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# On weighted balanced loss function under the Esscher principle and credibility premiums

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**Abstract**

This paper focuses on weighted balanced loss function under the Esscher principle (WBLF) of which we explore the modern practice of credibility theory and we generalize credibility premiums by using the WBLF. We obtain a distribution-free approach under the WBLF and the Esscher premium by using a minimization technique. Also, we discuss the consistency of the credibility premium generated by this distribution-free approach.

**Keywords:** Credibility premium, Esscher premium, loss function.

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## 1. Introduction and motivation

According to Rodermund (1989), the concept of credibility has been the casualty actuaries most important and enduring contribution to casualty actuarial science.

In this sense, credibility theory is used to determine the expected claims experience of an individual risk when those risks are not homogeneous, given that the individual risk belongs to a heterogeneous collective. The main objective of the credibility theory is to calculate the weight which should be assigned to the individual risk data to determine a fair premium to be charged, for recent detailed introductions to credibility theory, see Norberg (2004), Bühlmann and Gisler (2005).

Moreover, the credibility assumed that the individual risk, $X$, has a density $f(x \mid \theta)$ indexed by a parameter $\theta \in \Theta$ which has a prior distribution with density $\pi(\theta)$. Let, now, $\pi^x(\theta)$ be the posterior density when $x$ is observed. In actuarial science, the unknown risk

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premium $P_R^L \equiv P_R^L(\theta)$ is obtained by minimizing the expected loss $E_f\left[L(\theta,P)\right]$, with $p \in P$ for some loss function $L$. If experience is not available, the actuary chooses the collective premium $P_C^L$, which is given by minimizing the risk function, i.e. $E_\pi[L(P_R^L(\theta),P_C^L)]$. In addition, if experience is available, the actuary takes a sample $X$ from the random variables $X_i$, $i=\overline{1,t}$ and uses this information to the unknown risk premium $P_R^L(\theta)$, through the Bayes premium $P_B^L$, obtained by minimizing the Bayes risk, i.e. $E_{\pi^x}[L(P_R^L(\theta),P_B^L)]$.

According to Heilmann (1989), many credibility premiums were obtained under statistical decision theory from a Bayesian point of view and using the weighted squared error loss function ($WLF$ henceforth), $L_1(P,x) = h(x)\,(x-P)^2$, using different functional forms of $h(x)$ we have different premium principles (such as net premium principle, expected value premium principle, variance premium principle, standard deviation premium principle, proportional hazards premium principle, principle of equivalent utility, dutch premium principle, Wang's premium principle, exponential principle, mean value principle, zero utility principle, Swiss premium calculation principle, Orlicz principle, Esscher principle). For example, if we take $h(x)=1$ and $h(x)=e^{hx}$, $h>0$, we have the net and the Esscher premium principles, see Heilmann (1989), Gómez (2006), and others.

In today's point of view, it would be better to understand credibility premium as a simplified version of Bayes estimation of the individual pure premium. It is well known that the credibility premium can be written as a convex combination between the individual and the collective information. Under the case of the exponential family of distributions, exactly in the case of the pair: Poisson- Gamma, the Bayes Esscher premium can be written as a credibility formula in the form:

$$P_B^{L_1} = Z\left(t\right)g\left(\bar{x}\right) + \left(1-Z\left(t\right)\right)P_C^{L_1}$$

with (see Heilmann (1989) and Gómez (2006) for details):

$P_B^{L_1}$: the Bayes premium obtained under WLF;

$P_C^{L_1}$: the collective premium obtained under WLF;

$g\left(\bar{x}\right)$: a function of the observed data;

$Z\left(t\right)$: is the credibility factor, satisfying the condition $0 \le Z\left(t\right) \le 1$.

In this work, we use the weighted balanced loss function (WBLF) to obtain new credibility premiums, WBLF is a generalized loss function introduced by Zellner (1994) (see Gupta and Berger (1994), pp.371-390) and which appears also in Dey et al. (1999) and Farsipour and Asgharzadhe (2004). It is given by

$$L_2\left(P,x\right) = \omega h(x)(\delta_0(x)-P)^2 + \left(1-\omega\right)h(x)(x-P)^2$$

where $0 \le \omega \le 1$, $h(x)$ is a positive weight function, and $\delta_0(x)$ is a function of the observed data (see Jafari et al. (2006)). When $\omega$ is chosen to equal 0,This loss includes as a particular case the WLF, i.e.

$$\begin{aligned} L_2\left(P,x\right) &= 0h(x)(\delta_0(x)-P)^2 + \left(1-0\right)h(x)(x-P)^2 \\ &= h(x)\left(x-P\right)^2 = L_1(P,x) \end{aligned}$$

Moreover, our work is a generalization of Gómez Déniz (2008) and the results obtained here are very close to those obtained by Najafabadi et al. (2010) whose approximate the Bayes estimator with respect to a general loss function and general prior distribution by a convex combination of the observation mean and mean of prior, say, approximate credibility formula.

