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Herstein's theorem for generalized derivations in rings with involution

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Abstract

Let R be an associative ring. An additive map $F : R \to R$ is called a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [7], Herstein proved the following result: If R is a prime ring of $char(R) \neq 2$ admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. In the present paper, we shall study the above mentioned result for generalized derivations in rings with involution.

Keywords: Prime ring, involution, derivation, generalized derivation.

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1. Introduction, Notations and Results

Throughout the present paper, R always denotes an associative ring with center $Z(R)$, C is an extended centroid of R and U is a left Utumi quotient ring of R. A ring R is said to be 2-torsion free if $2x = 0$ (where $x \in R$) implies that $x = 0$. A ring R is called prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$, and is called semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. Following [6], an additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x \in R$. A ring equipped with involution is called ring with involution or $*$-ring. A ring R with involution is called normal if $xx^* = x^*x$ for all $x \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$.

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An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d is said to be inner if there exists an element $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. In [3], Brešar introduced the algebraic definition of generalized derivation as follows: an additive mapping $F : R \to R$ is called a generalized derivation of R if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $F(x) = ax + xb$ for some $a, b \in R$; such generalized derivations are called inner.

In [7], Herstein proved that if R is a prime ring of $char(R) \neq 2$ admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Later, Daif [4] extended Herstein's result for two sided ideals of a semiprime ring. Motivated by these results Dar and Ali [5] proved the following result: Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. If R admits a nonzero derivation d such that $[d(x), d(x^*)] = 0$ for all $x \in R$, then R is commutative. Further, in [2] Bell and Rehman generalized Heristein's theorem for generalized derivations. In particular, they showed that if a prime ring of $char(R) \neq 2$ with identity admitting a generalized derivation $F : R \to R$ such that $[F(x), F(y)] = 0$ for all $x, y \in R$, then either R is commutative or R is a 2×2 matrices over a field and $f(x) = ax + xa$ for all $x \in R$, where a is a fixed element of R.

In the present paper, we study the Dar and Ali's [5] result for generalized derivations in prime rings with involution. More precisely, we prove the following:

1.1. Theorem. *Let R be a non commutative prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero generalized derivation $F : R \to R$ such that $[F(x), F(x^*)] = 0$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center and $F(x) = ax + xb$ for all $x \in R$ and fixed $a, b \in U$ such that $a - b \in C$.*

We recall some well known facts which will be helpful in order to prove our results:

Fact 1. [1, Lemma 2.1] Let R be a prime ring with involution such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.

Fact 2. The center of a prime ring is free from zero devisors.

Fact 3. Let R be a ring with involution such that $char(R) \neq 2$. Then, every $x \in R$ can uniquely represented as $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

Fact 4. [8, Theorem 3] Let $n > 1$ be a fixed integer and R is a prime ring of $char(R) \neq 2, ..., n-1$. If $F : R \to R$ is a generalized derivation such that $(F(x))^n = 0$ for all $x \in R$, then $F = 0$.

Proof of Theorem 1.1. By the given hypothesis, we have

(1.1) $$[F(x), F(x^*)] = 0 \text{ for all } x \in R.$$

Replacing x by $h + k$ in (1.1), where $h \in H(R)$ and $k \in S(R)$, we have

(1.2) $$[F(k), F(h)] = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R).$$

Taking $h = k_1^2$ in (1.2), where $k_1 \in S(R) \cap Z(R)$, we get

(1.3) $$[F(k), F(k_1)]k_1 = 0 \text{ for all } k \in S(R) \text{ and } k_1 \in S(R) \cap Z(R).$$

In view of $Fact$ 2, we have

(1.4) $$[F(k), F(k_1)] = 0 \text{ for all } k \in S(R) \text{ and } k_1 \in S(R) \cap Z(R).$$

Replacing k by h_0k_1 in (1.4), where $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$, we get

$$[F(h_0), F(k_1)]k_1 + [h_0, F(k_1)]d(k_1) = 0 \tag{1.5}$$

for all $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Application of (1.2) yields that

$$[h_0, F(k_1)]d(k_1) = 0 \text{ for all } h_0 \in H(R) \text{ and } k_1 \in S(R) \cap Z(R). \tag{1.6}$$

