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GLOBAL BEHAVIOR OF A THREE-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS OF ORDER THREE

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Abstract. In this paper, we investigate the global behavior of the positive solutions of the system of difference equations

$$u_{n+1}=\frac{\alpha_1 u_{n-1}}{\beta_1+\gamma_1 v_{n-2}^{p}},\ v_{n+1}=\frac{\alpha_2 v_{n-1}}{\beta_2+\gamma_2 w_{n-2}^{q}},\ w_{n+1}=\frac{\alpha_3 w_{n-1}}{\beta_3+\gamma_3 u_{n-2}^{r}}$$

for $n\in\mathbb{N}_0$ where the initial conditions u_{-i},v_{-i},w_{-i} $(i=0,1,2)$ are non-negative real numbers and the parameters $\alpha_j,\beta_j,\gamma_j$ $(j=1,2,3)$ and p,q,r are positive real numbers, by extending some results in the literature.

1. Introduction

Recently, difference equations have gained a great importance. Most of the recent applications of these equations have appeared in many scientific areas such as biology, physics, engineering, economics. Particularly, rational difference equations and their systems have great importance in applications. So, it is very worthy to examine the behavior of solutions of a system of rational difference equations and to discuss the stability character of their equilibrium points. In recent years, many researchers have investigated global behavior of solutions of difference equations or their two dimensional systems and have suggested some diverse methods for the qualitative behavior of the their solutions. But, studies on three dimensional systems of difference equations in the literature are quite limited. For example, Kulenović and Nurkanović [12] studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+y_n},\ y_{n+1}=\frac{c+y_n}{d+z_n},\ z_{n+1}=\frac{e+z_n}{f+x_n},\ n\in\mathbb{N}_0,$$

where a, b, c, d, e, $f\in(0,\infty)$ and the initial conditions x_0, y_0 and z_0 are arbitrary non-negative numbers. Kurbanli [15] studied the behavior of solutions of the system

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of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1},\ z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1},\ n \in \mathbb{N}_0,$$

where the initial conditions $x_{-1},\ x_0,\ y_{-1},\ y_0,\ z_{-1},\ z_0$ are real numbers. See also [13, 14, 16]. Yazlik et al. [34] obtained the explicit solutions of a three-dimensional system of difference equations with multiplicative terms

$$x_{n+1} = \frac{x_n y_{n-1}}{a_0 x_n + b_0 y_{n-2}},\ y_{n+1} = \frac{y_n z_{n-1}}{a_1 y_n + b_1 z_{n-2}},\ z_{n+1} = \frac{z_n x_{n-1}}{a_2 z_n + b_2 x_{n-2}},\ n \in \mathbb{N}_0,$$

where the parameters a_i, b_i, and the initial conditions x_{-i}, y_{-i} $(i = 0, 1, 2)$ are real numbers. extending some results in literature. Also, by using explicit forms of the solutions, they studied the asymptotic behavior of well-defined solutions of the system. For more works related to difference equations and their two and three dimensional systems, see references [1, 2, 3, 5, 6, 4, 9, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 29, 31, 32, 33, 34].

In [7], El-Owaidy et al. investigated global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}},\ n \in \mathbb{N}_0, \tag{1}$$

with non-negative parameters and non-negative initial conditions. Gumus and Soykan [10] studied the dynamic behavior of the positive solutions for a system of rational difference equations of the following form

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_{n-2}^{p}},\ v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_{n-2}^{p}},\ n \in \mathbb{N}_0, \tag{2}$$

where the parameters and initial conditions are positive real numbers.

In the present paper, we investigate the global behavior of the positive solutions of the system of difference equations

$$u_{n+1} = \frac{\alpha_1 u_{n-1}}{\beta_1 + \gamma_1 v_{n-2}^{p}},\ v_{n+1} = \frac{\alpha_2 v_{n-1}}{\beta_2 + \gamma_2 w_{n-2}^{q}},\ w_{n+1} = \frac{\alpha_3 w_{n-1}}{\beta_3 + \gamma_3 u_{n-2}^{r}},\ n \in \mathbb{N}_0, \tag{3}$$

where the initial conditions u_{-i}, v_{-i}, w_{-i} $(i = 0, 1, 2)$ are non-negative real numbers and the parameters $\alpha_j, \beta_j, \gamma_j$ $(j = 1, 2, 3)$ and p, q, r are positive real numbers, by extending some results in the literature. System (3) is a natural extension of Eq. (1) and system (2). Note that system (3) can be written as