The paper is organized as follows. Section 2 describes the Esscher premium principle and its properties . Section 3 is dedicated to derive the Esscher credibility premiums under WBLF. Section 3 provides the main contribution of this work, i.e., the solutions under the distribution free approach. Finally, a small simulation is carried out to illustrate the theoretical conclusions and some remarks.

## 2. Properties of the Esscher premium principle

Let $\chi$ denote the set of non-negative random variables on the probability space $(\Omega, F, P)$. Goovaerts et al. (1984) describe the Esscher premium as the expected value of the risk $X$ after multiplying the density of $X$ by an increasing weight function, which of course makes the risk less attractive to the insurer. The Esscher premium of $X \in \chi$ is given by

$$H[X] = e^{hx}(x-P)^2, \; h > 0.$$

which $h$ reflects the risk averseness of the insurer. In fact, the distribution function $F_x$ of $X$ is replaced by its Esscher transform, denoted by $F_{X,h}$, where $h$ is a real parameter:

$$dF_{X,h}(x) = \frac{e^{hx} dF_X(x)}{\int_0^\infty e^{hx} dF_X(x)}$$

Clearly, $F_{X,h}$ is also a distribution function. So the Esscher premium of $X$, with parameter $h$ can be calculated as

$$H[X] = \int_0^\infty x dF_{X,h}(x)$$

Some properties of Esscher premium principle are listed as follows. The proofs can be easily checked.

- **Risk loading**: $H[X] > E[X]$ for all $X \in \chi$, and $h > 0$. In addition, when $h \to 0$, we have $H[X] \to E(X)$ which is equal to the net premium principle. Loading for risk is desirable because one generally requires a premium rule to charge at least the expected payout of the risk $X$, namely $E(X)$, in exchange for insuring the risk. Otherwise, the insurer will lose money on average.
- **No unjustified risk loading**: If a risk $X \in \chi$ is identically equal to a constant $c \geq 0$ (almost everywhere), then $H[c] = e^{hc}(c-P)^2 = c$. If we know for certain (with probability 1) that the insurance payout is $c$, then we have no reason to charge a risk loading because there is no uncertainty as to the payout.
- **Maximal loss (or no rip-off)**: $H[X] \leq esssup[X]$ for all $X \in \chi$.
- **Translation equivariance (or translation invariance)**: $H[X+c] = H[X]+c$ for all $X \in \chi$ and all $c \geq 0$. If we increase a risk $X$ by a fixed amount $c$, then the premium for $X + c$ should be the premium for $X$ increased by the fixed amount $c$.
- **Additivity for independent risks**: If $X, Y \in \chi$ are independent of each other, then $H[X+Y] = H[X] + H[Y]$.
- **Monotonicity**: If $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, then $H[X] \leq H[Y]$.
- **Preserving of first stochastic dominance (FSD) ordering**: If $S_X(t) \leq S_Y(t)$ for all $t \geq 0$, then $H[X] \leq H[Y]$.
- **Preserving of stop-loss (SL) ordering**: If $E[X-d]_+ \leq E[Y-d]_+$ for all $d \geq 0$, then $H[X] \leq H[Y]$.
- **Continuity**: Let $X \in \chi$, then $lim_{a \to 0^+} \; H[max(X-a, 0)] = H[X]$, and $lim_{a \to \infty} H[min(X, a)] = H[X]$.

Now, we give the individual premium, the collective premium and the Bayesian premium under the Esscher principle:

**The individual premium**

The individual premium of risk $X$ and parameter $\theta$ under Esscher premium principle is

$$P_R^L = E_f\left[L(\theta, P)\right] = \frac{E_f\left[Xe^{hx} \mid \theta\right]}{E_f\left[e^{hx} \mid \theta\right]}$$

**The collective premium**

If we have no information, the actuary charges the collective premium to the insured which is given by

$$P_C^L = \frac{E_\pi\left[P_R^L e^{hP_R^L}\right]}{E_\pi\left[e^{hP_R^L}\right]}$$

