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LEFT-HOM-SYMMETRIC AND HOM-POISSON DIALGEBRAS

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Abstract. The aim of this paper is to introduce left-Hom-symmetric dialgebras (which contain left-Hom-symmetric algebras or Hom-preLie algebras and Hom-dialgebras as special cases) and Hom-Poisson dialgebras. We give some examples and some construction theorems by using the composition construction. We prove that the commutator bracket of any left-Hom-symmetric dialgebra provides Hom-Leibniz algebra. We also prove that bimodules over Hom-dialgebras are closed under twisting. Next, we show that bimodules over Hom-dendriform algebras D extend to bimodules over the left-Hom-symmetric algebra associated to D. Finally, we give some examples of Hom-Poisson dialgebras and prove that the commutator bracket of any Hom-dialgebra structure map leads to Hom-Poisson dialgebra.

1. Introduction

Leibniz algebras are introduced by J.-L. Loday in [8] as a generalization of Lie algebras where the skew-symmetry of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. The author showed that the relationship between Lie algebras and associative algebras translates into an analogous relationship between Leibniz algebras and the so-called diassociative algebras or associative dialgebras, which are a generalization of associative algebras possessing two products. In particular, he showed that any dialgebra becomes a Leibniz algebra under the commutator bracket.

Otherwise, left-symmetric dialgebras appear in the work of R. Felipe [10] as an algebraic structure with two products containing dialgebras as particular case, and Poisson dialgebras are introduced in [7] as a vector space endowed with both dialgebra structure and Leibniz structure which are compatible in certain sense.

The purpose of this paper is to study Left-Hom-symmetric dialgebras and Hom-Poisson dialgebras. We define bimodules over Hom-dialgebras and Hom-dendriform algebras [2] and give some construction theorems. Next, we introduce Hom-Poisson

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dialgebras as Hom-type of Poisson dialgebras which are generalization of "non-commutative Poisson algebras".

The paper is organized as follows. In section 2, we recall some basic notions related to Hom-algebras, Hom-Lie algebras and Hom-Leibniz algebras. In Section 3, we show that one can obtain a left-Hom-symmetric algebra from a left-symmetric algebra and an algebra endomorphism. We prove that twisting a Hom-dialgebra module structure map by an endomorphism of Hom-dialgebras, we get another one. Next, we show that any left-Hom-symmetric dialgebra leads to Hom-Leibniz algebra via the Loday commutator. Finally, we introduce affine Hom-Leibniz structure on Hom-Leibniz algebras and point out that one may associate a left-Hom-symmetric algebra to any affine Hom-Leibniz algebra. In section 4, we introduce bimodules over Hom-dendriform algebras and prove that to any bimodule over a Hom-dendriform algebra D corresponds a module over the left-Hom-symmetric algebra associated to D. In section 5, we introduce Hom-Poisson dialgebras ; we give some examples and some construction theorems of Hom-Poisson dialgebras.

Throughout this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 2.1. [1] By a Hom-algebra we mean a triple $(A, [\cdot,\cdot], \alpha)$ in which A is a vector space, $[\cdot,\cdot] : A \otimes A \to A$ is a bilinear map (the multiplication) and $\alpha : A \to A$ is a linear map (the twisting map).

If in addition, $\alpha \circ [\cdot,\cdot] = [\cdot,\cdot] \circ (\alpha \otimes \alpha)$, then the Hom-algebra $(A, [\cdot,\cdot], \alpha)$ is said to be multiplicative.

A morphism $f : (A, [\cdot,\cdot], \alpha) \to (A', [\cdot,\cdot]', \alpha')$ of Hom-algebras is a linear map f of the underlying vector spaces such that $f \circ \alpha = \alpha' \circ f$ and $[\cdot,\cdot]' \circ (f \otimes f) = f \circ [\cdot,\cdot]$.

Remark 2.1. If $(A, [\cdot,\cdot])$ is a non-necessarily associative algebra in the usual sense, we also regard it as the Hom-algebra $(A, [\cdot,\cdot], Id_A)$ with identity twisting map.