$$x_{n+1} = \frac{a x_{n-1}}{1 + y_{n-2}^{p}},\ y_{n+1} = \frac{b y_{n-1}}{1 + z_{n-2}^{q}},\ z_{n+1} = \frac{c z_{n-1}}{1 + x_{n-2}^{r}},\ n \in \mathbb{N}_0, \tag{4}$$

by the change of variables $u_n = \left(\frac{\beta_3}{\gamma_3}\right)^{1/r} x_n,\ v_n = \left(\frac{\beta_1}{\gamma_1}\right)^{1/p} y_n,\ w_n = \left(\frac{\beta_2}{\gamma_2}\right)^{1/q} z_n$ with $a = \frac{\alpha_1}{\beta_1}$, $b = \frac{\alpha_2}{\beta_2}$ and $c = \frac{\alpha_3}{\beta_3}$. So, we will consider system (4) instead of system (3) from now.

2. Preliminaries

Let I_1, I_2, I_3 be some intervals of real numbers and $f : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_1$, $g : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_2$, $h : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_3$ be continuously differentiable functions. Then, for every initial conditions $(x_{-i}, y_{-i}, z_{-i}) \in I_1 \times I_2 \times I_3$, $i = 0, \dots, k$, the system of difference equations

$$\left\{ \begin{array}{l} x_{n+1} = f\left(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}\right) \\ y_{n+1} = g\left(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}\right) \\ z_{n+1} = h\left(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}\right) \end{array} \right. \quad \text{for } n \in \mathbb{N}_0 \tag{5}$$

has the unique solution $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$. Also, an equilibrium point of system (5) is a point $(\overline{x}, \overline{y}, \overline{z})$ that satisfies

$$\begin{array}{l} \overline{x} = f\left(\overline{x}, \dots, \overline{x}, \overline{y}, \dots, \overline{y}, \overline{z}, \dots, \overline{z}\right), \\ \overline{y} = g\left(\overline{x}, \dots, \overline{x}, \overline{y}, \dots, \overline{y}, \overline{z}, \dots, \overline{z}\right), \\ \overline{z} = h\left(\overline{x}, \dots, \overline{x}, \overline{y}, \dots, \overline{y}, \overline{z}, \dots, \overline{z}\right), \end{array}$$

We rewrite system (5) in the vector form

$$X_{n+1} = F\left(X_n\right), \ n \in \mathbb{N}_0, \tag{6}$$

where $X_n = \left(x_n, \dots, x_{n-k}, y_n, \dots, y_{n-k}, z_n, \dots, z_{n-k}\right)^T$, F is a vector map such that $F : I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1} \to I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1}$ and

$$F\left(\left(\begin{array}{c} x_0 \\ \vdots \\ x_k \\ y_0 \\ \vdots \\ y_k \\ z_0 \\ \vdots \\ z_k \end{array}\right)\right) = \left(\begin{array}{c} f\left(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k\right) \\ \vdots \\ x_{k-1} \\ g\left(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k\right) \\ \vdots \\ y_{k-1} \\ h\left(x_0, \dots, x_k, y_0, \dots, y_k, z_0, \dots, z_k\right) \\ \vdots \\ z_{k-1} \end{array}\right).$$

It is clear that if an equilibrium point of system (5) is $(\overline{x}, \overline{y}, \overline{z})$, then the corresponding equilibrium point of system (6) is the point $\overline{X} = \left(\overline{x}, \dots, \overline{x}, \overline{y}, \dots, \overline{y}, \overline{z}, \dots, \overline{z}\right)^T$.

In this study, we denote by $\|.\|$ any convenient vector norm and the corresponding matrix norm. Also, we denote by $X_0 \in I_1^{k+1} \times I_2^{k+1} \times I_3^{k+1}$ a initial condition of system (6).

Definition 1. *Let $\overline{X}$ be an equilibrium point of system (6). Then,*

i) *The equilibrium point $\overline{X}$ is called stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left\|X_0 - \overline{X}\right\| < \delta$ implies $\left\|X_n - \overline{X}\right\| < \epsilon$, for all $n \geq 0$. Otherwise the equilibrium point $\overline{X}$ is called unstable.*

ii) *The equilibrium point* $\overline{X}$ *is called local asymptotically stable if it is stable and there exists* $\gamma > 0$ *such that* $\left\|X_0 - \overline{X}\right\| < \gamma$ *and* $X_n \to \overline{X}$ *as* $n \to \infty$.
iii) *The equilibrium point* $\overline{X}$ *is called a global attractor if* $X_n \to \overline{X}$ *as* $n \to \infty$.
iv) *The equilibrium point* $\overline{X}$ *is called globally asymptotically stable if it is both local asymptotically stable and global attractor.*

The linearized system of system (6) evaluated at the equilibrium point $\overline{X}$ is

$$Z_{n+1} = J_F Z_n, \ n \in \mathbb{N}_0, \tag{7}$$

where J_F is the Jacobian matrix of the map F at the equilibrium point $\overline{X}$. The characteristic polynomial of system (7) about the equilibrium point $\overline{X}$ is

$$P(\lambda) = a_0 \lambda^{3(k+1)} + a_1 \lambda^{3k+2} + \cdots + a_{3k+2}\lambda + a_{3(k+1)}, \tag{8}$$

with real coefficients and $a_0 > 0$.