**The Bayesian premium**

To calculate the Bayesian premium, we use both the prior information about the parameters of the loss process and the actual loss experience observed during the policy period. The posterior density function is obtained using the prior density function and the data on actual losses from the Bayes' theorem. Let $f(x \mid \theta)$ be the probability function of $X$, and let $\pi(\theta)$ denote the prior density function of $\theta$. Let $\pi^X(\theta)$ be the posterior density function, then:

$$P_B^L = E_{\pi^x}\left[L(P_R^L, P_B^L)\right] = \frac{E_{\pi^x}\left[P_R^L e^{hP_R^L}\right]}{E_{\pi^x}\left[e^{hP_R^L}\right]}$$

## 3. Derivation of premiums under WBLF

In this section, we aim to use the WBLF to derive a new credibility formula under the Esscher premium principle. Next lemma is a generalization of Lemma 3.1 in Jafari et al. (2006).

**3.1. Lemma.** *Under WBLF and prior $\pi$, the risk , collective and Bayes premium are given by*

$$P_R^{L_2} = \omega \frac{E_{f(x|\theta)}\left[\delta_0(x)h(x)\,|\theta\right]}{E_{f(x|\theta)}\left[h(x)\,|\theta\right]} + (1-\omega)\,\frac{E_{f(x|\theta)}\left[Xh(x)\,|\theta\right]}{E_{f(x|\theta)}\left[h(x)\,|\theta\right]}$$

$$\begin{aligned} P_C^{L_2} &= \omega \frac{E_\pi\left[\delta_0(x)h(P_R^{L_2})\right]}{E_\pi\left[h(P_R^{L_2})\right]} + (1-\omega)\,\frac{E_\pi\left[P_R^{L_2}h(P_R^{L_2})\right]}{E_\pi\left[h(P_R^{L_2})\right]} \\ &= \omega\delta_0^* + (1-\omega)\,\frac{E_\pi\left[P_R^{L_2}h(P_R^{L_2})\right]}{E_\pi\left[h(P_R^{L_2})\right]} \end{aligned}$$

$$P_B^{L_2} = \omega\delta_0^* + (1-\omega)\,\frac{E_{\pi^x}\left[P_R^{L_2}h(P_R^{L_2})\right]}{E_{\pi^x}\left[h(P_R^{L_2})\right]},$$

*where $\delta_0^*$ is a target estimator for the risk (individual) premium $P_R^{L_2}$.*

*Proof.* The proof, which is similar to the one given by Dey et al. (1999).
Under WBLF, we minimize $E_{f(x|\theta)}[L_2(\theta, P_R^{L2})]$ with respect to $P_R^{L2}$.
Under WBLF , we minimize $E_{\pi(\theta)}[L_2(P_R^{L_1}, P_C^{L_2})]$ with respect to $P_C^{L_2}$.
We replace $\pi(\theta)$ by $\pi^x(\theta)$ to obtain the Bayes premium $P_B^{L_2}$. □

**3.2. Lemma.** *If the Bayes premium obtained under $L_1(P, x)$ is a credibility formula, the Bayes balanced premium obtained under WBLF is also a credibility formula in this form:*

$$P_B^{L_2} = \omega\delta_0^* + (1-\omega)\frac{E_{\pi^x}\left[\frac{e^{\frac{h}{h-\theta}}}{h-\theta}\right]}{E_{\pi^x}\left[e^{\frac{h}{h-\theta}}\right]}$$

*Proof.* Consider the case in which, the claim follows a Poisson distribution with parameter $\theta > 0$ and the prior is a gamma distribution $\pi(\theta) \propto \theta^{\alpha-1}e^{-\beta\theta}$, $\alpha > 0, \beta > 0$. Suppose also that the actuary chooses the $WLF$ to obtain the Esscher risk premium and the $WBLF$ to obtain the Esscher collective and Bayes premiums. Then, we have:

$$P_R^{L_1} = \frac{E_{f(x|\theta)}\left[Xe^{hx}/\theta\right]}{E_{f(x|\theta)}\left[e^{hx}/\theta\right]} = \theta e^h$$

$$P_C^{L_2} = \omega\delta_0^* + (1-\omega)\frac{\alpha e^h}{\beta - he^h}$$

$$P_B^{L_2} = \omega\delta_0^* + (1-\omega)\frac{(\alpha+t)e^h}{\beta+t-he^h} = z(t)\, l\left(P_C^{L_2}\right) + (1-z(t))\, l(e^h\bar{x})$$

where $z(t) = \dfrac{\beta - h^h}{\beta + t - he^h}$ and $l(x) = \omega\delta_0^* + (1-\omega)x$.
However, if we replace Poisson($\theta$) with the exponential distribution Exp($\theta$), $\theta > 0$, $P_B^{L_2}$ no longer has a credibility formula, because we have

