http://www.newtheory.org ISSN: 2149-1402

Journal of New Theory

Received: 26.04.2018 *Year*: **2018**, *Number*: **23**, *Pages*: **68-77**
Published: 12.07.2018 *Original Article*

On Some Identities and Symmetric Functions for Balancing Numbers

Ali Boussayoud <aboussayoud@yahoo.fr>

LMAM Laboratory and Department of Mathematics, University of MSB Jijel, Algeria.

Abstract - In this paper, we derive new generating functions of the product of balancing numbers, Lucas balancing numbers and the Chebychev polynomials of the second kind by making use of useful properties of the symmetric functions mentioned in the paper.

Keywords *- Balancing numbers, Lucas balancing number, Chebychev polynomials*.

1 Introduction and Preliminaries

Recently, Behera and Panda [1] introduced balancing numbers $n \in \mathbb{Z}_+$ as solutions of the Diophantine equation

$$1+2+...+(n-1)=(n+1)+(n+2)+...+(n+r). \qquad (1.1)$$

for some positive integer r which is called balancer or cobalancing number. For example $6;35;204;1189$ and 6930 are balancing numbers with balancers $2;14;84;492$ and 2870, respectively. If n is a balancing number with balancer r, then from (1.1) one has

$$\frac{n(n+1)}{2}=rn+\frac{r(r+1)}{2},$$

and so

$$r=\frac{-(2n+1)+\sqrt{8n^2+1}}{2} \text{ and } n=\frac{2r+1+\sqrt{8r^2+8r+1}}{2}.$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then

$$\begin{cases} B_{n+1}=6B_n-B_{n-1},\ n\geq 1 \\ B_0=0,\ B_1=1 \end{cases},$$

and

$$\begin{cases} b_{n+1} = 6b_n - b_{n-1} + 2,\ n \geq 2 \\ b_1 = 0,\ b_2 = 2 \end{cases}.$$

Definition 1.1. [14] The Lucas-balancing $\{C_n\}_{n\in\mathrm{N}^*}$ is defined recurrently by

$$\begin{cases} C_{n+1} = 6C_n - C_{n-1},\ n \geq 1 \\ C_0 = 1,\ C_1 = 3 \end{cases}.$$

The main purpose of this paper is to present some results involving the balancing number and Lucas-balancing number using define a new useful operator denoted by $\delta_{p_1 p_2}$ for which we can formulate, extend and prove new results based on our previous ones [3, 4, 5]. In order to determine generating functions of the product of balancing number, Lucas-balancing number and Chebychev polynomials of first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 1.2. [5] Let k and n be tow positive integer and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th elementary symmetric function $e_k(a_1, a_{2,} ..., a_n)$ is defined by

$$e_k(a_1, a_{2,} ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Definition 1.3. [5] Let k and n be tow positive integer and $\{a_1, a_2, ..., a_n\}$ are set of given variables the k-th complete homogeneous symmetric function $h_k(a_1, a, ..., a_n)$ is defined by

$$h_k(a_1, a_{2,} ..., a_n) = \sum_{i_1+i_2+...+i_n=k} a_1^{i_1} a_2^{i_2} ... a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n \geq 0.$

Remark 1.1. We set $e_0(a_1, a_{2,} ..., a_n) = 1$ and $h_k(a_1, a_{2,} ..., a_n) = 1$ (by convention). For $k > n$ or $k < 0,$ we set $e_0(a_1, a_{2,} ..., a_n) = 0$ and $h_k(a_1, a_{2,} ..., a_n) = 0.$

Definition 1.4. [7] Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form

$$E(-z)H(z) = \sum_{n=0}^{\infty} S_n(B-P)z^n,$$

with $H(z) = \Pi_{b\varepsilon B}(1-bz)^{-1},\ E(-z) = \Pi_{p\varepsilon P}(1-pz).$

Remark 1.2. $S_n(B-P)=0$ for $n<0.$

Definition 1.5. [5] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f)=\frac{f(p_1,\cdots,p_i,p_{i+1},\cdots p_n)-f(p_1,\cdots p_{i-1},p_{i+1,}p_i,p_{i+2}\cdots p_n)}{p_i-p_{i+1}}.$$