Definition 2.2. [1] Let $(A, [\cdot,\cdot], \alpha)$ be a Hom-algebra.

(1) The Hom-associator of A is the trilinear map $as_\alpha : A^{\otimes 3} \to A$ defined as
$$as_\alpha = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \alpha - \alpha \otimes [\cdot,\cdot]).$$
(2) The Hom-Jacobian of A is the trilinear map $J_\alpha : A^{\otimes 3} \to A$ defined as
$$J_\alpha = [\cdot,\cdot] \circ ([\cdot,\cdot] \otimes \alpha) \circ (Id_A + \sigma + \sigma^2),$$
where $\sigma : A^{\otimes 3} \to A^{\otimes 3}$ is the cyclic permutation $\sigma(x \otimes y \otimes z) = y \otimes z \otimes x$.

(3) The Hom-Leibnizator of A is a trilinear map $Leib_\alpha : A^{\otimes 3} \to A$ defined as
$$Leib_\alpha = [\cdot,\cdot](\alpha \otimes [\cdot,\cdot]) + [\cdot,\cdot]([\cdot,\cdot] \otimes \alpha) - [\cdot,\cdot]([\cdot,\cdot] \otimes \alpha)(Id_A \otimes \tau),$$
where τ is the twist isomorphism i.e. $\tau(x \otimes y) = y \otimes x$, for any $x, y \in A$.

Definition 2.3. [1] A Hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a linear space A, a $\mathbb{K}$-bilinear map $\cdot : A \times A \longrightarrow A$ and a linear map $\alpha : A \longrightarrow A$ satisfying

$$as_\alpha(x, y, z) = 0 \quad \text{(Hom-associativity)}, \tag{2.1}$$

for all $x, y, z \in A$.

Definition 2.4. [6] A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, a bilinear map $[\cdot,\cdot] : V \times V \longrightarrow V$ and a linear map $\alpha : V \longrightarrow V$ satisfying

$$[x, y] = -[y, x] \quad \text{(skew-symmetry)}, \tag{2.2}$$

$$J_\alpha(x, y, z) = 0 \quad \text{(Hom-Jacobi identity)}, \tag{2.3}$$

for all $x, y, z \in V$.

Remark 2.2. When $\alpha = Id_V$, we obtain the definition of Lie algebras.

Definition 2.5. [1] A Hom-algebra $(L, [\cdot,\cdot], \alpha)$ is said to be a Hom-Leibniz algebra if it satisfies the *Hom-Leibniz identity* i.e.

$$Leib_\alpha(x, y, z) = 0, \tag{2.4}$$

for all $x, y, z \in L$.

Remark 2.3.
1. When $\alpha = Id_L$, we recover the concept of Leibniz algebra.
2. If the bracket is skew-symmetric, then L is a Hom-Lie algebra. Therefore Hom-Lie algebras are particular cases of Hom-Leibniz algebras.

3. Left-Hom-symmetric dialgebras

We introduce modules over Hom-dialgebras and left-Hom-symmetric dialgebras.

3.1. Left-Hom-symmetric algebras.

Definition 3.1. [1] A left-Hom-symmetric algebra is a vector space S together with a bilinear map $\circ : S \otimes S \to S$ and a linear map $\alpha : S \to S$ such that the following *left-Hom-symmetry identity*

$$\alpha(x) \circ (y \circ z) - (x \circ y) \circ \alpha(z) = \alpha(y) \circ (x \circ z) - (y \circ x) \circ \alpha(z), \tag{3.1}$$

holds.

Remark 3.1.
1. When $\alpha = Id_S$, we recover the notion of left-symmetric algebras.
2. In terms of Hom-associators, the *left-Hom-symmetry identity* is
$$as_\alpha(x, y, z) = as_\alpha(y, x, z).$$

Example 3.1. Let $(S, \circ, \alpha_S)$ be a left-Hom-symmetric algebra and $(A, \cdot, \alpha_A)$ a commutative Hom-associative algebra. Then $(S \otimes A, \bullet, \alpha_{S\otimes A})$ is a left-Hom-symmetric algebra, with

$$\begin{array}{rcl}\alpha_{S\otimes A} & = & \alpha_S \otimes \alpha_A, \\ (x \otimes a) \bullet (y \otimes b) & = & (x \circ y) \otimes (a \cdot b),\end{array}$$

for all $x, y \in S, a, b \in A$.