Theorem 2. [11] *Assume that* $\overline{X}$ *is a equilibrium point of system (6). If all eigenvalues of the Jacobian matrix* J_F *evaluated at* $\overline{X}$ *lie in the open unit disk* $|\lambda| < 1$, *then* $\overline{X}$ *is locally asymptotically stable. If one of them has a modulus greater than one, then* $\overline{X}$ *is unstable.*

3. Stability of the system

In this section, we investigate the stability of the equilibrium points of system (4). When $a, b, c \in (0,1)$, the point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ is the unique nonnegative equilibrium point of system (4). When $a, b, c \in (1, \infty)$, the unique positive equilibrium point of system (4) is

$$(\overline{x}_2, \overline{y}_2, \overline{z}_2) = \left((c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q}\right).$$

In addition,

(i) if $a = b = c = 1$,then

$$(\overline{x}_3, \overline{y}_3, \overline{z}_3) = (c_1, 0, 0), \ (\overline{x}_4, \overline{y}_4, \overline{z}_4) = (0, c_2, 0) \text{ and } (\overline{x}_5, \overline{y}_5, \overline{z}_5) = (0, 0, c_3),$$

(ii) if $a = 1$ and $b, c \in (1, \infty)$, then

$$(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0) \text{ and } (\overline{x}_6, \overline{y}_6, \overline{z}_6) = \left((c-1)^{1/r}, 0, c_3\right),$$

(iii) if $a \neq 1$, $b = 1$ and $c \in (1, \infty)$, then

$$(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0) \text{ and } (\overline{x}_7, \overline{y}_7, \overline{z}_7) = \left(c_1, (a-1)^{1/p}, 0\right),$$

(iv) if $a, b \in (1, \infty)$ and $c = 1$, then

$$(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0) \text{ and } (\overline{x}_8, \overline{y}_8, \overline{z}_8) = \left(0, c_2, (b-1)^{1/q}\right),$$

(v) if $a = b = 1$ and $c \in (1, \infty)$, then

$$(\overline{x}_3, \overline{y}_3, \overline{z}_3) = (c_1, 0, 0), \ (\overline{x}_4, \overline{y}_4, \overline{z}_4) = (0, c_2, 0) \text{ and } (\overline{x}_6, \overline{y}_6, \overline{z}_6) = \left((c-1)^{1/r}, 0, c_3\right),$$

(vi) if $a = c = 1$ and $b \in (1, \infty)$, then

$$(\overline{x}_3, \overline{y}_3, \overline{z}_3) = (c_1, 0, 0), \ (\overline{x}_5, \overline{y}_5, \overline{z}_5) = (0, 0, c_3) \text{ and } (\overline{x}_8, \overline{y}_8, \overline{z}_8) = \left(0, c_2, (b-1)^{1/q}\right),$$

(vii) if $b = c = 1$ and $a \in (1, \infty)$, then

$$(\overline{x}_4, \overline{y}_4, \overline{z}_4) = (0, c_2, 0), \ (\overline{x}_5, \overline{y}_5, \overline{z}_5) = (0, 0, c_3) \text{ and } (\overline{x}_7, \overline{y}_7, \overline{z}_7) = \left(c_1, (a-1)^{1/p}, 0\right),$$

where c_1, c_2 and c_3 are real numbers.

Theorem 3. *The following statements hold:*

i) *If $a, b, c \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of system (4) is locally asymptotically stable.*
ii) *If $a, b, c \in (1, \infty)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of system (4) is unstable.*
iii) *If $a, b, c \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = \left((c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q}\right)$ of system (4) is unstable.*