$$P_R^{L_1} = \frac{1}{h-\theta},$$

$$P_C^{L_2} = \omega\delta_0^* + (1-\omega)\frac{E_{\pi}\left[\frac{e^{\frac{h}{h-\theta}}}{h-\theta}\right]}{E_{\pi}\left[e^{\frac{h}{h-\theta}}\right]},$$

and

$$P_B^{L_2} = \omega\delta_0^* + (1-\omega)\frac{E_{\pi^x}\left[\frac{e^{\frac{h}{h-\theta}}}{h-\theta}\right]}{E_{\pi^x}\left[e^{\frac{h}{h-\theta}}\right]},$$

which is not a credibility formula. □

**3.3. Remark.** *Under the exponential family, the Bayes balanced premium is linear only under the case Poisson-gamma.*

## 4. The distribution-free approach: Main results

According to Bühlmann (1967), the classical formula in credibility theory often calculates the premium as a weighted sum of the average experience of the policyholder and the average experience of the entire collection of policyholders. The main idea in this work is to change the exact credibility premium $H(\mu(\theta)|X_1, X_2, ..., X_n)$ by a linear expression of the form $c_0 + c_1 H_n[x]$ which $H_n[x] = \frac{\sum_{i=1}^n X_i e^{hX_i}}{\sum_{i=1}^n e^{hX_i}}$ indicates the empirical Esscher premium, depending on the past claims $X_i, i = \overline{1, n}$. Using the $WBLF$, we will suppose that the variables $X_1|\theta, X_2|\theta, ..., X_n|\theta$ are independently and identically distributed. To simplify the presentation, we use the following notations

$$\mu(\theta) = \omega \frac{E_{f(x|\theta)}\left[\delta_0(x)e^{hx}|\theta\right]}{E_{f(x|\theta)}\left[e^{hx}|\theta\right]} + (1-\omega)\,\frac{E_{f(x|\theta)}\left[Xe^{hx}|\theta\right]}{E_{f(x|\theta)}\left[e^{hx}|\theta\right]},$$

is the individual Esscher premium.

$$m = \omega\delta_0^* + (1-\omega)\,\frac{E_\pi\left[\mu(\theta)e^{h\mu(\theta)}\right]}{E_\pi\left[e^{h\mu(\theta)}\right]},$$

is the collective Esscher premium. Then, the coefficients $c_0$, $c_1$ must be determined by minimizing

$$\min_{c_0,c_1} E\left[\omega\left(\delta_0 - c_0 - c_1 H_n\left[x\right]\right)^2 e^{h\mu(\theta)} + (1-\omega)\left(\mu(\theta) - c_0 - c_1 H_n\left[x\right]\right)^2 e^{h\mu(\theta)}\right]. \tag{4.1}$$

In order to find the solution to (4.1), we write

$$m_h\left(\theta\right) = E\left[e^{h\mu(\theta)} \mid \theta\right]$$

$$m_h = E\left[m_h\left(\theta\right)\right] = \;\; E_\Pi\left[e^{h\mu(\theta)}\right]$$

$$f_n\left(\theta\right) = E\left[H_n\left[x\right] \mid \theta\right]$$

$$E^*\left[f_n\left(\theta\right)\right] = \frac{E\left[f_n\left(\theta\right)m_h\left(\theta\right)\right]}{E\left[m_h\left(\theta\right)\right]} = \frac{E\left[H_n\left[x\right]e^{h\mu(\theta)}\right]}{E\left[e^{h\mu(\theta)}\right]},$$

where $\pi^*(\theta) = \dfrac{\pi(\theta)m_h\left(\theta\right)}{m_h}$ is a probability distribution function. To achieve the minimum in (6), the derivative with respect to "$c_0$" must be set to zero, namely,

$$E\left[\omega e^{h\mu(\theta)}\left(\delta_0 - c_0 - c_1 H_n\left[x\right]\right)\right] + E\left[\left(1-\omega\right)e^{h\mu(\theta)}\left(\mu(\theta) - c_0 - c_1 H_n\left[x\right]\right)\right] = 0$$

$$\begin{aligned}&\omega\left(E\left[e^{h\mu(\theta)}\delta_0\right] - c_0 E\left[e^{h\mu(\theta)}\right] - c_1 E\left[e^{h\mu(\theta)}H_n\left[x\right]\right]\right) + \\ &\left(1-\omega\right)\left(E\left[e^{h\mu(\theta)}\mu(\theta)\right] - c_0 E\left[e^{h\mu(\theta)}\right] - c_1 E\left[e^{h\mu(\theta)}H_n\left[x\right]\right]\right) = 0\end{aligned}$$

