used for i-th product to develop the proposed models:*

Decision variables:

m_i	Number of cycles in a planning horizon
M_i	Mark-up for a perfect unit
P_i	Production rate in units per unit time
q_i	Level of quality of a product, $\beta_i \leq q_i \leq 1$ where β_i is the minimum quality level of i-th product, which manufacturer intends to maintain

Parameters:

$\bar{H}$	The length of the finite planning horizon which is random with a normal distribution(m_h, σ_h)
T_i	Cycle time in appropriate unit
τ_i	Time (measured from the commencement of production), at which defective unit production begins. i.e., the beginning time of the "out-of-control" state
c_{hi}	Holding cost per unit per unit time
c_{di}	Price of disposal for an imperfect unit
c_{ri}	Cost to rework an imperfect unit
λ_i	Constant production rate of defective units per unit production. The machine produces imperfect units at this rate when the machinery system is in "out-of-control" state
θ_i	Percentage of rework of defective units
r_{mi}	Cost of raw materials required to produce an unit
d_{i0}	Market based original / prime demand, not taking effects of its own and substitute product's prices and qualities
d_{i1}, d_{i2}	Measures of responsiveness of each product's consumer demand to its own price and competitor's price respectively
d_{i3}, d_{i4}	Measures of responsiveness of each product's consumer demand to its own quality and competitor's quality respectively

Dependent variables:

$I_i(t)$	Inventory level at time t
t_i	Production run-time in one cycle
$C_i(P_i, q_i)$	Unit production cost
Cs_{ij}	The set up cost for jth cycle
Cm_{ij}	The maintenance cost for jth cycle
N_i	Defective units in a production cycle
s_i	Selling price per unit perfect product. It is mark-up of raw material cost. i.e., $s_i = M_i r_{mi}$
D_i	Resultant demand in the market. This is the demand of a product after taking influence of prices and qualities of its own and substitute product
Q_i	Total inventory unit for a single production

HC_i, PC_i, RC_i, SC_i, MC_i and TC_i are the total holding, production, reworking, set-up, maintenance and relevant total costs during (0, $\bar{H}$) respectively.
PSR_i, DSR_i, TSR_i and TP_i are the sales revenue for perfect units, sales revenue for imperfect units which are not reworked, total sales revenue and total profit during (0, $\bar{H}$) respectively.
$ds_{p_i}(= d_{i1} - d_{i2}), ds_{q_i}(= d_{i4} - d_{i3})$ are proportional to Inverse Of Degree Of Substitutability (IODOS) due to price and quality respectively.
$Dp_i(= -d_{i1}s_i + d_{i2}s_j)$, $Dq_i(= d_{i3}q_i - d_{i4}q_j)$, where $j = 1, 2,\ j \neq i$ are amount of substitution demand rates due to price and quality. Here the above variables and parameters are taken in appropriate units.

2.2. Assumptions for the proposed models. *The following assumptions are used to develop the proposed models:*

(1) Multi-product imperfect production inventory models are considered. Products are substitutable depending on their prices and qualities jointly or either of these two. Here prices and qualities are assumed to be independent to each other.
(2) Finite time planning horizon (random) is considered.
(3) Production rate is finite and taken as a decision variable.
(4) Lead time is zero and no shortages are allowed.
(5) The inventory system considers price and quality dependent demand rate.
(6) The production process shifts from "In-control" state to "Out-of-control" state after a certain time. Imperfect units are produced at a constant rate per unit production in the "Out-of-control" state only.
(7) There is immediate partially reworking for the defective units at a cost and the defective units which are not reworked, are sold at a lower price.
(8) Unit production cost is dependent on raw material, labour and quality improvement cost and one part of it is also spent for environment protection.

(9) A maintenance cost is considered for the production system of each product to bring back the system to its initial condition by some maintenance operations (these may be mechanical, electrical, technical, replacement of parts, etc.) during the each time gap between the end of production and beginning of next production.

(10) "Fully substitution" means the loss of customers for a product is equal to the gain of its competitor product.

(11) The sum of resultant demands of all substitutable products after substitution does not exceed the total market based (i.e. prime) demands of the products.

(12) For any type of multi-product substitution, there is either loss of customers or fully substitution (i.e. no loss of sales) for the system if and only if the sum of resultant demands is either strictly less or equal to the total market based demand respectively.

(13) During substitution, demand of a product is more or equally sensitive to the changes due to its own price than the changes due to its competitor's price.

(14) During substitution, loss of customers of product -1 due to its own price is more or equal than the gain of customers of product-2 due to the price of product-1.

(15) During substitution, demand of a product is less or equally sensitive to the changes due to its own quality than the changes due to its competitor's quality.

(16) During substitution, loss of customers of product -1 due to its own quality is more or equal than the gain of customers of product -2 due to the quality of product -1.

2.3. Demands based on price dependent substitution. In the case of only price dependent substitutable products, original demand of a product decreases for the increase of its own price and at the same time, it gets some additional customers due to its competitor's price. Thus, the resultant demands of the two substitutable merchandises can be expressed as
$D_i(s_i, s_j) = d_{i0} - d_{i1}s_i + d_{i2}s_j,\ i, j = 1, 2,\ j \neq i.$
where D_i is the Resultant Demand (RD) for i-th product at price s_i given that the price of the other product j is s_j. Here, the range of selling price of i-th product is assumed as $r_{mi} \leq s_i \leq d_{i0}/d_{i1}$.

$$\begin{aligned} &i.e. \quad D_1(s_1, s_2) = d_{10} - d_{11}s_1 + d_{12}s_2, r_{m1} \leq s_1 \leq d_{10}/d_{11}, \\ &Similarly,\ D_2(s_2, s_1) = d_{20} - d_{21}s_2 + d_{22}s_1, r_{m2} \leq s_2 \leq d_{20}/d_{21}. \end{aligned} \tag{2.1}$$

where $d_{i0}s\,(> 0)$, i=1, 2; represent the market based prime demand of product i. d_{i1}, $d_{i2}\,(> 0)$, i=1, 2; denote the measures of the responsiveness of each product's consumer demand to its own price and to its competitor's price respectively. These parameters d_{i0}, d_{i1} and d_{i2} are mutually independent and non negative. According to assumptions 13 and 14, they satisfied the conditions $d_{11} \geq d_{12}$, $d_{21} \geq d_{22}$, $d_{11} \geq d_{22}$ and $d_{21} \geq d_{12}$. The difference $d_{11} - d_{12}(= ds_{p_1})$ is inversely related to the degree of substitutability (IODOS) of the 1st product with respect to the 2nd product. If this difference is smaller, then the product-1 is more substitutable with the 2nd product. i.e. product-1 is less differentiable. Hence the price of the product is higher. Same is true for the 2nd product with the difference $d_{21} - d_{22}(= ds_{p_2})$.

The ranges of limit of selling prices of i-th merchandise are determined on the basis of two realistic requirements- (i) It should be more than the raw material cost per unit product and (ii) less than $\frac{d_{i0}}{d_{i1}}$ as loss of customers due to i-th product's price should be less than or equal to its original demand $(d_{i0} - d_{i1}s_i \geq 0)$.

Proposition-1. For two products substitutable under price with demands (2.1), there is loss of sales (i.e. customers) or no loss in the system if and only if

$$s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) > 0 \text{ or } = 0 \text{ respectively.} \tag{2.2}$$

Proof. Necessary part: Let us assume that for any two substitutable products, only loss of sales or fully substitution case can arise. Therefore, from assumption 12 we have, Sum of resultant demands $\leq$ Total market based demand.

$$\begin{aligned}
&\text{i.e., } D_1 + D_2 \leq d_{10} + d_{20}\\
&\text{or, } d_{10} - d_{11}s_1 + d_{12}s_2 + d_{20} - d_{21}s_2 + d_{22}s_1 \leq d_{10} + d_{20}\\
&\text{or, } s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0.
\end{aligned}$$

Therefore, the necessary part is complete.
Sufficient part: Let $s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0$.

$$\begin{aligned}
\text{Sum of resultant demands} \quad &= D_1 + D_2\\
&= d_{10} - d_{11}s_1 + d_{12}s_2 + d_{20} - d_{21}s_2 + d_{22}s_1\\
&= d_{10} + d_{20} - [s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12})]\\
&\leq d_{10} + d_{20}, \text{ since } s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0.\\
\text{i.e., Sum of resultant demands} \quad &\leq \text{ Total market based demand.}
\end{aligned}$$

Thus it is concluded that $s_1(d_{11} - d_{22}) + s_2(d_{21} - d_{12}) \geq 0$ is the condition to be satisfied for the above assumption. Thus the sufficient part is complete.
Hence the Proposition.

2.4. Demands based on quality dependent substitution. In the case of only quality dependent substitution, demand of an product increases due to increase of its own quality and at same time, it looses some customers due to its competitor's quality. Thus, the RD functions for the two substitutable products are expressed as

$$D_1(q_1, q_2) = d_{10} + d_{13}q_1 - d_{14}q_2,$$

$$D_2(q_2, q_1) = d_{20} + d_{23}q_2 - d_{24}q_1; \beta_i \leq q_i \leq 1 \; for\, i = 1,\, 2. \tag{2.3}$$

where d_{i3}, d_{i4} (> 0), i=1, 2; denote the measures of the responsiveness of each product's consumer demand to its own quality and to its competitor's quality respectively. These parameters d_{i0}, d_{i3} and d_{i4} are mutually independent and non negative. According to assumptions 15 and 16, they satisfied the conditions $d_{13} \leq d_{14}$, $d_{23} \leq d_{24}$, $d_{13} \leq d_{24}$ and $d_{23} \leq d_{14}$. The difference $d_{14} - d_{13} (= ds_{q_1})$ is inversely related to the degree of substitutability (IODOS) of the 1st product with the 2nd product. If this difference is smaller, the the product -1 is more substitutable with the 2nd product. i.e. product -1 is less differentiable. Same is true for the 2nd product with the difference $d_{24} - d_{23} (= ds_{q_2})$. Here it is assumed that qualities q_1 and q_2 lies within $[\beta_i,\, 1.0]$.