Definition 1.6. The symmetrizing operator $\delta^k_{p_1p_2}$ is defined by

$$\delta^k_{p_1p_2}(g)=\frac{p_1^k g(p_1)-p_2^k g(p_2)}{p_1-p_2} \text{ for all } k\in\mathbb{N}.$$

Proposition 1.1. [6] Let $P=\{p_1,p_2\}$ an alphabet, we define the operator $\delta^k_{p_1p_2}$ as follows

$$\delta^k_{p_1p_2}g(p_1)=S_{k-1}(p_1+p_2)g(p_1)+p_2^k\partial_{p_1p_2}g\,(p_1),\ \textit{for all}\ k\in\mathbb{N}.$$

2 Main Results

In our main results, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 2.1. Let A and P be two alphabets, respectively, $\{a_1,a_2\}$ and $\{b_1,b_2\}$, then we have

$$\sum_{n=0}^{\infty} h_n(a_1,a_2)h_{n+k-1}(p_1,p_2)z^n = \frac{h_{k-1}(p_1,p_2)+p_1p_2(a_1+a_2)h_{k-2}(p_1,p_2)z-a_1a_2p_1p_2\delta_{p_1p_2}(p_2^{k-1})z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}, \tag{2.1}$$

for all $k\in\mathbb{N}$.

Proof. By applying the operator $\partial_{p_1p_2}$ to the series $f(p_1z)=\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n$, we have

$$\begin{aligned}\partial_{p_1p_2}\left(\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n\right)&=\frac{\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_1^{n+k}z^n-\sum_{n=0}^{\infty}h_n(a_1,a_2)\,p_2^{n+k}z^n}{p_1-p_2}\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2)\left(\frac{p_1^{n+k}-p_2^{n+k}}{p_1-p_2}\right)z^n\\&=\sum_{n=0}^{\infty}h_n(a_1,a_2)\,h_{n+k-1}(p_1,p_2)\,z^n.\end{aligned}$$

On the other hand,

$$\partial_{p_1p_2}\left(\frac{p_1^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}\right)=\frac{\frac{p_1^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}-\frac{p_2^k}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}}{p_1-p_2}$$

$$=\frac{p_1^k\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)-p_2^k\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)}{(p_1-p_2)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{p_1^k-p_2^k-(a_1+a_2)\left(p_1^kp_2-p_2^kp_1\right)z-a_1a_2\left(p_2^kp_1^2-p_1^kp_2^2\right)z^2}{(p_1-p_2)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{\frac{p_1^k-p_2^k}{p_1-p_2}-(a_1+a_2)p_1p_2\left(\frac{p_1^{k-1}-p_2^{k-1}}{p_1-p_2}\right)z-a_1a_2p_1p_2\left(\frac{p_1p_2^{k-1}-p_2p_1^{k-1}}{p_1-p_2}\right)z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}$$

$$=\frac{h_{k-1}(p_1,p_2)+p_1p_2(a_1+a_2)h_{k-2}(p_1,p_2)z-a_1a_2p_1p_2\delta_{p_1p_2}(p_2^{k-1})z^2}{\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_1z)^n\right)\left(\sum_{n=0}^{\infty}e_n(a_1,a_2)(-p_2z)^n\right)}.$$

Thus, this completes the proof.

3 **Generating Functions of Some Well-known Numbers**

We now derive new generating functions of the products of some well-known numbers. Indeed, we consider Theorem 2.1 in order to derive balancing numbers, Lucas balancing numbers and Tchebychev polynomials of second kind and the symmetric functions.