Proof. First, we can write system (4) in the form of system (6) such that

$$X_n = (x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}, z_n, z_{n-1}, z_{n-2})^T$$

the map F is

$$F\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ y_0 \\ y_1 \\ y_2 \\ z_0 \\ z_1 \\ z_2 \end{pmatrix}\right) = \begin{pmatrix} ax_1/(1+y_2^p) \\ x_0 \\ x_1 \\ by_1/(1+z_2^q) \\ y_0 \\ y_1 \\ cz_1/(1+x_2^r) \\ z_0 \\ z_1 \end{pmatrix}.$$

i) The linearized system of (4) about the equilibrium point $\overline{X}_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ is given

$$X_{n+1} = J_F(\overline{X}_0) X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ z_n \\ z_{n-1} \\ z_{n-2} \end{pmatrix} \text{ and } J_F(\overline{X}_0) = \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic equation of $J_F(\overline{X}_0)$ is given by

$$P(\lambda) = \lambda^9 - (a+b+c)\lambda^7 + (ab+ac+bc)\lambda^5 - abc\lambda^3 = 0 \tag{9}$$

or

$$P(\lambda) = \lambda^3 \left(\lambda^2 - a\right)\left(\lambda^2 - b\right)\left(\lambda^2 - c\right) = 0.$$

It is easy to see that if $a, b, c \in (0,1)$, then all the roots of the characteristic equation (9) lie in the open unit disk $|\lambda| < 1$. So, the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of (4) is locally asymptotically stable.

ii) It is clearly seen that if $a, b, c \in (1, \infty)$, then some roots of characteristic equation (9) have absolute value greater than one. In this case, the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0,0,0)$ of (4) is unstable.

iii) The linearized system of (4) about the positive equilibrium point

$$\overline{X}_{a,b,c} = \begin{pmatrix} (c-1)^{1/r} \\ (c-1)^{1/r} \\ (c-1)^{1/r} \\ (a-1)^{1/p} \\ (a-1)^{1/p} \\ (a-1)^{1/p} \\ (b-1)^{1/q} \\ (b-1)^{1/q} \\ (b-1)^{1/q} \end{pmatrix}$$

is given by

$$X_{n+1} = J_F(\overline{X}_{a,b,c})X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ z_n \\ z_{n-1} \\ z_{n-2} \end{pmatrix}, J_F(\overline{X}_{a,b,c}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

with

$$A = -\frac{p\,(a-1)^{(p-1)/p}\,(c-1)^{1/r}}{a}, \quad B = -\frac{q\,(b-1)^{(q-1)/q}\,(a-1)^{1/p}}{b}$$

and

$$C = -\frac{r\,(c-1)^{(r-1)/r}\,(b-1)^{1/q}}{c}.$$

The characteristic polynomial of $J_F(\overline{X}_{a,b,c})$ is given by

$$P(\lambda) = \lambda^9 - 3\lambda^7 + 3\lambda^5 - \lambda^3 + \frac{pqr\,(a-1)\,(b-1)\,(c-1)}{abc}. \tag{10}$$

It is clear that $P(\lambda)$ has a root in interval $(-\infty, -1)$, since

$$P(-1) = \frac{pqr\,(a-1)\,(b-1)\,(c-1)}{abc} > 0 \text{ and } \lim_{\lambda \to -\infty} P(\lambda) = -\infty.$$

So, from Theorem 2, we can say that if $a, b, c \in (1, \infty)$, then the positive equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = \left((c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q}\right)$ of system (4) is unstable.

□

Theorem 4. *If $a, b, c \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of system (4) is globally asymptotically stable.*

Proof. From Theorem 3, we know that if $a, b, c \in (0, 1)$, then the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (0, 0, 0)$ of system (4) is locally asymptotically stable. Hence, it suffices to show that

$$\lim_{n \to \infty} (x_n, y_n, z_n) = (0, 0, 0). \tag{11}$$

From system (4), we have that

$$0 \le x_{n+1} = \frac{a x_{n-1}}{1 + y_{n-2}^p} \le a x_{n-1},\ 0 \le y_{n+1} = \frac{b y_{n-1}}{1 + z_{n-2}^q} \le b y_{n-1},\ 0 \le z_{n+1} = \frac{c z_{n-1}}{1 + x_{n-2}^r} \le c z_{n-1} \tag{12}$$

for $n \in \mathbb{N}_0$. From (12), we have by induction

$$0 \le x_{2n-i} \le a^n x_{-i},\ 0 \le y_{2n-i} \le b^n y_{-i},\ 0 \le z_{2n-i} \le c^n z_{-i} \tag{13}$$

where x_{-i}, y_{-i}, z_{-i} $(i = 0, 1)$ are the initial conditions. Consequently, by taking limits of inequalities in (13) when $a, b, c \in (0, 1)$, then we have the limit in (11) which completes the proof. □

4. Oscillation behavior and existence of unbounded solutions

In the following result, we are concerned with the oscillation of positive solutions of system (4) about the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = \left((c-1)^{1/r}, (a-1)^{1/p}, (b-1)^{1/q}\right)$.