$$c_0 E\left[e^{h\mu(\theta)}\right] = \omega E\left[e^{h\mu(\theta)}\delta_0\right] + \left(1-\omega\right)E\left[e^{h\mu(\theta)}\mu(\theta)\right] - c_1 E\left[e^{h\mu(\theta)}H_n\left[x\right]\right]$$

$$
\begin{aligned}
c_0 &= \frac{\omega E\left[e^{h\mu(\theta)}\delta_0\right] + (1-\omega)\, E\left[e^{h\mu(\theta)}\mu(\theta)\right] - c_1 E\left[e^{h\mu(\theta)} H_n\,[x]\right]}{E\left[e^{h\mu(\theta)}\right]} \\
c_0 &= \omega\frac{E\left[\delta_0 e^{h\mu(\theta)}\right]}{E\left[e^{h\mu(\theta)}\right]} + (1-\omega)\,\frac{E\left[\mu(\theta) e^{h\mu(\theta)}\right]}{E\left[e^{h\mu(\theta)}\right]} - c_1\frac{E\left[H_n\,[x]\, e^{h\mu(\theta)}\right]}{E\left[e^{h\mu(\theta)}\right]} \\
c_0 &= m - c_1\frac{E\left[H_n\,[x]\, e^{h\mu(\theta)}\right]}{E\left[e^{h\mu(\theta)}\right]} \\
c_0 &= m - c_1\frac{E\left[f_n\,(\theta)\, m_h\,(\theta)\right]}{E\left[m_h\,(\theta)\right]}
\end{aligned}
$$

$$
c_0 = m - c_1 E^*\left[f_n\,(\theta)\right]. \tag{4.2}
$$

Now, the problem is equivalent to:

$$
\min_{c_1} E\left[\begin{array}{c} \omega\left(\delta_0 - m + c_1 E^*\left[f_n\,(\theta)\right] - c_1 H_n\,[x]\right)^2 e^{h\mu(\theta)} \\ +\,(1-\omega)\left(\mu(\theta) - m + c_1 E^*\left[f_n\,(\theta)\right] - c_1 H_n\,[x]\right)^2 e^{h\mu(\theta)} \end{array}\right] = 0
$$

$$
\min_{c_1} E\left[\begin{array}{c} \omega e^{h\mu(\theta)}\left(\delta_0 - m - c_1\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\right)^2 \\ +(1-\omega)e^{h\mu(\theta)}(\mu(\theta) - m - c_1\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right))^2 \end{array}\right] = 0
$$

Taking derivative to "$c_1$":

$$
\begin{aligned}
&2c_1 E\left[\omega e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2\right] - 2E\left[\omega e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\left(\delta_0 - m\right)\right] \\
&+2c_1 E\left[(1-\omega)\, e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2\right] \\
&-2E\left[(1-\omega)\, e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\left(\mu(\theta) - m\right)\right] = 0 \\
&c_1 E\left[e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2\right] = E\left[\omega e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\left(\delta_0 - m\right)\right] \\
&\qquad\qquad + E\left[(1-\omega)\, e^{h\mu(\theta)}\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\left(\mu(\theta) - m\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)\left(\omega\delta_0 - \omega\mu\,(\theta) + \mu\,(\theta) - m\right) e^{h\mu(\theta)}\right]}{E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 e^{h\mu(\theta)}\right]} \\
c_1 &= \frac{E\left[\left(\omega\delta_0 + (1-\omega)\mu\,(\theta) - m\right)\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right) e^{h\mu(\theta)}\right]}{E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 e^{h\mu(\theta)}\right]} \\
c_1 &= \frac{E\left[E\left[\left(\omega\delta_0 + (1-\omega)\mu\,(\theta) - m\right)\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right) e^{h\mu(\theta)} \mid \theta\right]\right]}{E\left[E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 e^{h\mu(\theta)} \mid \theta\right]\right]} \\
c_1 &= \frac{E\left[E\left[\left(\omega\delta_0 + (1-\omega)\mu\,(\theta) - m\right)\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right) e^{h\mu(\theta)} \mid \theta\right]\right]}{E\left[E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 e^{h\mu(\theta)} \mid \theta\right]\right]} \\
c_1 &= \frac{\frac{1}{m_h} E\left[E\left[\left(\omega\delta_0 + (1-\omega)\mu\,(\theta) - m\right)\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right) e^{h\mu(\theta)} \mid \theta\right]\right]}{\frac{1}{m_h} E\left[E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 e^{h\mu(\theta)} \mid \theta\right]\right]} \\
c_1 &= \frac{\frac{1}{m_h} E\left[\left(\omega\delta_0 + (1-\omega)\mu\,(\theta) - m\right)\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right) m_h\,(\theta)\right]}{\frac{1}{m_h} E\left[E\left[\left(H_n\,[x] - E^*\left[f_n\,(\theta)\right]\right)^2 m_h\,(\theta)\right]\right]}
\end{aligned}
$$