Proposition-2. For two substitutable products under quality with demands (2.3), there is loss of sales (i.e. customers) or no loss in the system if and only if

$$q_1(d_{13} - d_{24}) + q_2(d_{23} - d_{14}) < 0 \text{ or } = 0 \text{ respectively.} \tag{2.4}$$

Proof. Proceeding as proposition-1, this proposition can be proved.

2.5. Demands based on both price and quality dependent substitution. Here, we assume that price and quality of a product are independent to each other. Then, in the case of both price and quality dependent substitutable items, the original demand of an item is downward slopping in its own price and at same time, it gets some additional customers due to its competitor's price. It is reversed with respect to quality e.g. increases in its own quality and decreases for other's quality. Thus, RDs of the substitutable items on joint effect of price and quality can be expressed as

$$\begin{aligned} &D_1(s_1,s_2,q_1,q_2) = d_{10} - d_{11}s_1 + d_{12}s_2 + d_{13}q_1 - d_{14}q_2, \\ (2.5)\quad &D_2(s_1,s_2,q_1,q_2) = d_{20} - d_{21}s_2 + d_{22}s_1 + d_{23}q_2 - d_{24}q_1 \\ &\text{with } r_{mi} \le s_i \le \frac{d_{i0}}{d_{i1}} \text{ and } \beta_i \le q_i \le 1. \end{aligned}$$

where d_{i0}, d_{i1}, d_{i2}, d_{i3}, d_{i4} for i=1, 2 have the meanings as earlier. These parameters are mutually independent.

Proposition -3. For two substitutable products under both price and quality with demands (2.5), there is loss of sales (i.e. customers) or no loss in the system if and only if

(2.6) $[-s_1(d_{11}-d_{22})-s_2(d_{21}-d_{12})+q_1(d_{13}-d_{24})+q_2(d_{23}-d_{14})] < 0$ or $= 0$ respectively.

Proof. The proof is similar as propositions -1 and -2.

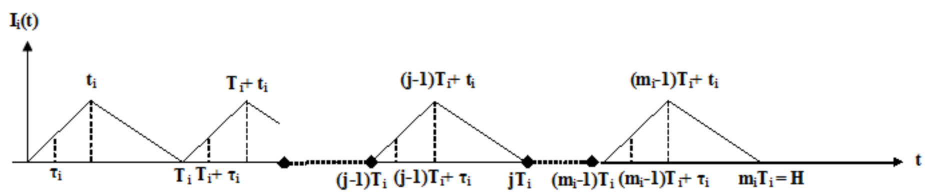

Figure 1. Inventory versus time for ith item.

2.6. Model development. In this investigation, an imperfect EPL model for i-th item is assumed over a finite random planning horizon of length $\bar{H}$ in which time m_i number of full cycles are completed. In this production process, for j-th cycle, the production starts with a rate P_i at time $t = (j-1)T_i$ and runs up to time $t = (j-1)T_i + t_i$. The system produces perfect quality units up to a certain time $(j-1)T_i+\tau_i$ (i.e., in-control state), after that, the production system shifts to an "out-of-control" state $[(j-1)T_i+\tau_i, (j-1)T_i+t_i]$. In this "out-of-control" state, some of the produced units are of non-conforming quality (i.e., defective units) and some of these defective units are reworked immediately. The inventory piles up, during the time interval $[(j-1)T_i, (j-1)T_i+t_i]$ adjusting demand D_i in the market and the production and reworking processes produce perfect product Q_i units upto time $t = (j-1)T_i + t_i$, i.e., when the system stops the production. The stock at $t = (j-1)T_i + t_i$ is depleted satisfying the demand D_i in the market and it reaches zero level at time jT_i (cf. Fig. 1). After the end of one production run, we assume that the machinery system is maintenanced against a cost and brought back to its original good condition before the next production.
For the multi-item imperfect production process with different demand functions, the governing differential equations for the j-th cycle of i-th (i=1,2) item are:

$$(2.7)\quad \frac{dI_i(t)}{dt} = \begin{cases} P_i - D_i, & (j-1)T_i \le t \le (j-1)T_i + \tau_i \\ P_i - D_i - (1-\theta_i)\lambda_i P_i, & (j-1)T_i + \tau_i \le t \le (j-1)T_i + t_i \\ -D_i, & (j-1)T_i + t_i \le t \le jT_i \end{cases}$$

with the boundary conditions

$$\begin{cases} I_i(t) = 0, & at \;\; t = (j-1)T_i \\ I_i(t) = 0, & at \;\; t = jT_i \end{cases}$$

The solutions of the above differential equations are :

$$(2.8) \qquad I_i(t) = \begin{cases} (P_i - D_i)\{t - (j-1)T_i\}, & (j-1)T_i \le t \le (j-1)T_i + \tau_i \\ (P_i - D_i)\{t - (j-1)T_i\} - (1-\theta_i)\lambda_i P_i\{t - (j-1)T_i - \tau_i\}, & \\ & (j-1)T_i + \tau_i \le t \le (j-1)T_i + t_i \\ D_i(jT_i - t) & (j-1)T_i + t_i \le t \le jT_i \end{cases}$$

where $t_i = \frac{D_i T_i - (1-\theta_i)\lambda_i P_i \tau_i}{P_i\{1-(1-\theta_i)\lambda_i\}}$ and $Q_i = P_i t_i - (1-\theta_i)\lambda_i P_i (t_i - \tau_i)$

2.6.1. *Holding cost.* The total holding cost in the time horizon H is $HC_i = \sum_{j=1}^{m_i} c_{hi}(Ih_{1i} + Ih_{2i} + Ih_{3i})$ where,

$$\begin{aligned} Ih_{1i} &= \int_{(j-1)T_i}^{(j-1)T_i+\tau_i} I_i(t)dt = \int_{(j-1)T_i}^{(j-1)T_i+\tau_i} (P_i - D_i)\{t - (j-1)T_i\}dt = \frac{P_i - D_i}{2}\tau_i^2. \\ Ih_{2i} &= \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} I_i(t)dt \\ &= \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} [(P_i - D_i)\{t - (j-1)T_i\} - (1-\theta_i)\lambda_i P_i\{t - (j-1)T_i - \tau_i\}]dt \\ &= \frac{P_i - D_i}{2}(t_i^2 - \tau_i^2) - \frac{(1-\theta_i)\lambda_i P_i}{2}(t_i - \tau_i)^2. \\ Ih_{3i} &= \int_{(j-1)T_i+t_i}^{jT_i} I_i(t)dt = \int_{(j-1)T_i+t_i}^{jT_i} D_i(jT_i - t,)dt = \frac{D_i}{2}(T_i - t_i)^2. \end{aligned}$$

2.6.2. *Rework cost.* The total rework cost (RC_i) in the time horizon H is $RC_i = \sum_{j=1}^{m_i} c_{ri}\theta_i N_i$, where N_i are the defective units during $[(j-1)T_i + \tau_i, (j-1)T_i + t_i]$ for i=1,2 and expressed as

$$N_i = \int_{(j-1)T_i+\tau_i}^{(j-1)T_i+t_i} \lambda_i P_i dt = \lambda_i P_i (t_i - \tau_i).$$

2.6.3. *Production cost.* Unit production cost is considered for i-th item (i=1,2) as

$$C_i(P_i, q_i) = r_{mi} + \frac{g_{1i}}{P_i} + \frac{g_{2i}q_i}{1 - a_i q_i} + g_{3i}P_i^{\frac{1}{2}},$$

where r_{mi} is the raw material cost per unit item, g_{1i} is the total labour/energy costs per unit time in a production system which is equally distributed over the unit item. So,$(\frac{g_i}{P_i})$ decreases with increases of P_i. The third term $\frac{g_{2i}q_i}{1-a_i q_i}$ is quality improvement cost, proportional to the positive power of quality of a product and the fourth term $g_{3i}P_i^{\frac{1}{2}}$ is environment protection cost assuming that the cost due to the measures taken for the environment protection is proportional to square root of production rate P_i, where the power term varies with the nature of production firms.
Therefore, the total production cost for i-th item is

$$PC_i = \sum_{j=1}^{m_i} C_i(P_i, q_i)P_i t_i.$$

2.6.4. *Setup cost.* Some researchers [5, 31, 8] considered the learning effect modelling into the set up cost in different forms. Here, the set up cost for j-th cycle ($j = 1, 2, ..., m_i$) of i-th item (i=1,2) is partly constant and partly decreases in each cycle due to learning effect of the employees and is of the form: $Cs_{ij} = C_{s0i} + C_{s1i}e^{-jc_i}$, where $c_i > 0$. Therefore total set up cost for m_i number of cycle is

$$SC_i = \sum_{j=1}^{m_i} Cs_{ij} = m_i C_{s0i} + C_{s1i}\frac{1 - e^{-m_i c_i}}{e^{c_i} - 1}.$$

2.6.5. *Maintenance cost.* Maintenance cost for the machinery system is used to bring the system to its original position after the end of each production. In Tarakci et al. [41], a manufacturer contracts to an external contractor who is responsible for scheduling and performing preventive maintenance and carrying out minimal repairs when the process fails. Here, learning occurs in both cost and time of preventive maintenance. For the first cycle no maintenance is required, but for the next cycles on wards, it is increased in each cycle due to the reuse of the system for several times. Maintenance cost for j-th cycle of the i-th item is taken as: $Cm_{ij} = C_{m0i}[1 - e^{-(j-1)c'_i}]$, where $c'_i > 0$. Therefore total maintenance cost for m_i number of cycle is

$$MC_i = \sum_{j=1}^{m_i} Cm_{ij} = C_{m0i}[m_i - \frac{1 - e^{-m_i c'_i}}{1 - e^{-c'_i}}].$$

2.6.6. *Total relevant model cost.* As a result, the total model cost = Holding cost + Rework cost + Production cost + Set-up cost + Maintenance cost.