Theorem 5. *Assume that $a, b, c \in (1, \infty)$ and let $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$ be a positive solution of system (4) such that*

$$x_{-2}, x_0 \ge \overline{x}_2,\ x_{-1} < \overline{x}_2,\quad y_{-2}, y_0 \ge \overline{y}_2,\ y_{-1} < \overline{y}_2,\quad z_{-2}, z_0 \ge \overline{z}_2,\ z_{-1} < \overline{z}_2 \tag{14}$$

or

$$x_{-2}, x_0 < \overline{x}_2,\quad x_{-1} \ge \overline{x}_2,\quad y_{-2}, y_0 < \overline{y}_2,\ y_{-1} \ge \overline{y}_2,\quad z_{-2}, z_0 < \overline{z}_2,\ z_{-1} \ge \overline{z}_2. \tag{15}$$

Then, $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$ oscillates about the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ with semi-cycles of length one .

Proof. Assume that (14) holds. (The case where (15) holds is similar and will be omitted.) From (4), we have

$$\begin{aligned} x_1 &= \frac{ax_{-1}}{1+y_{-2}^p} < \frac{a\overline{x}_2}{1+\overline{y}_2^p} = \overline{x}_2, \\ y_1 &= \frac{by_{-1}}{1+z_{-2}^q} < \frac{b\overline{y}_2}{1+\overline{z}_2^q} = \overline{y}_2, \\ z_1 &= \frac{cz_{-1}}{1+x_{-2}^r} < \frac{c\overline{z}_2}{1+\overline{x}_2^r} = \overline{z}_2 \end{aligned}$$

and

$$\begin{aligned} x_2 &= \frac{ax_0}{1+y_{-1}^p} \geqslant \frac{a\overline{x}_2}{1+\overline{y}_2^p} = \overline{x}_2, \\ y_2 &= \frac{by_0}{1+z_{-1}^q} \geqslant \frac{b\overline{y}_2}{1+\overline{z}_2^q} = \overline{y}_2, \\ z_2 &= \frac{cz_0}{1+x_{-1}^r} \geqslant \frac{c\overline{z}_2}{1+\overline{x}_2^r} = \overline{z}_2 \end{aligned}$$

then, the proof follows by induction. □

In the following theorem, we show the existence of unbounded solutions for system (4).

Theorem 6. *Assume that $a, b, c \in (1, \infty)$, then system (4) possesses an unbounded solution.*

Proof. From Theorem 5, we can assume that without loss of generality that the solution $\{(x_n, y_n, z_n)\}_{n=-2}^{\infty}$ of system (4) is such that $x_{2n-1} < \overline{x}_2$, $y_{2n-1} < \overline{y}_2$, $z_{2n-1} < \overline{z}_2$, $x_{2n} > \overline{x}_2$, $y_{2n} > \overline{y}_2$ and $z_{2n} > \overline{z}_2$ for $n \in \mathbb{N}_0$. Then, we have

$$\begin{aligned} x_{2n+2} &= \frac{ax_{2n}}{1+y_{2n-1}^p} > \frac{ax_{2n}}{1+\overline{y}_2^p} = \frac{ax_{2n}}{1+(a-1)} = x_{2n}, \\ y_{2n+2} &= \frac{by_{2n}}{1+z_{2n-1}^q} > \frac{by_{2n}}{1+\overline{z}_2^q} = \frac{by_{2n}}{1+(b-1)} = y_{2n}, \\ z_{2n+2} &= \frac{cz_{2n}}{1+x_{2n-1}^r} > \frac{cz_{2n}}{1+\overline{x}_2^r} = \frac{cz_{2n}}{1+(c-1)} = z_{2n} \end{aligned}$$

and

$$\begin{aligned} x_{2n+3} &= \frac{ax_{2n+1}}{1+y_{2n}^p} < \frac{ax_{2n+1}}{1+\overline{y}_2^p} = \frac{ax_{2n+1}}{1+(a-1)} = x_{2n+1}, \\ y_{2n+3} &= \frac{by_{2n+1}}{1+z_{2n}^q} < \frac{by_{2n+1}}{1+\overline{z}_2^q} = \frac{by_{2n+1}}{1+(b-1)} = y_{2n+1}, \\ z_{2n+3} &= \frac{cz_{2n+1}}{1+x_{2n}^r} < \frac{cz_{2n+1}}{1+\overline{x}_2^r} = \frac{cz_{2n+1}}{1+(c-1)} = z_{2n+1} \end{aligned}$$

from which it follows that

$$\lim_{n\to\infty}(x_{2n},y_{2n},z_{2n})=(\infty,\infty,\infty) \text{ and } \lim_{n\to\infty}(x_{2n+1},y_{2n+1},z_{2n+1})=(0,0,0)$$

which completes the proof. □

5. Periodicity

In this section, we investigate the existence of period two solution of system (4).