If $k=0,1$ and $A=\{1,0\}$, we deduce the following lemmas

Lemma 3.1. [2] Given an alphabet $P=\{p_1,p_2\}$, we have

$$\sum_{n=0}^{\infty}h_n(p_1,p_2)z^n=\frac{1}{(1-p_1z)(1-p_2z)}. \tag{3.1}$$

Lemma 3.2. [3] Given an alphabet $P=\{p_1,p_2\}$, we have

$$\sum_{n=0}^{\infty}h_{n-1}(p_1,p_2)z^n=\frac{z}{(1-p_1z)(1-p_2z)}. \tag{3.2}$$

Replacing p_2 by $(-p_2)$ in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{\infty} h_n(p_1,[-p_2])z^n = \frac{1}{(1-p_1 z)(1+p_2 z)}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])z^n = \frac{z}{(1-p_1 z)(1+p_2 z)}. \tag{3.4}$$

Choosing p_1 and p_2 such that

$$\begin{cases} p_1 p_2 = -1, \\ p_1 - p_2 = 6, \end{cases}$$

and substituting in (3.3) and (3.4) we end up with

$$\sum_{n=0}^{\infty} h_n(p_1,[-p_2])z^n = \frac{1}{1-6z+z^2}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])z^n = \frac{z}{1-6z+z^2} \text{ with } p_{1,2} = 3 \pm 2\sqrt{2}, \tag{3.6}$$

Which represents a generating function for balancing numbers, such that $B_n = S_{n-1}(p_1 + [-p_2])$.

Multiplying the equation (3.6) by (**-3**) and added to (3.5), we obtain

$$\sum_{n=0}^{\infty} (h_n(p_1,[-p_2]) - 3h_{n-1}(p_1,[-p_2]))z^n = \frac{1-3z}{1-6z+z^2} \ ,$$

which represents a generating function for Lucas-balancing numbers.

Corollary 3.1. For all $n \in \mathbb{N}$, we have

$$C_n = h_n(p_1,[-p_2]) - 3h_{n-1}(p_1,[-p_2]), \text{with } p_{1,2} = 3 \pm 2\sqrt{2}.$$

Theorem 3.1. For $n \in \mathbb{N}$, the generating function of the cobalancing numbers numbers is given by

$$\sum_{n=0}^{\infty} b_n z^n = \frac{2z^2}{(1-6z+z^2)(1-z)}.$$

Proof. The ordinary generating function associated is defined by $G(b_n, z) = \sum_{n=1}^{\infty} b_n z^n$.
Using the initial conditions, we get

$$\sum_{n=1}^{\infty} b_n z^n = b_1 z + b_2 z^2 + \sum_{n=3}^{\infty} b_n z^n$$
$$= 2z^2 + \sum_{n=3}^{\infty} (6b_n - b_{n-1} + 2) z^n.$$

Consider that $j = n-2$ and $p = n-1$. Then can be written by

$$\sum_{n=1}^{\infty} b_n z^n = 2z^2 + 6z \sum_{n=1}^{\infty} b_n z^n - z^2 \sum_{n=3}^{\infty} b_n z^n + 2z^3 \sum_{n=0}^{\infty} z^n,$$

which is equivalent to

$$(1-6z+z^2) \sum_{n=1}^{\infty} b_n z^n = 2z^2 + \frac{2z^3}{1-z},$$

Therefore

$$\sum_{n=0}^{\infty} b_n z^n = \frac{2z^2}{(1-6z+z^2)(1-z)}.$$

This completes the proof.

If $k=0,\ k=1$ and $A = \{a_1, a_2\}$, we deduce the following theorems

Theorem 3.2. [8] Given two alphabets $A = \{a_1, a_2\}$ and $P = \{p_1, p_2\}$ we have

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_{n-1}(p_1, p_2) z^n = \frac{(a_1 + a_2) z - a_1 a_2 (p_1 + p_2) z^2}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n \right) \left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n \right)}. \quad (3.7)$$

Theorem 3.3. [9] Given two alphabets $A = \{a_1, a_2\}$ and $P = \{p_1, p_2\}$ we have

$$\sum_{n=0}^{\infty} h_n(a_1, a_2) h_n(p_1, p_2) z^n = \frac{1 - a_1 a_2 p_1 p_2 z^2}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n \right) \left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n \right)}. \quad (3.8)$$

From (3.8) we can deduce

$$\sum_{n=0}^{\infty} h_{n-1}(a_1, a_2) h_{n-1}(p_1, p_2) z^n = \frac{z - a_1 a_2 p_1 p_2 z^3}{\left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_1 z)^n \right) \left(\sum_{n=0}^{\infty} e_n(a_1, a_2)(-p_2 z)^n \right)}. \quad (3.9)$$