Using the primeness of R we obtain, either $[h_0, F(k_1)] = 0$ or $d(k_1) = 0$. Assume that $[h_0, F(k_1)] = 0$ for all $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Replacing h_0 by k_0k_1 in the last expression, where $k_0 \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, we obtain $[k_0, F(k_1)]k_1 = 0$ for all $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. This further implies that $[k_0, F(k_1)] = 0$ for all $k_0 \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. In view of $Fact$ 3, we have $2[y, F(k_1)] = [2y, F(k_1)] = [h_0 + k_0, F(k_1)] = [h_0, F(k_1)] + [k_0, F(k_1)] = 0$. Since $char(R) \neq 2$, the last expression yields that $[y, F(k_1)] = 0$ for all $y \in R$ and $k_1 \in S(R) \cap Z(R)$. That is, $F(k_1) \in Z(R)$ for all $k_1 \in S(R) \cap Z(R)$. Next, substituting k by hk_1 in (1.2), where $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$, we get $[F(hk_1), F(h)] = [F(h)k_1, F(h)] + [hd(k_1), F(h)] = [h, F(h)]d(k_1) = 0$ for all $h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. Again primeness of R forces that either $[h, F(h)] = 0$ or $d(k_1) = 0$. Assume that $[h, F(h)] = 0$ for all $h \in H(R)$. Taking $h = kk_1$ for all $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$ and using $F(k_1) \in Z(R)$, we get $[k, d(k)]k_1^2 = 0$ for all $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. Application of $Fact$ 2 yields that $[k, d(k)] = 0$ for all $k \in S(R)$. Next, replacing k by hk_1 in the last relation, we get $[h, d(h)]k_1^2 = 0$. This implies that

$$[h, d(h)] = 0 \text{ for all } h \in H(R). \tag{1.7}$$

On linearizing (1.7), we obtain

$$[d(h), h_0] + [d(h_0), h] = 0 \text{ for all } h, h_0 \in H(R). \tag{1.8}$$

Which can be further written as

$$[d(h_0), h] = [h_0, d(h)] \text{ for all } h, h_0 \in H(R). \tag{1.9}$$

Substituting h^2 for h in above expression, we obtain

$$[d(h_0), h^2] = [h_0, d(h)]h + h[h_0, d(h)] + d(h)[h_0, h] + [h_0, h]d(h) \tag{1.10}$$

for all $h, h_0 \in H(R)$. Also, we have

$$[d(h_0), h^2] = [d(h_0), h]h + h[d(h_0), h] = [h_0, d(h)]h + h[h_0, d(h)] \tag{1.11}$$

for all $h, h_0 \in H(R)$. Combining (1.10) and (1.11), we obtain

$$d(h)[h_0, h] + [h_0, h]d(h) = 0 \text{ for all } h, h_0 \in H(R). \tag{1.12}$$

Now, taking $h_0 = kk_1$ in (1.12), where $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, and using the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$d(h)[k, h] + [k, h]d(h) = 0 \text{ for all } h \in H(R) \text{ and } k \in S(R). \tag{1.13}$$

Replacing h by $h + h_1$ in (1.13), where $h_1 \in H(R) \cap Z(R)$, we get

$$d(h)[k, h] + d(h_1)[k, h] + [k, h]d(h) + [k, h]d(h_1) = 0 \tag{1.14}$$

for all $h \in H(R)$ and $k \in S(R)$. In view of (1.13), the last relation reduces to

$$2[k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R).$$

Since $char(R) \neq 2$, the above expression gives us

$$[k, h]d(h_1) = 0 \text{ for all } h \in H(R), k \in S(R) \text{ and } h_1 \in H(R) \cap Z(R). \tag{1.15}$$

Since R is prime, this yields that either $[k, h] = 0$ or $d(h_1) = 0$. If $[k, h] = 0$ for all $h \in H(R)$ and $k \in S(R)$, then in view of $Fact$ 1, R must be commutative, this leads to a contradiction. Now assume that $d(h_1) = 0$ for all $h_1 \in H(R) \cap Z(R)$. This further implies that $d(k_1^2) = 0$ and hence $d(k_1) = 0$ for all $k_1 \in S(R) \cap Z(R)$. Replacing k by $h_0 k_1$ in (1.2), where $h_0 \in H(R)$ and $k_1 \in S(R) \cap Z(R)$ and using $d(k_1) = 0$, we get

$$[F(h_0), F(h)]k_1 = 0 \tag{1.16}$$

for all $h_0, h \in H(R)$ and $k_1 \in S(R) \cap Z(R)$. This further implies that

$$[F(h_0), F(h)] = 0 \text{ for all } h_0, h \in H(R). \tag{1.17}$$

In view of *Fact* 3, (1.2) and (1.17), we find that $2[F(x), F(h)] = [F(2x), F(h)] = [F(h_0 + k), F(h)] = [F(h_0), F(h)] + [F(k), F(h)] = 0$ for all $h, h_0 \in H(R)$, $k \in S(R)$ and $x \in R$. Since $char(R) \neq 2$, the last expression yields that $[F(x), F(h)] = 0$ for all $x \in R$ and $h \in H(R)$. Taking $h = kk_1$ in this relation, where $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$, yields $[F(x), F(k)] = 0$ for all $x \in R$ and $k \in S(R)$; and another application of *Fact* 3 gives

$$[F(x), F(y)] = 0 \text{ for all } x, y \in R. \tag{1.18}$$

In view of [9, Theorem 1.1], R is an order in a central simple algebra of dimension at most 4 over its center and $F(x) = ax + xb$ for all $x \in R$ and fixed $a, b \in U$ such that $a - b \in C$. This completes the proof. □