Theorem 7. *If $a=b=c=1$, then, system (4) possesses the prime period two solution*

$$...,(0,0,\varphi),(0,0,\psi),(0,0,\varphi),(0,0,\psi),...$$

with $\varphi,\psi>0$. Furthermore, every solution of system (4) converges to a period two solution.

Proof. Assume that $a=b=c=1$ and let $\{(x_n,y_n,z_n)\}_{n=-2}^{\infty}$ be a solution of system (4). Then, from system (4), we have

$$\begin{array}{lll} x_{2n+1}=\frac{x_{2n-1}}{1+y_{2n-2}^{p}} & \text{and} & x_{2n+2}=\frac{x_{2n}}{1+y_{2n-1}^{p}} \\ y_{2n+1}=\frac{y_{2n-1}}{1+z_{2n-2}^{q}} & \text{and} & y_{2n+2}=\frac{y_{2n}}{1+z_{2n-1}^{q}} \\ z_{2n+1}=\frac{z_{2n-1}}{1+x_{2n-2}^{r}} & \text{and} & z_{2n+2}=\frac{z_{2n}}{1+x_{2n-1}^{r}} \end{array} \tag{16}$$

for $n\in\mathbb{N}_0$. From (16), we get

$$\begin{array}{lll} x_{2n-1}=x_{-1}\prod_{i=0}^{n-1}\left(\frac{1}{1+y_{2i-2}^{p}}\right) & \text{and} & x_{2n}=x_0\prod_{i=0}^{n-1}\left(\frac{1}{1+y_{2i-1}^{p}}\right) \\ y_{2n-1}=y_{-1}\prod_{i=0}^{n-1}\left(\frac{1}{1+z_{2i-2}^{q}}\right) & \text{and} & y_{2n}=y_0\prod_{i=0}^{n-1}\left(\frac{1}{1+z_{2i-1}^{q}}\right) \\ z_{2n-1}=z_{-1}\prod_{i=0}^{n-1}\left(\frac{1}{1+x_{2i-2}^{r}}\right) & \text{and} & z_{2n}=z_0\prod_{i=0}^{n-1}\left(\frac{1}{1+x_{2i-1}^{r}}\right) \end{array} \tag{17}$$

for $n\in\mathbb{N}_0$. If $(x_{-1},x_0)=(0,0)$ and $(y_{-1},y_0)=(0,0)$, then $(x_n,y_n)=(0,0)$ and $(z_{2n-1},z_{2n})=(z_{-1},z_0)$ for $n\in\mathbb{N}_0$.Therefore,

$$...,(0,0,\varphi),(0,0,\psi),(0,0,\varphi),(0,0,\psi),...$$

is a period two solution of system (4) with $z_{-2},z_0=\varphi>0$ and $z_{-1}=\psi>0$. Furthermore, from (16), we have

$$\begin{array}{l} x_{2n+1}-x_{2n-1}=-\frac{x_{2n-1}y_{2n-2}^{p}}{1+y_{2n-2}^{p}}\leq 0, \\ y_{2n+1}-y_{2n-1}=-\frac{y_{2n-1}z_{2n-2}^{q}}{1+z_{2n-2}^{q}}\leq 0, \\ z_{2n+1}-z_{2n-1}=-\frac{z_{2n-1}x_{2n-2}^{r}}{1+x_{2n-2}^{r}}\leq 0 \end{array} \tag{18}$$

and

$$\begin{aligned} x_{2n+2} - x_{2n} &= -\tfrac{x_{2n} y_{2n-1}^{p}}{1+y_{2n-1}^{p}} \leq 0, \\ y_{2n+2} - y_{2n} &= -\tfrac{y_{2n} z_{2n-1}^{q}}{1+z_{2n-1}^{q}} \leq 0, \\ z_{2n+2} - z_{2n} &= -\tfrac{z_{2n} x_{2n-1}^{r}}{1+x_{2n-1}^{r}} \leq 0. \end{aligned} \tag{19}$$

From (18) and (19), we get

$$\begin{aligned} x_{2n+1} &\leq x_{2n-1}, \quad y_{2n+1} \leq y_{2n-1}, \quad z_{2n+1} \leq z_{2n-1} \text{ and} \\ x_{2n+2} &\leq x_{2n}, \quad y_{2n+2} \leq y_{2n}, \quad z_{2n+2} \leq z_{2n}. \end{aligned}$$

That is, the sequences $(x_{2n-1}, y_{2n-1}, z_{2n-1})$ and (x_{2n}, y_{2n}, z_{2n}) are non-increasing. Hence, while the odd-index terms tend to one periodic point and the even-index terms tend to another periodic point. This completes the proof. □

6. Numerical examples

In this section, we support our theoretical results related to system (4) with some numerical examples.