(2.9) $\quad$ i.e. $TC_i = HC_i + RC_i + PC_i + SC_i + MC_i.$

2.6.7. *Total sale revenue.*
Revenue for perfect units: Total sales revenue of perfect products for m_i number of cycles is

$$PSR_i = \sum_{j=1}^{m_i} s_i \int_{(j-1)T_i}^{jT_i} D_i dt = \sum_{j=1}^{m_i} s_i D_i T_i,$$

where $s_i = M_i r_{mi}$ is the selling price of each product which is mark-up of raw material cost r_{mi}. **Sales Revenue for imperfect units:** The defective products which are not to be reworked is disposed by a lower price and total sales revenue for m_i number of cycles is
$DSR_i = \sum_{j=1}^{m_i} c_{di}(1 - \theta_i)N_i$, where $c_{di} = x_i s_i$, $0 < x_i < 1$.
Therefore, total sales revenue for this model is

(2.10) $\quad TSR_i = PSR_i + DSR_i.$

2.6.8. *Total profit.* Total profit during the whole planning horizon for i-th item is

(2.11)

$$\begin{aligned}
&TP_i = TSR_i - TC_i = \textstyle\sum_{j=1}^{m_i} M_i r_{mi}[D_i T_i + x_i(1 - \theta_i)\lambda_i P_i(t_i - \tau_i)] \\
&- \textstyle\sum_{j=1}^{m_i} c_{hi}[\frac{P_i - D_i}{2}t_i^2 - \frac{(1-\theta_i)\lambda_i P_i}{2}(t_i - \tau_i)^2 + \frac{D_i}{2}(T_i - t_i)^2] - \sum_{j=1}^{m_i} c_{ri}\theta_i\lambda_i P_i(t_i - \tau_i) \\
&- \textstyle\sum_{j=1}^{m_i}[r_{mi} + \frac{g_{1i}}{P_i} + \frac{g_{2i}q_i}{1 - a_i q_i} + g_{3i}P_i^{\frac{1}{2}}]P_i t_i - [m_i C_{s0i} + C_{s1i}\frac{1-e^{-m_i c_i}}{e^{c_i} - 1}] \\
&- C_{m0i}[m_i - \frac{1 - e^{-m_i c'_i}}{1 - e^{-c'_i}}]
\end{aligned}$$

2.7. Model constraints.

2.7.1. *Chance constraint for random time horizon.* For the random time horizon, we consider two constraints as $\bar{H} \geq m_i T_i$ for i=1, 2. In this consideration, constraints are expressed as Chance constraints which are
$Pr(\bar{H} \geq m_i T_i) \geq r$, for i=1, 2; where $r \in (0, 1)$ is a specified permissible probability.

$$\text{or} m_i T_i \leq m_h + \sigma_h \Phi^{-1}(1-r), \text{ for i=1, 2 (cf. Rao, [34])} \tag{2.12}$$

where m_h and σ_h are the expectation and standard deviation of normally distributed random variable $\bar{H}$ respectively and $\Phi^{-1}(x)$ denotes inverse function of standard normal distribution of standard normal variate $\frac{\bar{H}-m_h}{\sigma_h}$.

2.7.2. *Demand function constraints.* In reality, the consumer demands $D_i(s_i, q_i)$ are non negative. Sum of RDs of all substitutable items under any type substitution does not exceed the total market based demand of those items. Thus,

$$\begin{array}{l} D_i(s_i, q_i) > 0, \ \ for\ i = 1, 2; \\ and \sum_{i=1}^{2} D_i(s_i, q_i) \leq d_{10} + d_{20}. \end{array} \tag{2.13}$$

2.7.3. *Ranges of mark-up and quality.* According to Yao and Wu [42], we have ranges of the best prices for i=1, 2 as

$$r_{mi} \leq s_i \leq d_{i0}/d_{i1} \text{or}, \ r_{mi} \leq M_i r_{mi} \leq d_{i0}/d_{i1} \text{or}, \ 1 \leq M_i \leq \frac{d_{i0}}{d_{i1} r_{mi}} \tag{2.14}$$

From our earlier assumption, we take the ranges of quality as

$$\beta_i \leq q_i \leq 1 \text{ for i=1, 2} \tag{2.15}$$

2.8. Optimization problems.

2.8.1. *Model 1.* Considering the demand is measured only on selling price, the problem for multi-items inventory model is finally reduced to the maximization of total profit subject to Chance constraints on the Random Time Horizon and Demand constraints. Hence the problem is reduced to

$$\left\{ \begin{array}{l} \text{Maximize } Z_1 = \sum_{i=1}^{2} TP_i(m_1, m_2, M_1, M_2, P_1, P_2) \\ \text{with constraints (2.12), (2.13) and (2.14).} \end{array} \right. \tag{2.16}$$

where $D_i(s_i)$ is given by the equation (2.1) and

$$\begin{array}{l} TP_i = TSR_i - TC_i = \sum_{j=1}^{m_i} M_i r_{mi}[D_i T_i + x_i(1-\theta_i)\lambda_i P_i(t_i - \tau_i)] \\ - \sum_{j=1}^{m_i} c_{hi}[\frac{P_i - D_i}{2} t_i^2 - \frac{(1-\theta_i)\lambda_i P_i}{2}(t_i - \tau_i)^2 + \frac{D_i}{2}(T_i - t_i)^2] - \sum_{j=1}^{m_i} c_{ri}\theta_i \lambda_i P_i(t_i - \tau_i) \\ - \sum_{j=1}^{m_i}[r_{mi} + \frac{g_{1i}}{P_i} + g_{3i} P_i^{\frac{1}{2}}] P_i t_i - [m_i C_{s0i} + C_{s1i}\frac{1-e^{-m_i c_i}}{e^{c_i} - 1}] - C_{m0i}[m_i - \frac{1-e^{-m_i c_i'}}{1-e^{-c_i'}}] \end{array} \tag{2.17}$$

2.8.2. *Model 2.* Similarly, considering the demand is measured only on quality, the problem is reduced to

$$\left\{ \begin{array}{l} \text{Maximize } Z_2 = \sum_{i=1}^{2} TP_i(m_1, m_2, P_1, P_2, q_1, q_2) \\ \text{with constraints (2.12), (2.13) and (2.15).} \end{array} \right. \tag{2.18}$$

$D_i(q_i)$ and TP_i are given by the equations (2.3) and (2.11) respectively.

2.8.3. *Model 3.* Considering the demand is measured on the joint effect of selling price and quality, the problem for multi-items inventory model is finally reduced to

$$\left\{ \begin{array}{l} \text{Maximize } Z_3 = \sum_{i=1}^{2} TP_i(m_1, m_2, M_1, M_2, P_1, P_2, q_1, q_2) \\ \text{with constraints (2.12), (2.13), (2.14) and (2.15).} \end{array} \right. \tag{2.19}$$

where $D_i(s_i, q_i)$ and TP_i are given by the equations (2.5) and (2.11) respectively.

3. Solution methodology

Genetic algorithm: Now-a-days Genetic Algorithm (GA) (Michalewicz [26]; Mondal and Maiti, [28]) is extensively used to solve complex decision making problems in different fields of science and technology. Following Last and Eyal [19], here, a GA (Roy et al. [36]; Maiti [23] with varying population size is used where diversity of the chromosomes in the initial population is maintained using entropy originating from information theory and chromosomes are classified into young, middle age and old (in fuzzy sense) according to their age and lifetime. Following comparison of fuzzy numbers using possibility theory (Liu & Iwamura [21]) here crossover probability is measured as a function of parent's age interval (a fuzzy rule base on parents age limit is also used for this purpose). General structure of this GA is presented below:

Algorithm:

1. **Start**
2. Set iteration counter t=0, Maxsize=200, $\epsilon = 0.0001$ and $p_m(0) = 0.9$.
3. Randomly generate **Initial** population P(t), where diversity in the population is maintained using entropy originating from information theory.
4. **Evaluate** initial population P(t).
5. Set Maxfit= Maximum fitness in P(t) and Avgfit=Average fitness of P(t).
6. **While** $(Maxfit - Avgfit \leq \epsilon)$ **do**
7. $t = t + 1$.
8. Increase age of each chromosome.
9. **For** each pair of parents **do**
10. Determine probability of crossover $\tilde{p_c}$ for the selected pair of parents using fuzzy rule base and possibility theory
11. Perform crossover with probability $\tilde{p_c}$.
12. **End** for
13. **For** each offspring perform mutation with probability p_m **do**
14. Store offsprings into offspring set.
15. **End** for
16. Evaluate P(t).
17. Remove from P(t) all individuals with age greater than their lifetime.
18. Select a percent of better offsprings from the offspring set and insert into P(t), such that maximum size of the population is less than Maxsize.
19. Remove all offsprings from the offspring set.
20. Reduce the value of the probability of mutation p_m.
21. **End While**
22. Output: Best chromosome of P(t).
23. **End** algorithm.

3.1. GA procedures for the proposed model.

Representation: A "n dimensional real vector", $X_i = (x_{i1}, x_{i2}, ..., x_{in})$, is used to represent i-th solution, where $x_{i1}, x_{i2}, ..., x_{in}$ represent n decision variables of the decision making problem under consideration. X_i is called i-th chromosome and x_{ij} is called j-th gene of i-th chromosome.

Initialization: N such solutions $X_i = (x_{i1}, x_{i2}, ..., x_{in})$, i = 1, 2, ...,N are randomly generated by random number generator within the boundaries of each variable $[B_{jl}, B_{jr}]$, j= 1, 2, ..., n. These bounds are calculated from the nature of the problem and previous experience. Initialize(P(1)) sub-function is used for this purpose.

Constraint Checking: For constrained optimization problems, at the time of generation of each individuals X_i of P(1), constraints are checked using a separate sub-function check constraint(X_i), which returns 1 if X_i satisfies the constraints otherwise returns 0. If check constraint (X_i) =1, then X_i is included in P(1) otherwise X_i is again generated and it continues until constraints are satisfied.
Diversity Preservation: At the time of generation of P(1) diversity is maintained using entropy originating from information theory. Following steps are used for this purpose.