$$c_1 = \frac{E\left[\omega(\delta_0-\delta_0^*)+(1-\omega)(\mu\left(\theta\right)-\frac{E_\pi\left[\mu(\theta)e^{h\mu(\theta)}\right]}{E_\pi\left[e^{h\mu(\theta)}\right]})\left(f_n\left(\theta\right)-E^*\left[f_n\left(\theta\right)\right]\right)\frac{m_h(\theta)}{m_h}\right]}{E\left[E\left[\left(H_n\left[x\right]-E^*\left[f_n\left(\theta\right)\right]\right)^2\frac{m_h(\theta)}{m_h}\right]\right]}$$

$$c_1 = \frac{(1-\omega)E\left[(\mu\left(\theta\right)-\frac{E_\pi\left[\mu(\theta)e^{h\mu(\theta)}\right]}{E_\pi\left[e^{h\mu(\theta)}\right]})\left(f_n\left(\theta\right)-E^*\left[f_n\left(\theta\right)\right]\right)\frac{m_h(\theta)}{m_h}\right]}{E\left[E\left[\left(H_n\left[x\right]-E^*\left[f_n\left(\theta\right)\right]\right)^2\frac{m_h(\theta)}{m_h}\right]\right]}$$

$$c_1 = \frac{(1-\omega)E\left[(\mu\left(\theta\right)-\frac{E_\pi\left[\mu(\theta)e^{h\mu(\theta)}\right]}{E_\pi\left[e^{h\mu(\theta)}\right]})\left(f_n\left(\theta\right)-E^*\left[f_n\left(\theta\right)\right]\right)\frac{m_h(\theta)}{m_h}\right]}{var^*\left[f_n\left(\theta\right)\right]+E^*\left[var\left[H_n\left[x\right]\mid\theta\right]\right]}$$

$$c_1 = \frac{(1-\omega)cov^*\left(\mu(\theta), f_n\left(\theta\right)\right)}{var^*\left[f_n\left(\theta\right)\right]+E^*\left[var\left[H_n\left[x\right]\mid\theta\right]\right]}. \tag{4.3}$$

Using (4.2) and (4.3), we find

$$\begin{aligned} H(\mu(\theta)|X_1,X_2,...,X_n) &= c_0+c_1H_n\left[x\right] \\ &= m-c_1E^*\left[f_n\left(\theta\right)\right]+c_1H_n\left[x\right] \\ &= c_1H_n\left[x\right]+\left(1-\frac{c_1E^*\left[f_n\left(\theta\right)\right]}{m}\right)m \end{aligned}$$

## 5. Numerical simulation

This section is made in order to illustrate the convergence of the empirical premium to the individual Esscher premium using a numerical simulation. We assume that $X$ follows a Poisson distribution with parameter $\theta$, and the prior is a gamma distribution. Taking $\delta_0(x)=\bar{x}e^h$

$$H(\mu(\theta)|X_1,X_2,...,X_n) = c_1H_n\left[x\right]+\left(1-\frac{c_1E^*\left[f_n\left(\theta\right)\right]}{m}\right)m$$

With:

$$c_1 = \frac{(1-\omega)cov^*\left(\mu(\theta), f_n\left(\theta\right)\right)}{var^*\left[f_n\left(\theta\right)\right]+E^*\left[var\left[H_n\left[x\right]\mid\theta\right]\right]}$$