Example 8. *In the following figures, we illustrate the solution which corresponds to the initial conditions* $x_{-2} = 0.1$, $x_{-1} = 1.2$, $x_0 = 0.17$, $y_{-2} = 0.11$, $y_{-1} = 1.12$, $y_0 = 2.17$, $z_{-2} = 3.1$, $z_{-1} = 2.12$, $z_0 = 0.1$ *and* $p = 3$, $q = 2$, $r = 4$ *of (4) for difference values of the parameters* a , b, c.

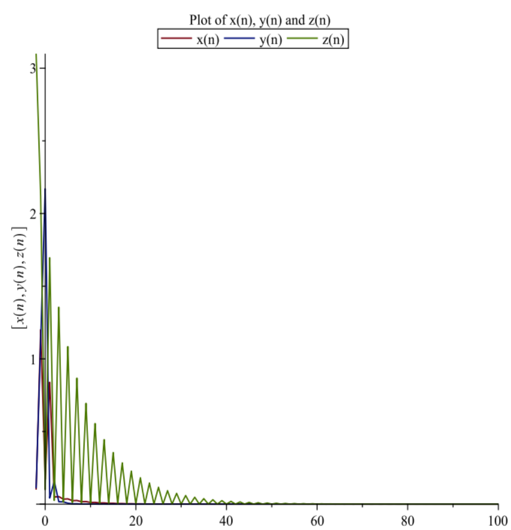

Figure 1. $a = 0.7$, $b = 0.4$, $c = 0.8$ and $p = 3$, $q = 2$, $r = 4$

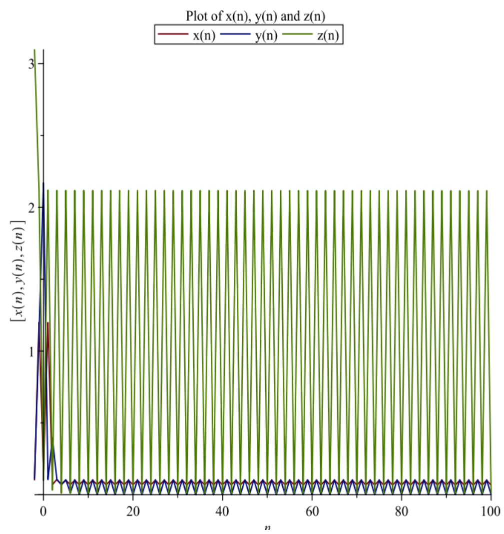

FIGURE 2. $a = b = c = 1$ and $p = 3$, $q = 2$, $r = 4$

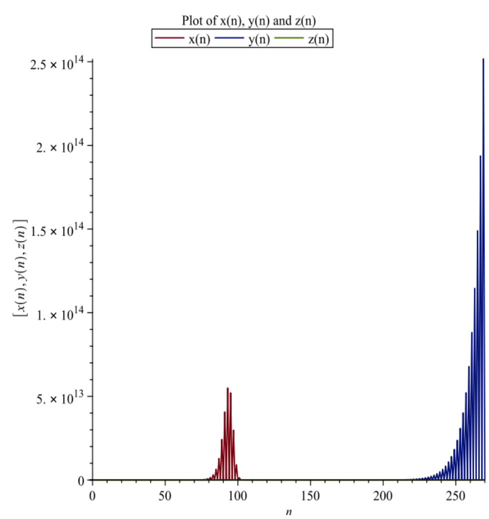

FIGURE 3. $a = 2.1$, $b = 1.3$, $c = 1.1$ and $p = 3$, $q = 2$, $r = 4$