(i) Probability, pr_{jk}, that the value of the i-th gene (variable) of the j-th chromosome is different from the i-th gene of the k-th chromosome is calculated using the formula $pr_{jk} = 1 - \frac{x_{ji}-x_{ki}}{B_{jr}-B_{jl}}$ where $[B_{jl}, B_{jr}]$ is the variation domain of the i-th gene.
(ii) Entropy of the i-th gene, $E_i(M)$, i=1, 2, ..., n is calculated using the formula: $E_i(M) = \sum_{j=1}^{M-1} \sum_{k=j+1}^{M} -pr_{jk} log pr_{jk}$, where M is the size of the current population.
(iii) Average entropy of the current population is calculated by the formula: $E(M) = \frac{1}{n} \sum_{i=1}^{n} E_i(M)$
(iv) Incorporating the above three steps a separate sub-function check diversity(X_i) is developed. Every time a new chromosome X_i is generated, the entropy between this one and previously generated individuals is calculated. If this information quantity is higher than a threshold, E_T, fixed at the beginning, X_i is included in the population otherwise X_i is again generated until diversity exceeds the threshold, E_T . This method induces a good distribution of initial population.

Determination of fitness and lifetime: Value of the objective function due to the solution X_i, is taken as fitness of X_i. Let it be Z(X_i). At the time of initialization age of each solution is set to zero. Following Michalewicz [26] at the time of birth life-time of X_i is computed using the following formula:
If Avgfit$\geq Z(X_i)$, lifetime(X_i)=Minlt+$\frac{K(Z(X_i)-Minfit)}{Avgfit-Minfit}$,
If Avgfit$< Z(X_i)$, lifetime(X_i)=$\frac{Minlt+Maxlt}{2} + \frac{K(Z(X_i)-Avgfit)}{Maxfit-Avgfit}$.
where Maxlt and Minlt are maximum and minimum allowed lifetime of a chromosome, $K = (Maxlt - Minlt)/2$. Maxfit, Avgfit, Minfit represent the best, average and worst fitness of the current population. To optimize objective functions it is assumed that Maxlt=7 and Minlt=1, N=10. According to the age, a chromosome can belongs to any one of age intervals-young, middle-age or old, whose membership functions are presented in Figure-2. For a small positive number δ given by the user, the common fuzzy age (a,b,c) is described by Eq. (3.1).

$$Age = \begin{cases} Young, & \text{for } a \leq age < b-\delta \\ Middle, & \text{for } b-\delta \leq age \leq b+\delta \\ Old, & \text{for } b+\delta < age \leq c \end{cases} \tag{3.1}$$

Crossover:
Determination of probability of crossover($\tilde{p_c}$ *)*: Probability of crossover $\tilde{p_c}$, for a pair of parents (X_i, X_j) is determined as:

(i) Following Maiti [23](§2), at first age intervals (young, middle-age, old) of X_i and X_j are determined by making possibility measure of fuzzy numbers young, middle-age, old with respect to their age.
(ii) After determination of age intervals of the parents their crossover probability ($\tilde{p_c}$) is determined as a linguistic variable (low, medium or high) using a fuzzy rule base as presented in Table-1. Membership function of these linguistic variables are presented in Figure-3.

Table 1. *Fuzzy rule base for crossover probability*

Parent-2	Parent-1		
	Young	Middle-age	Old
Young	Low	Medium	Low
Middle-age	Medium	High	Medium
Old	Low	Medium	Low

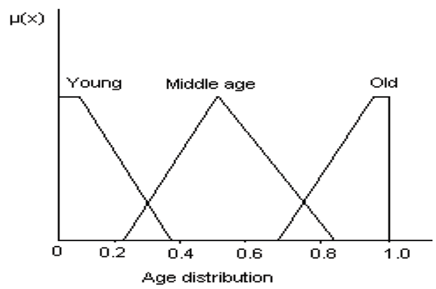

Figure 2. Membership functions of age intervals.

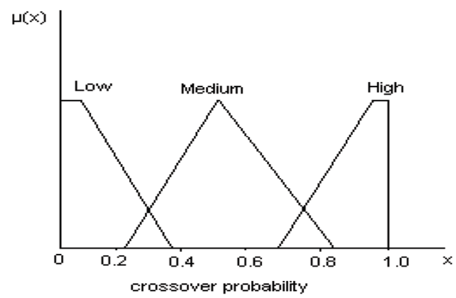

Figure 3. Membership functions of crossover probabilities.

Crossover process:
For each pair of parent solutions X_i, X_j a random number c is generated from the range $[0, 1]$ and if $Nes(c < \tilde{p_c}) > \beta$ (cf. § 2 of Maiti [23]), crossover operation is made on X_i, X_j where "Nes means necessity measure and β $(0 < \beta < 1)$ is a predefined necessity level. For the proposed model it is assumed that $\beta = 0.5$. To made crossover operation on each pair of coupled solutions X_i, X_j a random number c_1 is generated from the range $[0, 1]$ and their offsprings Y_1 and Y_2 are determined by the formula: $Y_1 = c_1X_i + (1 - c_1)X_j$, $Y_2 = c_1X_j + (1 - c_1)X_i$.
For constrained optimization problems, if a child solution satisfies the constraints of the problem then it is included in the offspring set otherwise it is not included in the offspring set.

Mutation:

(i) *Selection for mutation:* For each offspring generate a random number r from the range [0, 1]. If $r < p_m$ then the solution is taken for mutation, where p_m is the probability of mutation.

(ii) *Mutation process:* To mutate a solution $X = (x_1, x_2, ..., x_n)$ a random integer I in the range [1,n] has to be selected. Then replace x_i by randomly generated value within the boundary $[B_{il}, B_{ir}]$ of i-th component of X. New solution (if satisfies constraints of the problem) replaces the parent solution. If child solution does not satisfy the constraint then parent solution will not be replaced by child solution. Constraint checking of a child solution C_i is made using check constraint (C_i) Function.

Reduction process of p_m: According to real world demand as generation increases, p_m will decrease smoothly since the search space was more wide initially and after some iterations, it should move towards the convergence. This concept lead us to reduce the

value of p_m in each generation. Let $p_m(0)$ is the initial value of p_m. Then probability of mutation in T-th generation $p_m(T)$ is calculated by the formula $p_m(T) = p_m(0)$ $exp(-T/\alpha_1)$, where α_1 is calculated so that the final value of p_m is small enough (10^{-2} in our case). So $\alpha_1 = Maxgen/log[\frac{p_m(0)}{10^{-2}}]$, where Maxgen is the expected number of generations that the GA can run for convergence.
Selection of offsprings: Maximum population growth in a generation is assumed as forty percent. So not all offsprings are taken into the parent set for next generation. At first offspring set is arranged in descending order in fitness. Then better solutions are selected and entered into parent set such that population size does not exceeds Maxsize.
Termination Condition: Algorithm terminates when difference between maximum fitness (Maxfit) of chromosome, i.e., fitness of the best solution of the population and average fitness (Avgfit) of the population becomes negligible.
Implementation: With the above function and values the algorithm is implemented using C-programming language

4. Numerical experiments

4.1. Input data. We consider the proposed EPL models(Model-1, -2 and -3) with following inputs parameters in appropriate units:
$m_h = 25$, $\sigma_h = 2.0$, $r = 0.70$;
$C_{s01} = 1000$, $C_{s11} = 200$, $c_1 = 0.70$, $C_{m01} = 210$, $c_1' = 0.75$, $c_{h1} = 1.80$, $c_{r1} = 2.50$, $\theta_1 = 0.75$, $\lambda_1 = 0.35$, $x_1 = 0.50$, $\tau_1 = 0.75$, $d_{10} = 55$, $\beta_1 = 0.50$;
$C_{s02} = 1150$, $C_{s12} = 225$, $c_2 = 0.75$, $C_{m02} = 220$, $c_2' = 0.80$, $c_{h2} = 1.75$, $c_{r2} = 2.75$, $\theta_2 = 0.70$, $\lambda_2 = 0.30$, $x_2 = 0.45$, $\tau_2 = 0.80$, $d_{20} = 60$, $\beta_2 = 0.50$ for Models -1, -2 and -3.
$C_1(P_1, q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20P_1^{\frac{1}{2}}$, $C_2(P_2, q_2) = 22 + \frac{460}{P_2} + \frac{8.50q_2}{1-0.55q_2} + 0.18P_2^{\frac{1}{2}}$ for Models-2 and -3 and $C_1(P_1) = 20 + \frac{450}{P_1} + 0.20P_1^{\frac{1}{2}}$, $C_2(P_2) = 22 + \frac{460}{P_2} + 0.18P_2^{\frac{1}{2}}$ for Model-1. The bounds of decision variables M_i and q_i are considered using constraints (2.14) and (2.15) and the bounds of other decision variables are considered as $P_i \in [50, 250]$ and $m_i \in [1, 8]$.

4.2. Near-optimum results. With the above parameters and expressions, the Models-1, -2 and -3 are formulated and optimized using above mentioned GAVP. The corresponding i-th item's near-optimum values - number of cycles (m_i^*), Mark-ups (M_i), production rates(P_i^*), qualities (q_i), selling prices s_i^* per unit perfect product, amount of substitution demand rates due to price($Dp_i^* = -d_{i1}s_i^* + d_{i2}s_{3-i}^*$)and quality ($Dq_i^* = d_{i3}q_i^* - d_{i4}q_{3-i}^*$), resultant demand ($D_i$) and production run time(t_i^*), defective units(N_i^*), total produced good inventories (Q_i^*) for each production cycle and maximum total profit (Z_1^*, Z_2^* and Z_3^*) for whole time horizon are evaluated for the different set values of $d_{i1}, d_{i2}, d_{i3}\, and\, d_{i4}$ which are satisfied the assumptions 13 to16 and proposition 1 to 3. For every set of these parameter, we treat it as a case of the corresponding model. The obtained results are presented in Tables-2, -3, -4, -5, -6, -7 and -8.