Case 1: Replacing p_2 by $(-p_2)$ and a_2 by $(-a_2)$ in (3.7) and (3.9) yields

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{(a_1-a_2)z + a_1a_2(p_1-p_2)z^2}{(1-a_1p_1z)(1+a_2p_1z)(1+a_1p_2z)(1-a_2p_2z)}, \qquad (3.10)$$

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z - a_1a_2p_1p_2z^3}{(1-a_1p_1z)(1+a_2p_1z)(1+a_1p_2z)(1-a_2p_2z)}. \qquad (3.11)$$

This case consists of four related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1a_2 = 1, \end{cases} \text{and} \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.10) give

$$\sum_{n=0}^{\infty} h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+6z^2}{1-6z+35z^2+6z^3+z^4},$$

which represents a new generating function for product of Fibonacci numbers with balancing numbers, such that $F_nB_n = h_n(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_{1,2} = \frac{1\pm\sqrt{5}}{2}$, $p_{1,2} = 3\pm 2\sqrt{2}$.

Secondly, the substitution of

$$\begin{cases} a_1 - a_2 = 6, \\ a_1a_2 = -1, \end{cases} \text{and} \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z-z^3}{1-36z+2z^2-36z^3+z^4},$$

which represents a new generating function for balancing numbers of second order, such that $B_n^2 = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_{1,2} = p_{1,2} = 3\pm 2\sqrt{2}$.

Thirdly, the substitution of

$$\begin{cases} a_1 - a_2 = 1, \\ a_1a_2 = 2, \end{cases} \text{and} \begin{cases} p_1 - p_2 = 6, \\ p_1p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+2z^3}{1-6z-13z^2+12z^3+4z^4},$$

which represents a new generating function for product of Jacobsthal numbers with balancing numbers, such that $J_nB_n = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_1 = 2,\ a_2 = -1$ and $p_{1,2} = 3\pm 2\sqrt{2}$.

Finally, the substitution of

$$\begin{cases} a_1 - a_2 = 2, \\ a_1 a_2 = 1, \end{cases} \text{and} \begin{cases} p_1 - p_2 = 6, \\ p_1 p_2 = -1, \end{cases}$$

in (3.11) give

$$\sum_{n=0}^{\infty} h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{z+z^3}{1-12z+38z^2+12z^3+z^4}.$$

which represents a new generating function for product of Pell numbers with balancing numbers, such that $P_n B_n = h_{n-1}(a_1,[-a_2])h_{n-1}(p_1,[-p_2])$ with $a_1 = 1 \pm \sqrt{2}$ and $p_{1,2} = 3 \pm 2\sqrt{2}.$

Case 2: Replacing p_2 by $(-p_2)$ and a_1 by $2a_1$ and a_2 by $(-2a_2)$ in (3.10) yields

$$\sum_{n=0}^{\infty} h_n(2a_1,[-2a_2])h_{n-1}(p_1,[-p_2])z^n = \frac{2(a_1-a_2)z + 4a_1a_2(p_1-p_2)z^2}{(1-2a_1p_1z)(1+2a_2p_1z)(1+2a_1p_2z)(1-2a_2p_2z)}, \quad (3.12)$$

The substitution of

$$\begin{cases} p_1 - p_2 = 6, \\ p_1 p_2 = -1, \\ a_1 a_2 = \frac{-1}{4}, \end{cases}$$

in (3.12) and set for ease on notations $x = a_1 - a_2$, we reach

$$\sum_{n=0}^{\infty} h_n(2a_1,[-2a_2])h_{n-1}(p_1,[-p_2])z^n = \sum_{n=0}^{\infty} B_n U_n(x) z^n$$
$$= \frac{2xz - 6z^2}{1-12xz+2(17+2x)z^2-12xz^3+z^4},$$

which corresponds to a new generating function for the combined balancing numbers and Tchebychev polynomials of the second kind.