It is quite difficult to work out a closed form of $c_1$ due to the obstacle in the analytic calculation of $f_n\left(\theta\right)=E\left[H_n\left[x\right]\mid\theta\right]$ where $H_n\left[x\right]=\frac{\sum_{i=1}^n X_ie^{hX_i}}{\sum_{i=1}^n e^{hX_i}}$. Thus, instead, we use a Monte Carlo method to compute numerically $c_1$. The algorithm is described as follows:
1- Randomly sample 4 values, $\theta_k$, $k=1,2,3,4$ from distribution with density $\pi^*(\theta)\sim gamma(\alpha,\beta-he^h)$.
2- For each $\theta_k$, we produce 1000 repetitions of sampling data, each of which consists of $n$ independent and identically distributed values.
3- For each $\theta_k$, we find the vector $H_j$ (the empirical premium), according to this vector, we calculate: $U_k, V_k, W_k$ ,i.e., compute:

$$H_j = \frac{\sum_{i=1}^n X_{ij}e^{hX_{ij}}}{\sum_{i=1}^n e^{hX_{ij}}}$$

$$U_k = \frac{\sum_{j=1}^{1000} H_j}{1000}$$

$$V_k = \frac{\sum_{j=1}^{1000}\left(H_j-U_k\right)^2}{1000-1}$$

$$W_k = \frac{\sum_{j=1}^{1000}\sum_{i=1}^{n} X_{ij} e^{hX_{ij}}}{\sum_{j=1}^{1000}\sum_{i=1}^{n} e^{hX_{ij}}}.$$

4- We calculate:

$$\begin{aligned}
A &= \frac{1}{4-1}\sum_{k=1}^{4}\left(U_k-\bar{U}\right)\left(W_k-\bar{W}\right) = cov^*\left(\mu(\theta), f_n(\theta)\right) \\
B &= \frac{1}{4-1}\sum_{k=1}^{4}\left(U_k-\bar{U}\right)^2 = var^*\left[f_n(\theta)\right] \\
C &= \frac{1}{4}\sum_{k=1}^{4} V_k = E^*\left[var\left[H_n[x] \mid \theta\right]\right] \\
D &= \frac{1}{4}\sum_{k=1}^{4} U_k = E^*\left[f_n(\theta)\right]
\end{aligned}$$

then,

$$c_1 = \frac{(1-\omega)A}{B+C}.$$

**Simulation I**

we take $h = 0.8$, $\omega = 0.9$, $\alpha = 2$ and $\beta = 6$. In addition, four different values of $\theta$ are given. Furthermore, two sample sizes are considered: $n = 100$ and $n = 150$. The corresponding simulation results are listed in the following tables:

| $n$=100 | $\mu(\theta)$ | $c_1$ | $m = P_c^{L_2}$ | $\bar{H}$ | $sd_{H(\mu(\theta)\mid X_1,X_2,\ldots,X_n)}$ |
|---|---|---|---|---|---|
| $\theta$=0.2 | 0.445108 | 0.0880289 | 0.503881 | 0.4467347 | 0.01009818 |
| $\theta$=0.4 | 0.890216 | 0.0880289 | 0.903917 | 0.8855569 | 0.00539665 |
| $\theta$=0.6 | 1.335300 | 0.0880289 | 1.312647 | 1.332238 | 0.2863931 |
| $\theta$=0.8 | 1.780400 | 0.0880289 | 1.707736 | 1.763652 | 0.17487 |

Table 1. Simulation results, $n = 100$

| $n$=150 | $\mu(\theta)$ | $c_1$ | $m = P_c^{L_2}$ | $\bar{H}$ | $sd_{H(\mu(\theta)\mid X_1,X_2,\ldots,X_n)}$ |
|---|---|---|---|---|---|
| $\theta$=0.2 | 0.445108 | 0.08798384 | 0.5012096 | 0.4436123 | 0.00149569 |
| $\theta$=0.4 | 0.890216 | 0.08798384 | 0.9060136 | 0.887553 | 0.00266295 |
| $\theta$=0.6 | 1.335300 | 0.08798384 | 1.3071587 | 1.326706 | 0.01095288 |
| $\theta$=0.8 | 1.780400 | 0.08798384 | 1.707462 | 1.76525 | 0.04263834 |

Table 2. Simulation results, $n = 150$

**Simulation II**

To proving the closeness of this new credibility premium, we make another simulation by Taking 9 values of $\theta$ with the following parameters: $h = 0.004$, $\omega = 0.9$, $\alpha = 2$ and $\beta = 8$, the same sample sizes are considered in this simulation.