5. Discussion

5.1. Effect of IODOSs (with respect to price) on profit for Model-1.

(i) We perform some experiments with Model-1 in which substitutability occurs due price only and the results with different marks-up for the sale of the items are presented in Table-2. Here, the different mark-ups for the products -1 and -2 are bounded by the expression (2.14) i.e. $1 \le M_i \le d_{i0}/(d_{i1}r_{mi})$, i=1,2. As d_{i0} and

Table 2. Results (near-optimum quantities) for Model-1 with different mark-ups

Case	Responsiveness		IODOS	Near-optimum results									
	d_{11} / d_{21}	d_{12} / d_{22}	ds_{p_1} / ds_{p_2}	m_1^* / m_2^*	M_1^* / M_2^*	P_1^* / P_2^*	Z_1^*	s_1^* / s_2^*	Dp_1^* / Dp_2^*	D_1^* / D_2^*	t_1^* / t_2^*	N_1^* / N_2^*	Q_1^* / Q_2^*
1P11	0.50	0.25	0.25	3	5.50	120	110382	110	-22	33	2.32	66	261
	0.45	0.20	0.25	2	5.94	121		131	-37	23	2.46	60	279
1P12	0.50	0.25	0.25	3	5.49	118	123869	110	-22	33	2.40	68	267
	0.45	0.25	0.20	3	6.05	125		133	-32	28	1.86	40	220
1P13	0.50	0.15	0.35	3	5.26	133	86029	105	-35	20	1.26	24	161
	0.45	0.20	0.25	3	5.38	136		118	-32	28	1.72	38	223
1P14	0.50	0.25	0.25	3	5.50	122	96429	110	-25	30	2.05	55	236
	0.50	0.20	0.30	3	5.37	137		118	-37	23	1.39	24	184
1P15	0.50	0.25	0.25	3	5.49	124	108216	110	-25	30	2.06	57	241
	0.50	0.25	0.25	3	5.45	135		120	-33	27	1.71	37	219
1P16	0.50	0.15	0.35	3	5.05	137	74920	101	-35	20	1.23	23	164
	0.50	0.20	0.30	3	4.85	147		107	-33	27	1.53	32	215
1P17	0.40	0.25	0.15	3	6.85	106	146451	137	-22	33	2.70	72	268
	0.45	0.20	0.25	3	6.04	124		133	-32	28	1.87	40	221
1P18	0.40	0.25	0.15	3	6.87	104	163852	137	-22	33	2.73	72	267
	0.45	0.25	0.20	3	6.05	117		133	-26	34	2.51	60	275
1P19	0.40	0.15	0.25	3	6.60	115	111396	132	-34	21	1.57	33	172
	0.45	0.20	0.25	3	5.84	127		128	-31	29	1.90	42	229
1P110	0.50	0.25	0.25	3	5.49	111	192123	110	-17	38	2.90	83	300
	0.40	0.40	0.00	3	6.81	98		150	-16	44	3.87	90	352
1P111	0.40	0.40	0.00	1	6.85	59	203829	137	-2	53	23.93	475	1282
	0.45	0.20	0.25	3	6.05	124		133	-32	28	1.87	40	220
1P21	0.50	0.50	0.00	1	5.50	66	160715	110	5	60	23.88	533	1439
	0.50	0.20	0.30	2	5.45	129		120	-38	22	2.18	53	265
1P22	0.50	0.45	0.05	1	5.50	66	167830	110	5	60	23.84	532	1437
	0.45	0.20	0.25	2	6.06	120		133	-39	22	2.36	56	266
1P23	0.50	0.45	0.05	1	5.47	66	236598	109	5	60	23.99	534	1441
	0.45	0.45	0.00	3	6.04	88		133	-11	49	4.87	107	396
1P31	0.50	0.25	0.25	3	5.49	123	171945	110	-25	30	2.06	57	240
	0.50	0.50	0.00	1	5.45	61		120	-5	55	23.80	419	1320
1P32	0.40	0.25	0.15	3	6.86	111	219616	137	-22	33	2.57	71	267
	0.45	0.40	0.05	1	6.05	60		133	-5	55	23.93	419	1319
1P33	0.40	0.40	0.00	1	6.87	59	276713	137	-2	53	23.88	475	1281
	0.45	0.40	0.05	1	6.06	62		133	-5	55	23.12	418	1319
1P41	0.50	0.50	0.00	1	5.46	56	206095	109	-4	51	23.90	451	1217
	0.50	0.50	0.00	1	4.57	72		101	4	64	23.61	490	1543

r_{mi} are constants, mark-up changes with d_{i1}, i=1,2, i.e. the measure of responsiveness of the products to their own prices. Here, the responsivenesses have been assumed to be less than 1 (i.e. $0 < d_{i1} < 1$) and therefore, **smaller the resposiveness, larger the mark-up**. The near-optimum mark-ups change the respective prices and as a consequence, alter the demands, production rates and finally the maximum profits. Here, for the cases -1P17, -1P18, -1P111, -1P32 and -1P33, mark-ups are almost same. Similarly, cases -1P11 and -1P12 have nearly same mark-up. Comparing these two cases, the IODOS of the 2nd product is reduced from 0.25 to 0.20 where IODOS of 1st product remains same(i.e. 0.25) and as a consequence, RD of the 2nd product increases from 23 to 28 units whereas the RD of 1st item remains unaltered at 33 units. It can be seen from the values of $d_{11}s_1, d_{12}s_2, d_{21}s_2, d_{22}s_1$ as (=55, 33, 59, 22) and (=55, 33, 60, 27)

Table 3. Results (near-optimum quantities) for Model-1 with same mark-ups in each case

Case	Responsiveness		IODOS	Near-optimum results									
	d_{11}	d_{12}	ds_{p_1}	m_1^*	M^*	P_1^*	Z_1^*	s_1^*	Dp_1^*	D_1^*	t_1^*	N_1^*	Q_1^*
	d_{21}	d_{22}	ds_{p_2}	m_2^*		P_2^*		s_2^*	Dp_2^*	D_2^*	t_2^*	N_2^*	Q_2^*
1P11	0.50	0.25	0.25	3	5.50	122	109546	110	-25	30	2.10	58	242
	0.45	0.20	0.25	3		134		121	-32	28	1.73	37	221
1P12	0.50	0.25	0.25	3	5.50	122	121704	110	-25	30	2.10	58	242
	0.45	0.25	0.20	3		128		121	-27	33	2.20	53	264
1P13	0.50	0.15	0.35	3	5.32	132	85984	106	-36	19	1.21	21	155
	0.45	0.20	0.25	3		135		117	-31	29	1.78	40	229
1P14	0.50	0.25	0.25	3	5.45	123	96277	109	-25	30	2.10	58	244
	0.50	0.20	0.30	2		129		120	-38	22	2.15	52	262
1P15	0.50	0.25	0.25	3	5.45	123	108011	109	-25	30	2.10	58	244
	0.50	0.25	0.25	3		135		120	-33	27	1.70	36	218
1P16	0.50	0.15	0.35	3	4.94	139	74768	99	-33	22	1.31	27	175
	0.50	0.20	0.30	3		146		109	-35	25	1.46	29	203
1P17	0.40	0.25	0.15	3	6.06	100	139991	121	-15	40	3.42	94	319
	0.45	0.20	0.25	3		126		133	-36	24	1.62	31	194
1P18	0.40	0.25	0.15	3	6.06	101	155147	121	-15	40	3.40	93	319
	0.45	0.25	0.20	3		124		133	-30	30	2.07	47	243
1P19	0.40	0.15	0.25	3	6.05	119	109585	121	-28	27	1.88	47	212
	0.45	0.20	0.25	3		125		133	-36	24	1.63	31	194
1P110	0.50	0.25	0.25	3	5.49	123	173838	110	-25	30	2.09	58	242
	0.40	0.40	0.00	1		61		121	-5	55	23.75	420	1321
1P111	0.40	0.40	0.00	1	6.05	66	189661	121	5	60	23.85	532	1436
	0.45	0.20	0.25	3		126		133	-36	24	1.61	31	195
1P21	0.50	0.50	0.00	1	5.45	67	159833	109	5	60	23.65	537	1451
	0.50	0.20	0.30	2		129		120	-38	22	2.16	52	262
1P22	0.50	0.45	0.05	1	5.50	60	161339	110	-1	54	23.68	484	1307
	0.45	0.20	0.25	3		135		121	-32	28	1.72	37	221
1P23	0.50	0.45	0.05	1	5.49	62	225059	110	-1	54	23.16	484	1307
	0.45	0.45	0.00	1		60		121	-5	55	23.98	420	1321
1P31	0.50	0.25	0.25	3	5.45	124	171231	109	-25	30	2.09	58	244
	0.50	0.50	0.00	1		60		120	-5	55	23.93	416	1309
1P32	0.40	0.25	0.15	3	6.05	100	203436	121	-15	40	3.41	93	319
	0.45	0.40	0.05	1		54		133	-11	49	23.78	370	1164
1P33	0.40	0.40	0.00	1	6.05	66	253083	121	5	60	23.95	532	1436
	0.45	0.40	0.05	1		55		133	-11	49	23.14	369	1164
1P41	0.50	0.50	0.00	1	5.45	67	234912	109	5	60	23.66	537	1451
	0.50	0.50	0.00	1		60		120	-5	55	23.75	416	1309

for the cases -1P11 and -1P12 respectively that more customers of the 1st product adapt for the 2nd product i.e. **2nd product is more substitutable**. Same observation can be made from the cases -(1P14 and 1P15) and the cases -(1P17 and 1P18). The opposite observations are observed in cases -(1P32 and 1P33). Here, from the values of $d_{11}s_1, d_{12}s_2, d_{21}s_2, d_{22}s_1$ as (=55, 33, 60,55) and (=55, 53, 60, 55) for the cases -1P32 and -1P33 respectively and it can be said as before that **the 1st product is more substitutable**. Thus it can be concluded that **lower IODOS increase the corresponding RD and vice versa**. i.e. it makes the products more substitutable. If the mark-ups remain same, **lower IODOS fetches more profit**. This observation is also substantiated from the following cases. The cases -1P31 and -1P21 with respective IODOSs (0.25, 0.00) and (0.00, 0.30) furnish that the RDs of 1st and 2nd products in the case -1P21 respectively increase and decrease than those of the case -1P31. This is because the values of D_i's change from 30 to 60 units for the 1st product and from 55 to 22 units for the 2nd product. But when the mark-ups are different in two cases, it is difficult to predict the behaviour of RDs. This can be seen from the cases -1P15 and -1P19. In these cases, both IODOSs are (0.25, 0.25), but the RDs are different and as a result, profits are different. This is because, in these cases, mark-ups are different. From this table, it is also observed that when IODOSs are high, the production rates for the products are high, but the production times are much small (cases -1P13,-1P14,-1P15 and -1P16). On the other hand, for the cases with low IODOSs, the product rates are much small but the production time are very high (cases -1P23, -1P33, -1P41, et). Total defective products are more for the cases with higher profits (cases -1P33, -1P41, -1P23, etc) and in these cases, salvage amounts contribute more than the rework costs.