The following theorem allows to obtain left-Hom-symmetric algebras from left-symmetric algebras.

Theorem 3.1. *Let $(S, \bullet)$ be a left-symmetric algebra and $\alpha : S \to S$ be an endomorphism. Then, $S_\alpha = (S, \bullet_\alpha, \alpha)$, where $x \bullet_\alpha y = \alpha(x \bullet y)$, is a left-Hom-symmetric algebra.*

Moreover, suppose that $(S', \bullet')$ is another left-symmetric algebra and $\alpha' : S' \to S'$ is an algebra endomorphism. If $f : S \to S'$ is a left-symmetric algebra morphism that satisfies $f \circ \bullet = \bullet' \circ f$ then $f : S_\alpha \to S'_{\alpha'}$ is a morphism of left-Hom-symmetric algebras.

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned}
\alpha(x) \bullet_\alpha (y \bullet_\alpha z) - (x \bullet_\alpha y) \bullet_\alpha \alpha(z) &= \alpha(x) \bullet_\alpha (\alpha(y) \bullet \alpha(z)) - (\alpha(x) \bullet \alpha(y)) \bullet_\alpha \alpha(z) \\
&= \alpha^2(x) \bullet (\alpha^2(y) \bullet \alpha^2(z)) - (\alpha^2(x) \bullet \alpha^2(y)) \bullet \alpha^2(z) \\
&= (\alpha^2)^{\otimes 3}((x \bullet y) \bullet z) - (x \bullet y) \bullet z) \\
&= (\alpha^2)^{\otimes 3}(y \bullet (x \bullet z) - (y \bullet x) \bullet z) \\
&= \alpha(y) \bullet_\alpha (x \bullet_\alpha z) - (y \bullet_\alpha x) \bullet_\alpha \alpha(z).
\end{aligned}$$

For the second part, we have

$$f \circ \bullet_\alpha = f \circ \alpha \circ \bullet = \alpha' \circ f \circ \bullet = \alpha' \circ \bullet' \circ (f \otimes f) = \bullet'_{\alpha'} \circ (f \otimes f).$$

This completes the proof. □

Example 3.2. : *Left-Hom-symmetric algebra of vector fields*
First we need some definitions. Let M be a differential manifold, and let $\bigtriangledown$ be the covariant operator associated to a connection on the tangent bundle TM. The covariant derivation is a bilinear operator on vector fields (i.e. two sections of the tangent bundle) $(X, Y) \mapsto \bigtriangledown_X Y$ such that the following axioms are fulfilled :

$$\begin{aligned}
\bigtriangledown_{fX} Y &= f \bigtriangledown_X Y, \\
\bigtriangledown_X (fY) &= f \bigtriangledown_X Y + (X \cdot f) Y \quad \text{(Leibniz rule)}.
\end{aligned}$$

The torsion of the connection τ is defined by :

$$\tau(X, Y) = \bigtriangledown_X Y - \bigtriangledown_Y X - [X, Y], \tag{3.2}$$

and the curvature tensor is defined by :

$$R(X, Y) = [\bigtriangledown_X, \bigtriangledown_Y] - \bigtriangledown_{[X,Y]}. \tag{3.3}$$

The connection is flat if the curvature R vanishes identically, and torsion-free if $\tau = 0$.