If we replace commutator by the anti-commutator in Theorem 1.1, then we obtain the following result:

1.2. Theorem. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ such that $F(x) \circ F(x^*) = 0$ for all $x \in R$, then $F = 0$.*

Proof. We have

$$F(x) \circ F(x^*) = 0 \text{ for all } x \in R. \tag{1.19}$$

That is,

$$F(x)F(x^*) + F(x^*)F(x) = 0 \text{ for all } x \in R. \tag{1.20}$$

On linearizing (1.20), we get

$$F(x)F(y^*) + F(y)F(x^*) + F(y^*)F(x) + F(x^*)F(y) = 0 \tag{1.21}$$

for all $x, y \in R$. Replacing y by yh_1 in (1.21), where $h_1 \in H(R) \cap Z(R)$, we get

$$\begin{aligned} 0 = \; & F(x)F(y^*)h_1 + F(x)y^*d(h_1) + F(y)F(x^*)h_1 \\ & + d(h_1)yF(x^*) + F(x^*)F(y)h_1 \\ & + F(x^*)yd(h_1) + F(y^*)F(x)h_1 \\ & + d(h_1)yF(x) \text{ for all } x, y \in R \text{ and } h_1 \in H(R) \cap Z(R). \end{aligned} \tag{1.22}$$

In view of (1.21), we conclude that

$$(F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x))d(h_1) = 0 \tag{1.23}$$

for all $x, y \in R$ and $h_1 \in H(R) \cap Z(R)$. Application of *Fact* 2 yields that either $F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x) = 0$ or $d(h_1) = 0$. Suppose that $d(h_1) = 0$ for all $h_1 \in H(R) \cap Z(R)$. This further implies that $d(k_1) = 0$ for all $k_1 \in S(R) \cap Z(R)$. Replacing y by yk_1 in (1.21), where $k_1 \in S(R) \cap Z(R)$ and using the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$-F(x)F(y^*) + F(y)F(x^*) - F(y^*)F(x) + F(x^*)F(y) = 0 \tag{1.24}$$

for all $x, y \in R$. On combining (1.21) and (1.24), we get

$$F(x)F(y^*) + F(y^*)F(x) = 0 \text{ for all } x, y \in R. \tag{1.25}$$

Taking $y = x^*$ and using the fact that $char(R) \neq 2$, we have

$$(F(x))^2 = 0 \text{ for all } x \in R. \tag{1.26}$$

In view of $Fact$ 4, we get $F = 0$. Next, we suppose that

$$F(x)y^* + yF(x^*) + F(x^*)y + y^*F(x) = 0 \text{ for all } x, y \in R. \tag{1.27}$$

Taking $y = h_1$ in above expression, where $h_1 \in H(R) \cap Z(R)$ and using the fact that $char(R) \neq 2$, we obtain

$$F(x + x^*)h_1 = 0 \text{ for all } x \in R \text{ and } h_1 \in H(R) \cap Z(R). \tag{1.28}$$

This implies that

$$F(x + x^*) = 0 \text{ for all } x \in R. \tag{1.29}$$

That is,

$$F(x) = -F(x^*) \text{ for all } x \in R. \tag{1.30}$$

This reduces (1.20) into

$$(F(x))^2 = 0 \text{ for all } x \in R. \tag{1.31}$$

Again in view of $Fact$ 4, we get $F = 0$. Thereby proof of the theorem is completed. $\square$

At the end, let us write an example which shows that the restriction of second kind involution in Theorem 1.1 is not superfluous.

1.3. Example. Let $\mathbb{F}$ be any field. Consider $R = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| \, a_1, a_2, a_3, a_4 \in \mathbb{F} \right\}$. Of course, R with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings $F : R \longrightarrow R$, $d : R \longrightarrow R$, and $* : R \longrightarrow R$ such that $F\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & -2a_2 \\ 2a_3 & 0 \end{pmatrix}$, $d\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & -2a_2 \\ 2a_3 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^* = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix}$. Obviously, $Z(R) = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} \middle| \, a_1 \in \mathbb{F} \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, F is a nonzero generalized derivation with associated derivation d and satisfies the condition $[F(x), F(x^*)] = 0$ for all $x \in R$. In this case, F is of the form $F(x) = ax+xb$, where $a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. However, $a - b \notin C$. Hence, the hypothesis of the second kind involution is crucial in Theorem 1.1.

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