(ii) We evaluate the profits of Model-1 with the same marks-up for the sale of the units and the near-optimum results are presented in Table-3. Here, the expression (14) is modified as $1 \leq M \leq Min[d_{10}/(d_{11}r_{m1}),\, d_{20}/(d_{21}r_{m2})]$. The observation made from Table-2 are also true with respect to Table-3. In addition it is observed in this table that lower IODOSs are related to less number of cycles required for the system. For example, the cases -1P23, -1P33 and -1P41 with corresponding IODOSs (0.05, 0.00), (0.00, 0.05) and (0.00, 0.00) have the single time cycle for both products. i.e. $m_1 = 1 = m_2$. The cases -1P22, -1P32, -1P110, -1P31, -1P111 and -1P21 with corresponding IODOSs (0.05, 0.25), (0.15, 0.05), (0.25, 0.00), (0.25, 0.00), (0.00, 0.25) and (0.00, 0.30)have the cycles for 1st and 2nd products as (1,3), (3, 1), (3, 1), (3, 1), (1, 3) and (1, 2) respectively. The other cases in Table-3 with higher IODOSs are having no. of cycles as (3, 3) for both products. Though in the cases -1P110 and -1P31, d_{22} and d_{21} are different, their IODOSs are same (0.25, 0.00) and all near-optimum parameters are almost same. Here the cases (-1P11, -1P12, -1P14, -1P15, -1P22, -1P110, -1P23, -1P31, -1P41) with same or almost same mark-ups have the different near-optimum parametric values with different IODOSs. Comparing the Tables-2 and -3, case -1P110 have the same IODOS (0.25, 0.00), but all other results are different including the cycle numbers as (3, 3) and (3, 1). The Table-3 with same mark-up fetches the lower profits in all cases than the corresponding profits in Table-2 with different mark-ups. The main reasons for this are that the mark-ups in Table-3 are selected following modified (2.14) as mentioned above.

5.2. Effect of IODOSs (with respect to quality) on profit for Model-2. For the Model-2, which is developed substitutability due to qualities, some experiments like Model-1 are performed and the near-optimum results are presented in Table-4. Here, mark-ups are same (5.00, 5.00), because mark-ups are related to selling prices only. For all cases, number of cycles are less, most of the cases are having only one cycle. With these values of d_{i3}, d_{i4}, $i = 1, 2$; profits are more than those in Tables-2 and -3 except few cases. In all cases, quality level goes down to the lowest value as a cost is involved for the improvement of quality of the products. This is a part of unit production cost. Here, losses of sales are minimum, rates of production are moderate and durations of production are high in most of all the cases. For the cases -1q22 and -1q32 with IODOSs (5, 10) and (10, 5) respectively, there is a single number of cycle in both cases but the losses of sales due to qualities are just reversed as expected. All other observations made for the IODOSes with respect to prices in Tables-2 and -3 are also true in this case.

Table 4. Results (near-optimum quantities) for Model-2 with same mark-ups(5.0, 5.0)

Case	Responsiveness		IODOS	Near-optimum results								
	d_{13} / d_{23}	d_{14} / d_{24}	ds_{q_1} / ds_{q_2}	m_1^* / m_2^*	P_1^* / P_2^*	q_1^* / q_2^*	Z_2^*	Dq_1^* / Dq_2^*	D_1^* / D_2^*	t_1^* / t_2^*	N_1^* / N_2^*	Q_1^* / Q_2^*
1q11	20	30	10	1	55	0.50	173260	-5	50	23.82	444	1199
	15	25	10	1	61	0.50		-5	55	23.93	419	1320
1q12	20	30	10	1	56	0.50	177847	-5	50	23.40	444	1201
	20	25	05	1	63	0.50		-3	57	2386	438	1379
1q13	10	30	20	3	102	0.50	162794	-10	45	3.82	109	360
	15	25	10	1	60	0.50		-5	55	23.95	419	1319
1q14	20	30	10	3	83	0.50	166088	-5	50	5.20	130	400
	15	30	15	1	58	0.50		-8	52	23.72	400	1259
1q15	20	30	10	1	55	0.51	173201	-5	50	23.91	445	1202
	20	30	10	1	60	0.50		-5	55	23.95	418	1316
1q16	10	30	20	3	95	0.50	157471	-10	45	4.10	111	360
	15	30	15	1	58	0.50		-8	52	23.64	399	1257
1q17	20	30	10	1	55	0.50	161139	-5	50	23.96	445	1202
	15	35	20	3	100	0.50		-10	50	4.29	105	398
1q18	20	30	10	1	55	0.50	168190	-5	50	23.78	444	1200
	20	35	15	1	58	0.50		-8	52	23.89	400	1259
1q19	10	30	20	1	52	0.50	151841	-10	45	22.88	399	1078
	15	35	20	3	99	0.50		-10	50	4.36	106	400
1q110	20	30	10	3	84	0.50	180863	-5	50	5.14	129	400
	25	25	00	1	66	0.50		0	60	23.85	457	1440
1q111	30	30	00	1	60	0.50	170117	0	55	23.94	489	1321
	15	35	20	3	99	0.50		-10	50	4.34	105	400
1q21	30	30	00	1	60	0.50	174700	0	55	23.88	489	1321
	15	30	15	3	90	0.50		-8	52	5.02	114	419
1q22	25	30	05	1	58	0.50	177396	-2	53	23.90	467	1262
	15	25	10	1	61	0.50		-5	55	23.50	418	1318
1q23	25	30	05	1	57	0.50	187521	-2	53	24.00	467	1261
	25	25	00	1	66	0.50		0	60	23.86	457	1439
1q31	20	30	10	1	55	0.50	183048	-5	50	23.98	443	1196
	30	30	00	1	66	0.50		0	60	23.99	458	1443
1q32	20	30	10	1	55	0.50	177999	-5	50	23.95	445	1201
	30	35	05	1	64	0.50		-3	57	23.69	438	1379
1q33	30	30	00	1	60	0.50	186827	0	55	23.99	486	1313
	30	35	05	1	64	0.51		-2	58	23.75	441	1387
1q41	30	30	00	1	56	0.50	188047	-4	51	23.97	453	1222
	30	30	00	1	72	0.64		4	64	23.49	488	1538

5.3. Effect of IODOSs (with respect to both prices and qualities) on profits for Model-3 and its comparison with other models.

(i) In Model-3, the substitutability among the items are due to both prices and qualities. By changing both these parameters, near-optimum parameters of the Model-3 are evaluated and presented in Table-5. Here, it is assumed that the customers who adopt for substitution on the basis of prices are not influenced by the quality and vice versa. Due to this assumption, Dp_1, Dp_2, Dq_1 and Dq_2 all are not positive and in such cases, there is loss of sales. Depending on the relations amongst the responsivenesses due to prices and qualities jointly, there will be in total 324 cases. Here results of some cases are presented in Table-5.

Table 5. Results (near-optimum quantities) for Model-3 (case-A)

Case	Responsiveness				Near-optimum results											
	d_{11} / d_{21}	d_{12} / d_{22}	d_{13} / d_{23}	d_{14} / d_{24}	m_1^* / m_2^*	M_1^* / M_2^*	P_1^* / P_2^*	q_1^* / q_2^*	Z_3^*	s_1^* / s_2^*	Dp_1^* / Dp_2^*	Dq_1^* / Dq_2^*	D_1^* / D_2^*	t_1^* / t_2^*	N_1^* / N_2^*	Q_1^* / Q_2^*
A1	0.50	0.25	20	30	3	5.49	135	0.50	82174	110	-23	-5	27	1.66	43	214
	0.45	0.20	15	25	2	5.73	137	0.50		126	-35	-5	20	1.85	43	241
A2	0.50	0.25	10	30	3	5.45	140	0.50	73146	109	-23	-10	22	1.29	26	174
	0.45	0.20	15	25	2	5.68	137	0.50		125	-34	-5	21	1.89	45	246
A3	0.50	0.25	20	30	3	5.49	137	0.50	72018	110	-25	-5	25	1.55	38	203
	0.45	0.20	15	35	2	5.49	147	0.50		121	-32	-10	17	1.49	30	210
A4	050	0.25	10	30	3	5.43	139	0.50	62973	109	-24	-10	21	1.24	24	167
	0.45	0.20	15	35	2	5.50	146	0.50		121	-33	-10	17	1.48	30	207
A5	0.50	0.25	20	30	3	5.46	139	0.50	65808	109	-26	-5	24	1.42	33	189
	0.50	0.20	15	30	2	5.14	156	0.50		113	-35	-7	18	1.43	29	213
A6	0.50	0.25	10	30	3	5.21	146	0.50	56938	104	-24	-10	21	1.17	21	165
	0.50	0.20	15	30	2	5.05	158	0.50		111	-35	-7	18	1.40	29	213
A7	0.50	0.15	20	30	2	4.90	150	0.50	52315	98	-34	-5	16	1.36	32	197
	0.50	0.20	15	25	3	4.67	174	0.50		103	-32	-5	23	1.09	15	186
A8	0.50	0.25	10	30	3	5.11	148	0.50	52869	102	-24	-10	21	1.16	21	166
	0.50	0.20	15	35	2	4.90	162	0.50		108	-33	-10	17	1.27	23	199
A9	0.50	0.25	10	30	3	5.49	137	0.50	68026	110	-24	-10	21	1.27	25	167
	0.45	0.20	15	30	2	5.60	144	0.50		123	-34	-7	19	1.66	37	228
A10	0.50	0.25	20	30	3	5.50	136	0.50	77108	110	-24	-5	26	1.60	41	208
	0.45	0.20	15	30	2	5.64	140	0.50		124	-34	-7	19	1.68	37	224
A11	0.50	0.25	20	30	3	5.48	140	0.50	93572	110	-22	-9	24	1.46	35	196
	0.45	0.20	30	30	3	6.02	144	0.63		132	-38	4	26	1.52	31	209
A12	0.50	0.25	30	30	2	5.49	135	0.50	86239	110	-24	0	31	1.94	56	247
	0.45	0.20	15	30	2	5.60	144	0.50		123	-34	-7	19	1.66	37	228
A13	0.50	0.25	20	30	3	5.49	137	0.50	136769	110	-25	-5	25	1.53	37	200
	0.50	0.50	15	25	3	5.44	96	0.50		120	-5	-5	50	4.48	106	400
A14	0.50	0.50	20	30	1	5.49	61	0.50	128684	110	5	-5	55	23.56	487	1316
	0.50	0.20	15	25	2	5.43	150	0.50		119	-38	-5	17	1.43	28	206
A15	0.50	0.25	20	30	3	5.50	136	0.50	133443	110	-25	-5	25	1.52	37	198
	0.50	0.50	15	30	1	5.42	53	0.50		119	-5	-7	48	23.94	365	1149
A16	0.50	0.50	20	30	1	5.49	59	0.50	123473	110	4	-5	54	23.89	480	1295
	0.50	0.20	15	30	2	5.35	115	0.50		118	-37	-7	16	1.25	21	187
A17	0.50	0.25	20	30	3	5.43	140	0.50	79743	109	-25	-5	25	1.52	38	204
	0.50	0.20	30	30	3	5.42	149	0.50		119	-38	0	22	1.22	19	177
A18	0.50	0.15	20	30	2	4.98	148	0.50	60485	100	-34	-5	16	1.37	32	196
	0.50	0.20	30	30	3	4.90	163	0.50		108	-34	0	26	1.32	26	208
A19	0.50	0.25	20	30	3	5.47	140	0.50	148763	109	-25	-5	25	1.50	37	200
	0.50	0.50	30	30	1	5.41	63	0.50		119	-5	0	55	23.17	420	1325
A20	0.50	0.50	20	30	1	5.49	61	0.50	138471	110	5	-5	55	23.62	487	1315
	0.50	0.20	30	30	3	5.43	149	0.50		119	-38	0	22	1.23	19	178