Now, let M be a smooth manifold endowed with a flat torsion-free connection $\bigtriangledown$, $\chi(M)$ the space of vector fields and $\varphi : M \to M$ a smooth map such that $d\varphi(\bigtriangledown_X Y) = \bigtriangledown_{d\varphi(X)} d\varphi(Y)$. Then $(\chi(M), \circ, d\varphi)$ is a left-Hom-symmetric algebra, with the left-Hom-symmetric product given by :

$$X \circ Y = \bigtriangledown_X Y.$$

3.2. **Modules over Hom-dialgebras.**

Definition 3.2. A Hom-dialgebra is a vector space D equipped with a linear map $\alpha : D \to D$ and two *Hom-associative products*

$$\begin{aligned}
\dashv: D \times D &\to D \\
\vdash: D \times D &\to D.
\end{aligned}$$

satisfying the identities :

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{3.4}$$

$$(x \vdash y) \dashv \alpha(z) = \alpha(x) \vdash (y \dashv z), \tag{3.5}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z). \tag{3.6}$$

If in addition, α is an endomorphism with respect to $\dashv$ and $\vdash$, then D is said to be a multiplicative Hom-dialgebra.

Remark 3.2. For any x, y, z in a Hom-dialgebra, one has

$$\alpha(x) \star (y \star z) = \alpha(x) \star (z \star y) \text{ (right commutativity)}$$

where $x \star y = x \dashv y + y \vdash x$.

Here are some examples of Hom-dialgebras.

Example 3.3. Any dialgebra is a Hom-dialgebra with $\alpha = Id$.

Example 3.4. If (A, μ, α) is a Hom-associative algebra, then $(D, \dashv, \vdash, \alpha)$ is a Hom-dialgebra in which $\dashv = \mu = \vdash$.

Example 3.5. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra. Then $(D, \dashv', \vdash', \alpha)$ is also a Hom-dialgebra, with

$$x \dashv' y := y \vdash x \quad \text{and} \quad x \vdash' y := y \dashv x.$$

Example 3.6. Let $(D, \dashv_D, \vdash_D, \alpha_D)$ and $(D', \dashv_{D'}, \vdash_{D'}, \alpha_{D'})$ be two Hom-dialgebras. The tensor product $D \otimes D'$ is also a Hom-dialgebra with

$$\begin{array}{rcl} \alpha_{D\otimes D'}(x \otimes y) & := & \alpha_D(x) \otimes \alpha_{D'}(x'), \\ (x \otimes x') \dashv (y \otimes y') & := & (x \dashv_D y) \otimes (x' \dashv_{D'} y'), \\ (x \otimes x') \vdash (y \otimes y') & := & (x \vdash_D y) \otimes (x' \vdash_{D'} y'). \end{array}$$

Example 3.7. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. Then, for any positive integer n, $A^n = A \times A \times \cdots \times A$ (n times) is a Hom-dialgebra, with

$$\begin{array}{rcl} \alpha_{A^n} & := & (\alpha, \alpha, \dots, \alpha), \\ (x \dashv_{A^n} y)_i & := & x_i \cdot (\sum y_j), \\ (x \vdash_{A^n} y)_i & := & (\sum x_j) \cdot y_i, \end{array}$$

for any $1 \leq i, j \leq n$.

Example 3.8. The Hom-dialgebra arising from a bimodule over Hom-associative algebra and morphism of Hom-bimodules is exposed in [4].

Now, we have the following definitions.

Definition 3.3. [5] A Hom-module is a pair (M, β) in which M is a vector space and $\beta : M \longrightarrow M$ is a linear map.

Definition 3.4. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra and let (M, β) be a Hom-module. A bimodule structure on M consists of :

(1) a left A-action, $\prec: A \otimes M \to M$ ($x \otimes m \mapsto x \prec m$), and
(2) a right A-action, $\succ: M \otimes A \to M$ ($m \otimes x \mapsto m \succ x$)

such that the following conditions hold for $x, y \in A$ and $m \in M$:

$$\beta(x \prec m) = \alpha(x) \prec \beta(m), \tag{3.7}$$

$$\beta(m \succ x) = \beta(m) \succ \alpha(x), \tag{3.8}$$

$$\alpha(x) \prec (y \prec m) = (x \cdot y) \prec \beta(m), \tag{3.9}$$

$$(m \succ x) \succ \alpha(y) = \beta(m) \succ (x \cdot y), \tag{3.10}$$

$$\alpha(x) \prec (m \succ y) = (x \prec m) \succ \alpha(y). \tag{3.11}$$

Definition 3.5. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra and (M, β) be a Hom-module. Assume that M is endowed with two operations $\prec: D \otimes M \to M$ and $\succ: M \otimes D \to M$. We say that $(M, \prec, \succ, \beta)$ is a bimodule over the Hom-dialgebra $(D, \dashv, \vdash, \alpha)$ if, for any $x, y \in D, m \in M$, the following identities are satisfied :

$$\beta(x \prec m) = \alpha(x) \prec \beta(m), \tag{3.12}$$

$$\beta(m \succ x) = \beta(m) \succ \alpha(x) \tag{3.13}$$

$$(x \prec m) \succ \alpha(y) = \alpha(x) \prec (m \succ y), \tag{3.14}$$

$$\beta(m) \succ (x \dashv y) = (m \succ x) \succ \alpha(y) = \beta(m) \succ (x \vdash y) \tag{3.15}$$

$$(x \dashv y) \prec \beta(m) = \alpha(x) \prec (y \prec m) = (x \vdash y) \prec \beta(m) \tag{3.16}$$

Remark 3.3. (1) (a) Axioms (3.12) and (3.13) can be interpreted as the multiplicativity in the Hom-modules theory.
(b) Axiom (3.15) (resp. (3.16)) is the left-module (resp. right-module) condition.
(c) Axiom (3.14) is the compatibility condition of left and right modules.
(2) Taking $M = D$ (as vector space), $\prec = \dashv$ and $\succ = \vdash$, we see that any Hom-dialgebra is a bimodule over itself.

We have the following result.

Proposition 3.1. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra. Then $(M, \prec, \succ, \beta)$ is a bimodule over $(D, \dashv, \vdash, \alpha)$ if and only if it is a bimodule over (D, μ, α), where $\dashv = \mu = \vdash$.*

Proof. The proof follows from Definition 3.4 and Definition 3.5. $\square$

The following theorem asserts that bimodules over Hom-dialgebras are closed under twisting.

Theorem 3.2. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dialgebra and $(M, \prec, \succ, \beta)$ be a bimodule over D. Define the maps*

$$\prec_\alpha := \prec \circ (\alpha^2 \otimes Id_M) : D \otimes M \to M, \quad x \otimes m \mapsto \alpha^2(x) \prec m \tag{3.17}$$

$$\succ_\alpha := \succ \circ (Id_M \otimes \alpha^2) : M \otimes D \to M, \quad m \otimes x \mapsto m \succ \alpha^2(x). \tag{3.18}$$

Then $(M, \prec_\alpha, \succ_\alpha, \beta)$ is a bimodule over D.

Proof. We shall only prove (3.12) and (3.14). For any $x, y \in D, m \in M$,

$$\beta(x \prec_\alpha m) \overset{(3.17)}{=} \beta(\alpha^2(x) \prec m) \overset{(3.12)}{=} \alpha^3(x) \prec \beta(m) \overset{(3.17)}{=} \alpha(x) \prec_\alpha \beta(m),$$

and,

$$\begin{aligned}(x \prec_\alpha m) \succ_\alpha \alpha(y) - \alpha(x) \prec_\alpha (m \succ_\alpha y) \quad &\overset{(3.17)}{=} \quad (\alpha^2(x) \prec m) \succ \alpha^3(y) \\ &\qquad -\alpha^3(x) \prec (m \succ \alpha^2(y)) \overset{(3.14)}{=} 0.\end{aligned}$$

All the rest of equalities are proved analogously. $\square$

Proposition 3.2. *Let $(M, \prec, \succ, \beta)$ be a bimodule over the Hom-dialgebra $(D, \dashv, \vdash, \alpha)$. Then, we have the following identities :*

$$[x, y] \prec \beta(m) = \alpha(x) \prec (y \prec m) - \alpha(y) \prec (x \prec m), \tag{3.19}$$

$$\beta(m) \succ [x, y] = (x \prec m) \succ \alpha(y) + \alpha(x) \prec (m \succ y), \tag{3.20}$$

where, $[x, y] = x \dashv y - y \vdash x$.