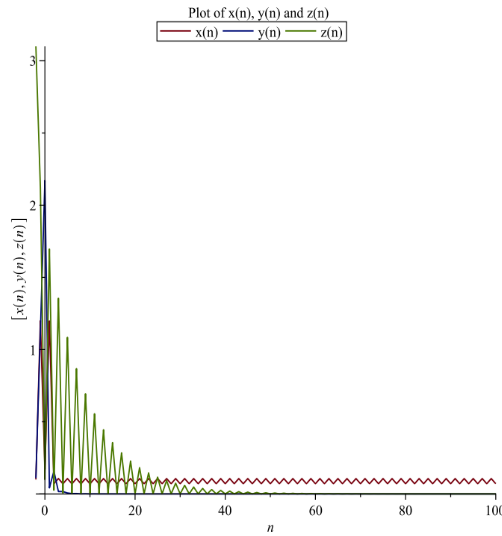

FIGURE 4. $a = 1$, $b = 0.4$, $c = 0.8$ and $p = 3$, $q = 2$, $r = 4$

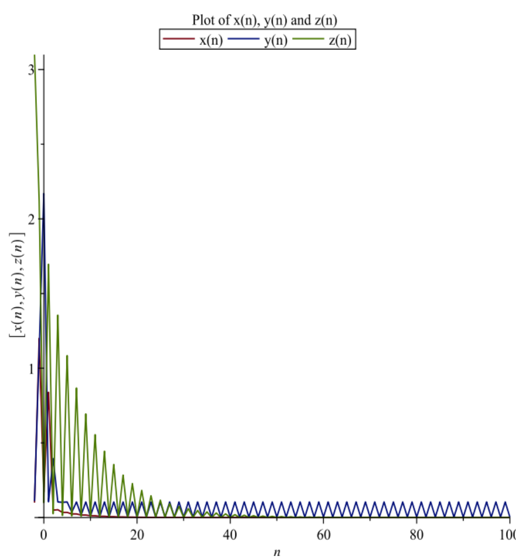

FIGURE 5. $a = 0.7$, $b = 1$, $c = 0.8$ and $p = 3$, $q = 2$, $r = 4$

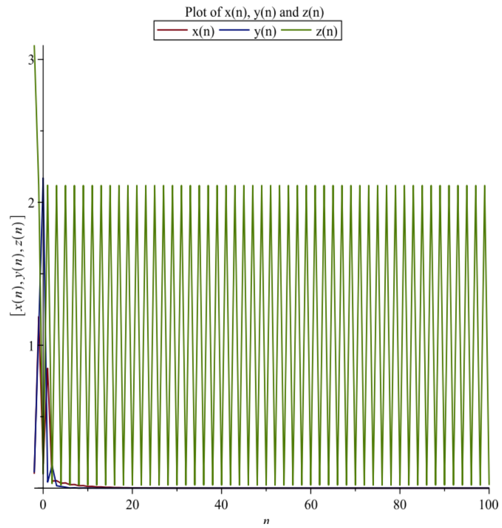

FIGURE 6. $a = 0.7$, $b = 0.4$, $c = 1$ and $p = 3$, $q = 2$, $r = 4$

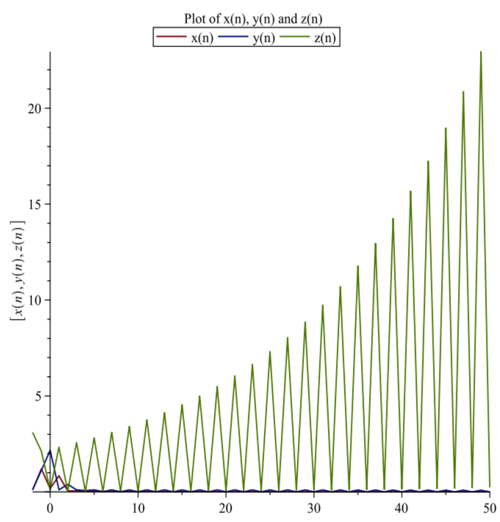

FIGURE 7. $a = 0.7$, $b = 1$, $c = 1.1$ and $p = 3$, $q = 2$, $r = 4$

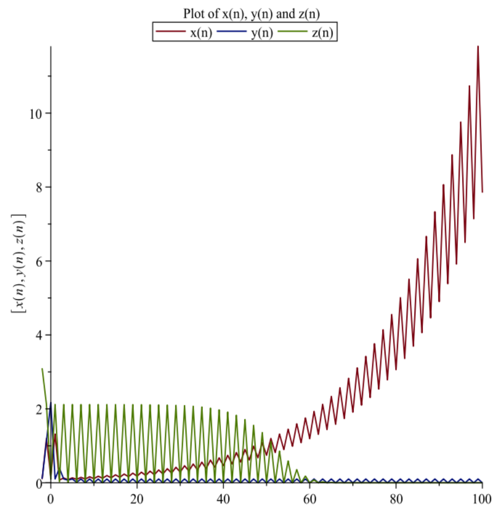

FIGURE 8. $a = 1.1$, $b = 1$, $c = 1$ and $p = 3$, $q = 2$, $r = 4$

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