In this table, profit of all cases are less than those of the corresponding cases in Table-2 in which only prices have been considered for substitution. This is because in the combined (both price and quality) effect on substitution, the effect of quality reduces the profit, whereas in Table-2, this effect is not considered. But, against the profit values in Table-4, no conclusion can be made as in this case(Table-4), mark-ups, instead of being calculated, have been assumed and taken as 5.00. For this reason, in some cases, profits in Table-5 are less than the corresponding profits in Table-4 and in few cases (cases -A13, -A14), it does not hold.

(ii) The customers who are attracted or buck away due to prices may be attracted by the qualities of the products. In this case, some of Dp_1, Dp_2, Dq_1 and Dq_2 may be positive. On the basis of this assumption, near-optimum results are presented in Table-6 taking both prices and qualities into consideration for

Table 6. Results (near-optimum quantities) for Model-3 (case-B)

Case	Responsiveness				Near-optimum results											
	d_{11} d_{21}	d_{12} d_{22}	d_{13} d_{23}	d_{14} d_{24}	m_1^* m_2^*	M_1^* M_2^*	P_1^* P_2^*	q_1^* q_2^*	Z_3^*	s_1^* s_2^*	Dp_1^* Dp_2^*	Dq_1^* Dq_2^*	D_1^* D_2^*	t_1^* t_2^*	N_1^* N_2^*	Q_1^* Q_2^*
B1	0.50 0.45	0.15 0.20	20 25	15 25	3 3	5.33 6.06	143 148	0.50 0.91	87721	107 133	-33 -39	-4 10	18 31	1.04 1.79	15 44	145 252
B2	0.50 0.45	0.15 0.20	35 15	20 20	3 3	5.50 5.03	129 163	0.87 0.50	91063	110 111	-38 -28	20 -10	37 22	2.44 1.13	76 16	296 179
B3	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.46 6.01	132 154	0.78 0.92	108212	109 132	-35 -38	14 7	34 29	2.17 1.62	66 38	271 238
					Results using Wolfram Mathematica 9.0											
					3 3	5.46 6.02	133 154	0.78 0.92	108198	109 132	-35 -38	13 7	34 29	2.16 1.62	65 38	270 238
B4	0.50 0.45	0.50 0.20	20 15	20 25	1 2	5.48 6.00	72 133	0.50 0.50	157470	110 132	11 -37	0 -5	66 18	23.97 1.66	588 34	1587 211
B5	0.50 0.45	0.50 0.20	20 25	15 25	1 3	5.49 6.04	70 141	0.50 0.84	178342	110 133	12 -39	-3 9	64 30	23.99 1.84	568 44	1533 246
B6	0.50 0.45	0.50 0.20	35 15	20 20	1 2	5.48 6.03	84 135	0.58 0.50	178136	110 133	12 -38	10 -4	77 18	23.99 1.71	682 37	1841 219
B7	0.50 0.45	0.50 0.20	35 25	15 20	1 3	5.48 6.03	83 146	0.62 0.80	198014	110 133	11 -38	10 8	76 30	23.99 1.72	677 40	1828 238
B8	0.50 0.45	0.15 0.55	20 15	20 25	3 1	5.50 6.01	141 62	0.50 0.50	160170	110 132	-35 1	0.0 -5	20 56	1.16 23.70	20 426	159 1343
B9	0.50 0.45	0.15 0.55	20 25	15 25	3 1	5.48 6.05	141 71	0.51 0.70	177995	110 133	-35 0	0 5	20 65	1.17 23.98	121 496	160 1560
B10	0.50 0.45	0.15 0.55	35 15	20 20	3 1	5.48 6.06	134 63	0.51 0.51	180587	110 133	-35 0	8 -3	28 57	1.75 23.95	47 440	223 1385
B11	0.50 0.45	0.15 0.55	35 25	15 20	3 1	5.44 6.02	134 72	0.55 0.66	196960	109 132	-34 0	9 6	30 66	1.87 23.95	53 502	237 1580
B12	0.50 0.45	0.50 0.55	20 15	20 25	1 1	4.79 4.42	63 63	0.96 0.97	133937	96 97	0 9.0	0 -9	55 60	23.13 23.78	491 455	1328 1432
B13	0.50 0.45	0.50 0.55	20 25	15 25	No Feasible Solution											
	0.50 0.45	0.50 0.55	20 25	15 35	1 1	5.33 3.96	63 63	0.98 0.50	152543	107 87	-10 19	12 -22	57 57	23.74 23.95	510 439	1376 1383
B14	0.50 0.45	0.50 0.55	35 15	20 20	No Feasible Solution											
	0.50 0.45	0.50 0.55	35 15	35 20	1 1	4.93 5.58	59 69	0.50 0.93	193061	99 123	12 -1	-15 4	52 63	23.35 23.93	463 479	1250 1510

substitution. For this reason, in all cases, the quality levels do not reach to the bottom level of their values (0.50). In case -B3, contribution of qualities to the demand functions for two products are positive i.e. $Dq_1 = 14$ and $Dq_2 = 7$. This is because of the contributions of prices i.e. $Dp_1 = -35$ and $Dp_2 = -38$ (these buck-aways customers due to prices again go back to the items due to qualities and for that $Dq_1 = 14$, $Dq_2 = 7$) and as a result, resultant demands are $D_1 = 34$, $D_2 = 29$ which are less than the prime demands $d_{10} = 55$, $d_{20} = 60$ units. It is interesting to note that in case -B12, the contribution of prices and qualities are reversed (+9 and -9 due to prices and qualities respectively) and sum total of both contribution is zero. As a result, the RDs are equal to the prime / base demands (55 and 60 units). In this case, qualities are almost equal to 1. In the cases -B4, -B5, -B6, -B7, -B9 and -B11, one of the RDs is more than the corresponding base demand but the sum total of RDs is less than that of base demands of two products. This condition holds good due to assumption -11. Due to this assumption, for some values of $d_{ij}, i = 1, 2; j = 1, 2, 3, 4$, there is no feasible solution for same cases (-B13 and -B14).

5.4. Comparison of near-optimum results by two methods. Near-optimum results of the system for different cases with different parametric values have been evaluated by the proposed Genetic Algorithm (cf. Table-6). To verify the results, the problem given by case-B3 of the Model-3 have been solved by Wolfram Mathematica 9.0 (Random Search Method) and the results are presented in Table-6. It is seen that the proposed GA gives better result than the Mathematica.

Table 7. Results (near-optimum) without learning effect for Model-3 (case-B3)

Case	Responsiveness				Near-optimum results											
	d_{11} d_{21}	d_{12} d_{22}	d_{13} d_{23}	d_{14} d_{24}	m_1^* m_2^*	M_1^* M_2^*	P_1^* P_2^*	q_1^* q_2^*	Z_3^*	s_1^* s_2^*	Dp_1^* Dp_2^*	Dq_1^* Dq_2^*	D_1^* D_2^*	t_1^* t_2^*	N_1^* N_2^*	Q_1^* Q_2^*
B3					Results without learning effect on set-up cost											
	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.48 6.01	1323 156	0.78 0.91	1 07448	110 132	-35 -38	14 7	34 29	2.17 1.59	66 37	270 237
					Results without learning effect on maintenance cost											
					3 3	5.49 5.94	128 154	0.80 0.91	107635	110 131	-35 -37	15 7	35 30	2.28 1.62	68 38	278 238
					Results without learning effect on both set-up and maintenance cost											
					3 3	5.49 6.00	131 155	0.79 0.91	106762	110 132	-35 -38	14 7	34 29	2.20 1.60	67 37	272 236

5.5. Effect of learning parameter on Model-3 (case-B3). To evaluate the effect of learning parameter introduced in the set-up and maintenance costs, we took the most general Model-3. The Model-3 was evaluated with and without learning effects in the above costs and the near-optimum results are presented in Table-7. It is observed that as expected, profits are less in all cases without learning effects.