| $n$=100 | $\mu(\theta)$ | $c_1$ | $m=P_c^{L2}$ | $\bar{H}$ | $sd_{H(\mu(\theta)\|X_1,X_2,\ldots,X_n)}$ |
|---|---|---|---|---|---|
| $\theta$=0.01 | 0.01002 | 0.059246202 | 0.021610547 | 0.01922807 | 0.00101804 |
| $\theta$=0.02 | 0.02004 | 0.059246202 | 0.030393611 | 0.02858928 | 0.008943467 |
| $\theta$=0.03 | 0.03006 | 0.059246202 | 0.040062203 | 0.03889427 | 0.008181747 |
| $\theta$=0.04 | 0.04008 | 0.059246202 | 0.048971771 | 0.04839034 | 0.009120648 |
| $\theta$=0.05 | 0.0501001 | 0.059246202 | 0.057808875 | 0.05780403 | 0.007527759 |
| $\theta$=0.06 | 0.06012012 | 0.059246202 | 0.066727051 | 0.06682108 | 0.001030524 |
| $\theta$=0.07 | 0.07014014 | 0.059246202 | 0.077531796 | 0.07676451 | 0.008697686 |
| $\theta$=0.08 | 0.08016016 | 0.059246202 | 0.083552218 | 0.08335358 | 0.00742907 |
| $\theta$=0.09 | 0.09036072 | 0.059246202 | 0.09637804 | 0.0947857 | 0.003017266 |

Table 3. Simulation results, $n = 100$

| $n$=150 | $\mu(\theta)$ | $c_1$ | $m=P_c^{L2}$ | $\bar{H}$ | $sd_{H(\mu(\theta)\|X_1,X_2,\ldots,X_n)}$ |
|---|---|---|---|---|---|
| $\theta$=0.01 | 0.01004008 | 0.06906078 | 0.021616573 | 0.0187785 | 0.009932746 |
| $\theta$=0.02 | 0.02008016 | 0.06906078 | 0.030670718 | 0.02855261 | 0.008946811 |
| $\theta$=0.03 | 0.03012024 | 0.06906078 | 0.039429681 | 0.03800808 | 0.00718741 |
| $\theta$=0.04 | 0.04016032 | 0.06906078 | 0.048875386 | 0.04820519 | 0.008981859 |
| $\theta$=0.05 | 0.0502004 | 0.06906078 | 0.057242788 | 0.05723828 | 0.006831935 |
| $\theta$=0.06 | 0.06024048 | 0.06906078 | 0.066706573 | 0.06745428 | 0.00106605 |
| $\theta$=0.07 | 0.07028056 | 0.06906078 | 0.073971576 | 0.07529701 | 0.006379061 |
| $\theta$=0.08 | 0.08032064 | 0.06906078 | 0.084158238 | 0.08629355 | 0.008426812 |
| $\theta$=0.09 | 0.09036072 | 0.06906078 | 0.093061777 | 0.09590581 | 0.001535335 |

Table 4. Simulation results, $n = 150$

where, $\bar{H}(\mu(\theta)|X_1, X_2, ..., X_n)$ is denoted by $\bar{H}$.

Here, $\bar{H}(\mu(\theta)|X_1, X_2, ..., X_n)$ is the average of 1000 repetitions of$H(\mu(\theta)|X_1, X_2, ..., X_n)$, $sd_{H(\mu(\theta)|X_1,X_2,...,X_n)}$ denotes the standard deviation of $H(\mu(\theta)|X_1, X_2, ..., X_n)$, and $\mu(\theta)$ is the individual Esscher premium. The simulation shows better closeness of $\bar{H}(\mu(\theta)|X_1, X_2, ..., X_n)$ to $\mu(\theta)$ than $m$. Moreover, the simulation results show that the new exact credibility premium $H(\mu(\theta)|X_1, X_2, ..., X_n)$ is much closed to the individual Esscher premium. Also, we observe if $\theta \to 1$, the new exact credibility premium $H(\mu(\theta)|X_1, X_2, ..., X_n)$ is more much closed to the individual Esscher premium. This closeness proves the consistency of this credibility premium.

**5.1. Remark.** In this work, we take only the single insurance contract, which is valid only when the collective premium is given.

## 6. Conclusion

This study investigated weighted balanced loss function under the Esscher principle and generalized credibility premiums. It then derived distribution-free credibility premiums. More precisely, it employed a distribution free approach under WBLF to obtain a simple and new credibility premium which it is a combination of the collective premium and the individual Esscher premium.

Using a numerical simulation approach, it obtained empirical premiums and illustrated whether the empirical premiums converged to the Esscher premium. Its numerical simulation provides evidence of the convergence of the new credibility premiums derived in the study, and shows a simple way to calculate credibility premiums for actuaries.

However, our study is limited in the case of Poisson-gamma in which the Esscher premium has a linear formula, but under the other combinations of the exponential family, the Esscher premium does not hold.

Another research topic should include the other cases of the exponential family to generalizing this study and giving to insurer the choice between distributions.

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