Theorem 3.4. For $n \in \mathbb{N}$, the new generating function of the product of balancing numbers B_n and Tchebychev polynomials of first kind is given by

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \frac{xz^3 - 6z^2 + 2xz - x}{1-12xz+2(17+2x)z^2-12xz^3+z^4}.$$

Proof . We see that

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])\left(h_n(2a_1,[-2a_2]) - x h_{n-1}(2a_1,[-2a_2])\right) z^n$$
$$= \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) h_n(2a_1,[-2a_2]) z^n - x \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) h_{n-1}(2a_1,[-2a_2]) z^n$$
$$= \sum_{n=0}^{\infty} B_n U_n(x) z^n - \frac{x}{2(a_1+a_2)} \sum_{n=0}^{\infty} S_{n-1}(p_1+[-p_2])((2a_1)^n - (-2a_2)^n) z^n$$
$$= \sum_{n=0}^{\infty} B_n U_n(x) z^n - \frac{x}{2(a_1+a_2)} \begin{pmatrix} \sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2])(2a_1 z)^n - \\ \sum_{n=0}^{\infty} h_{n-1}(p_1+[-p_2])(-2a_2 z)^n \end{pmatrix}.$$

On the other hand, we know that

$$\sum_{n=0}^{\infty} h_{n-1}(p_1,[-p_2]) z^n = \sum_{n=0}^{\infty} B_n z^n = \frac{z}{1-6z+z^2},$$

from which it follows

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \frac{2xz-6z^2}{1-12xz+2(17+2x)z^2-12xz^3+z^4}$$
$$- \frac{x}{2(a_1+a_2)} \left(\frac{2a_1 z}{1-12a_1 z+4a_1^2 z^2} + \frac{2a_2 z}{1+12a_2 z+4a_2^2 z^2} \right),$$

therefore

$$\sum_{n=0}^{\infty} B_n T_n(x) z^n = \frac{xz^3-6z^2+2xz-x}{1-12xz+2(17+2x)z^2-12xz^3+z^4}.$$

This completes the proof.

Acknowledgements

We would like to thank the referees for useful suggestions and several comments witch involve the quality of the paper.

References

[1] A. Behera, G. K. Panda, On the Square Roots of Triangular Numbers. *The Fibonacci Quarterly*. **37**, 98-105, (1999).

[2] A. Boussayoud, M. Kerada, N. Harrouche, On the *k*-Lucas numbers and Lucas Polynomials, *Turkish Journal of Analysis and Number*. **5**(3) 121-125, (2017).

[3] A. Boussayoud, M. Bolyer, M. Kerada, On Some Identities and Symmetric Functions for lucas and pell numbers, *Electron. J. Math. Analysis Appl.* 5(1), 202-207, (2017).

[4] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers,

Online.J. Anal. Comb. **12** 1-10, (2017).
[5] A. Boussayoud, N. Harrouche, Complete Symmetric Functions and *k*-Fibonacci Numbers, *Commun. Appl. Anal*. **20**, 457-467, (2016).
[6] A. Boussayoud, M. Boulyer, M. Kerada, A simple and accurate method for determination of some generalized sequence of numbers, *Int. J. Pure Appl. Math*. **108**, 503-511, (2016).
[7] A. Boussayoud, A. Abderrezzak, M. Kerada, Some applications of symmetric functions, *Integers*. **15**, A#48, 1-7, (2015).
[8] A. Boussayoud, M. Kerada, R. Sahali , W. Rouibah, Some Applications on Generating Functions, *J. Concr. Appl. Math*. **12**, 321-330, (2014).
[9] A. Boussayoud, M. Kerada, Symmetric and Generating Functions, *Int. Electron. J. Pure Appl. Math*. **7**, 195-203 (2014).
[10] P. Catarino, H. Campos, P. Vasco, On the Mersenne sequence, *Ann. Math. Inform.* 46, 37- 53, (2016).
[11] V. E. Hoggatt, Fibonacci and Lucas Numbers. A publication of the Fibonacci Association. University of Santa Clara, Santa Clara, Houghton Mifflin Company, 1969.
[12] T. Koshy, Fibonacci and Lucas Numbers with Applications, John Wiley, New York, 2001.
[13] T. Koshy, Z. Gao, Catalan numbers with Mersenne subscripts, *Math. Sci*. 38, 86-91, (2013).
[14] A. Ozkoc, A. Tekcan, On *k*-balancing numbers, *Noteson Number Theory and Discrete Mathematics*. 20, 38-52, (2017).