Table 8. Results of Model-3 (case-B3) without p_m reduction

p_m	Responsiveness				Near-optimum results											
	d_{11} d_{21}	d_{12} d_{22}	d_{13} d_{23}	d_{14} d_{24}	m_1^* m_2^*	M_1^* M_2^*	P_1^* P_2^*	q_1^* q_2^*	Z_3^*	s_1^* s_2^*	Dp_1^* Dp_2^*	Dq_1^* Dq_2^*	D_1^* D_2^*	t_1^* t_2^*	N_1^* N_2^*	Q_1^* Q_2^*
0.90	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.44 5.92	125 168	0.78 0.89	107884	109 130	-35 -37	14 7	34 30	2.31 1.48	68 34	273 239
0.70	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.48 5.92	116 188	0.78 0.89	108149	110 130	-35 -37	14 7	34 30	2.46 1.32	70 3	269 240
0.50	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.48 5.92	136 169	0.78 0.92	108186	110 130	-35 -37	14 7	34 30	2.10 1.48	64 34	269 240
0.30	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.48 5.90	119 156	0.80 0.89	108164	110 130	-35 -37	15 6	35 29	2.44 1.60	71 37	274 238
0.10	0.50 0.45	0.15 0.20	35 25	15 20	3 3	5.48 5.91	121 149	0.80 0.89	108152	110 130	-35 -37	15 6	35 29	2.42 1.67	70 39	274 238

5.6. Effect of p_m reduction on near-optimum profit for Model-3 in case B3. It is difficult to choose the system parameters of a GA. Normally, probability of mutation for a problem is assumed to be low ($\leq$ 0.50). We performed the optimization of the case-B3 of Model-3 with different values p_m from 0.90 to 0.10 (cf. Table-8). It is seen that as p_m reduces from 0.90 to 0.10 by 0.20, the near-optimum value of Z_3 (objective) increases initially and becomes maximum at $p_m = 0.50$ and then decreases. Thus the near-optimum value of p_m for the present model is 0.50. It may be noted that the near-optimum results are obtained with a particular value of p_m throughout the optimization of the system. But, in our proposed GA, the value of p_m has been reduced at each

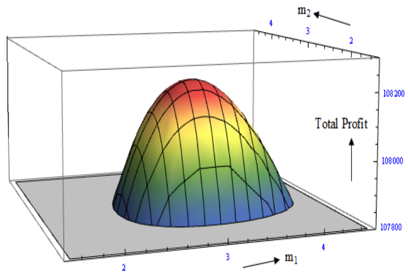

Figure 4. Total profit against the number of cycles.

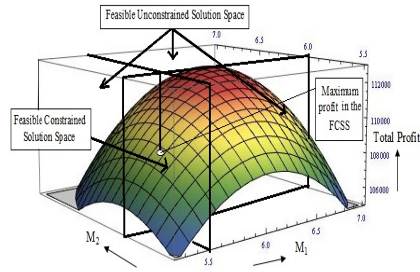

Figure 5. Total profit against the Mark-ups.

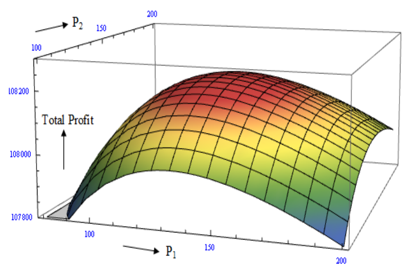

Figure 6. Total profit against the Production Rates.

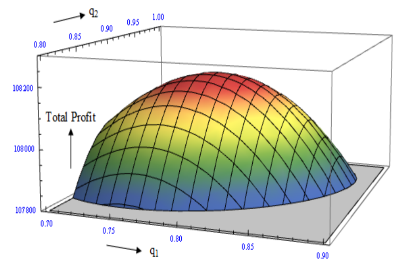

Figure 7. Total profit against the Quality levels.

iteration of the execution of the optimization process from 0.90 to 0.01 and it yields better result (cf. Table-6) than the result obtained with fixed p_m (cf. Table-8).

5.7. Pictorial representations of near-optimum results for Model-3.

(i) Considering the case-B3 from Table-6 of the Model-3, near-optimum profit $Z_3^* =$ 108212 units is obtained for $m_1^* = 3$, $m_2^* = 3$, $M_1^* = 5.46$, $M_2^* = 6.01$, $P_1^* = 132$, $P_2^* = 154$, $q_1^* = 0.78$ and $q_2^* = 0.92$. Taking number of cycles as variable and others by their near-optimum values, the total profit for the Model-3 is plotted in Fig-4 against the different values of m_1 and m_2. In the similar fashion Fig-5 is plotted against the mark-ups (M_1, M_2) of two products. In this figure, it is noted that global near-optimum values (Z_1^*=112341, M_1^*=6.44, M_2^*=6.46) lie on Feasible Unconstrained Solution Space(FUSS) but within Feasible Constrained Solution Space(FCSS) region Z_1^*=108212 units is the local near-optimum for M_1^*=5.46, M_2^*=6.01. Figs-6 and -7 are plotted for the total profit against production rates (P_1, P_2) and quality levels (q_1, q_2) as variables and others as constant by their near-optimum values respectively. These figures show that the objective function is concave.

(ii) Fig-8 is obtained by plotting the unit production cost $C_1(P_1, q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20P_1^{\frac{1}{2}}$ against the different values of production rate and quality of

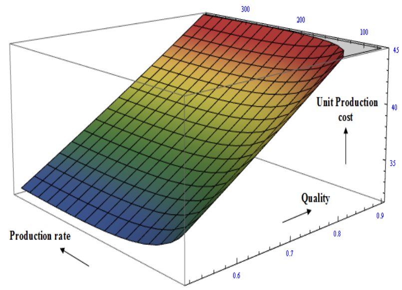

Figure 8. Unit production cost against the production rate and quality level of a product.

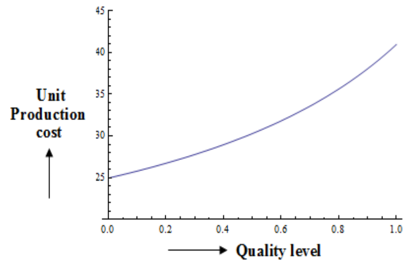

Figure 9. Unit production cost against the quality level of a product.

product-1. This unit production cost is a convex function against production rate only (cf. Fig-10 and 11).

(iii) Fig-9 represents unit production cost $C_1(q_1) = 20 + \frac{450}{P_1} + \frac{8.00q_1}{1-0.50q_1} + 0.20\sqrt{P_1}$ against the quality of product-1 when production rate $P_1 = 272$. This figure suggests that unit production cost is increasing function with respect to quality.

(iv) Fig-10 and Fig-11 represent the unit production cost against the production rate with and without quality improvement cost in the unit production cost respectively. Here $C_1(P_1) = 20 + \frac{450}{P_1} + \frac{8.00*q_1}{1-0.50*q_1} + 0.20P_1^{\frac{1}{2}}$ (taking $q_1 = 0.90$) for Fig-10 and $C_1(P_1) = 20 + \frac{450}{P_1} + 0.20P_1^{\frac{1}{2}}$ for Fig-11 are considered. In these figures, unit production cost is a convex function with respect to production rate. $C_1(P_1^*)$ have the minimum values 38.04 and 24.95 at $P_1^* = 272$ for the above two cases respectively. Though normally it is assumed that minimum value of unit production cost $(C_1(P_1))$ leads to maximum profit, in this case, the above value of P_1^* is not equal to the corresponding optimal values obtained by optimizing total profits. (example-$P_1^* = 129$ for the case -B2 in Table-6).

6. Practical implication

In a sugar mill where two types of sugar- good quality sugar and low quality sugar are produced or in the rice mills where two types of rice- fine quality and raw quality rices are produced, the products are substitutable and the customers (i.e. retailers) very often change the brand on the basis of prices and qualities. This analysis will be helpful for the production managers of the said mills to fix the optimum prices, qualities, production rates, etc for maximum profit. The responsiveness parameters $(d_{11}, d_{12}, d_{13}, d_{14}, d_{21}, d_{22}, d_{23}, d_{24})$ to prices and qualities can be obtained from the experts or may be calculated from past data. The present problem can also be applied for the managers of big departmental stores like Big Bazar, Pentaloons, etc, where several substitutable products are sold. In these stores also, customers of one brand very often change over to other brand. Here, the replenishment may be considered as procurements/ productions with infinite rate.

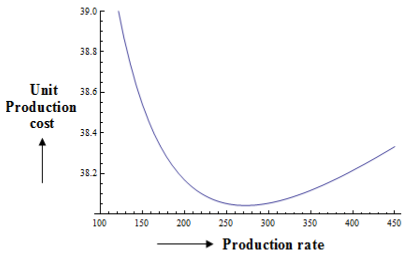

Figure 10. Unit production cost including quality improvement cost against production rate.

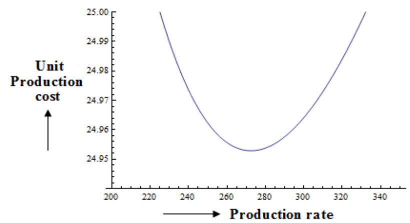

Figure 11. Unit production cost without quality improvement cost against production rate.

7. Conclusion

In this paper, production cum sale of imperfect products substitutable on the basis of the prices and qualities are considered over a random time horizon and the optimum prices, qualities, production rates and cycles are determined so that total profit is maximum. Joint and separate effects of price and quality on the substitution are taken into account in this production-marketing system. Here it is assumed that price and quality of a product are independent. It is fact that in the present competitive global marketing scenario, there is not much scope of having prices different from the similar products of sister companies. Therefore, now-a-days, price of a product has only little variations. But within that variation, the quality of the product has to be improved for existence or to be on the top at the market. Thus, price and quality of the products may be considered as independent for substitution within them. Variable production cost and the expenditure against the environment protection are included in the system. This investigation will be helpful for the managers of stores or production cum sale companies where substitutable products are produced and sold. The virgin ideas presented in this paper are (i) imperfect production cum sale of two substitutable products with the provision of repair of imperfect products, (ii) substitutability of the products on the basis of selling prices and qualities separately and jointly, (iii) allotment of some expenditure against improvement of quality and environment protection and (iv) uncertain planning horizon with normal distribution. This paper can be extended to include (i) reliability for the production process, (ii) more substitutable products and (iii) supply-chain system incorporating retailers and customers. New investigation also can be performed introducing price discounts (AUD/IQD) on the substitutable products, taking imprecise time horizon, etc.

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