Proof. The first equality is proved by using (3.16). For the second equality, we have, for any $x, y \in D, m \in M$,

$$\begin{aligned} &\beta(m) \succ [x,y] - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y) = \\ &= \beta(m) \succ (x \dashv y - y \vdash x) - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y) = \\ &= \beta(m) \succ (x \dashv y) - \beta(m) \succ (y \vdash x) - (x \prec m) \succ \alpha(y) - \alpha(x) \prec (m \succ y). \end{aligned}$$

The last line vanishes by (3.14) and (3.15). □

3.3. **Left-Hom-symmetric dialgebras.**

Definition 3.6. A Left-Hom-symmetric dialgebra is a vector space S equipped with two bilinear products

$$\begin{aligned} \dashv: S \times S &\to S, \\ \vdash: S \times S &\to S, \end{aligned}$$

satisfying the identities

$$\alpha(x) \dashv (y \dashv z) = \alpha(x) \dashv (y \vdash z), \tag{3.21}$$

$$(x \vdash y) \vdash \alpha(z) = (x \dashv y) \vdash \alpha(z), \tag{3.22}$$

$$\alpha(x) \dashv (y \dashv z) - (x \dashv y) \dashv \alpha(z) = \alpha(y) \vdash (x \dashv z) - (y \vdash x) \dashv \alpha(z), \tag{3.23}$$

$$\alpha(x) \vdash (y \vdash z) - (x \vdash y) \vdash \alpha(z) = \alpha(y) \vdash (x \vdash z) - (y \vdash x) \vdash \alpha(z). \tag{3.24}$$

Remark 3.4. The identities (3.23) and (3.24) can be written as

$$\begin{aligned} L^{\dashv}_{\alpha(x)} L^{\dashv}_y - L^{\vdash}_{\alpha(y)} L^{\dashv}_x &= L^{\dashv}_{[x,y]} \alpha, \\ L^{\vdash}_{\alpha(x)} L^{\vdash}_y - L^{\vdash}_{\alpha(y)} L^{\vdash}_x &= L^{\vdash}_{[x,y]} \alpha, \end{aligned}$$

where, $L^{\dashv}_x$ and $L^{\vdash}_x$ are defined respectively by $L^{\dashv}_x y = x \dashv y$ and $L^{\vdash}_x y = x \vdash y$, and $[x, y] = x \dashv y - y \vdash x$.

Now we give some examples of left-Hom-symmetric dialgebras.

Example 3.9. Any Hom-dialgebra is a left-Hom-symmetric dialgebra.

Example 3.10. Any left-Hom-symmetric algebra is a left-Hom-symmetric dialgebra in which $\vdash=\dashv$.

Example 3.11. Let $(S, \dashv, \vdash, \alpha_S)$ be a left-Hom-symmetric dialgebra and $(A, \cdot, \alpha_A)$ be a left-Hom-symmetric algebra, then $S \times A$ is a left-Hom-symmetric dialgebra with

$$\begin{aligned} \alpha_{S\times A} &:= (\alpha_S, \alpha_A), \\ (x, a) \dashv_{S\times A} (y, b) &:= (x \dashv y, a \cdot b), \\ (x, a) \vdash_{S\times A} (y, b) &:= (x \vdash y, a \cdot b). \end{aligned}$$

We have the following result whose ordinary case is Proposition 4 in [10].

Proposition 3.3. *A left-Hom-symmetric dialgebra S is a Hom-dialgebra if and only if both products of S are Hom-associative.*

Proof. If a left-Hom-symmetric dialgebra S is a Hom-dialgebra, then both products $\dashv$ and $\vdash$ defined over S are Hom-associative according to Definition 3.2. Conversely, if each product of a left-Hom-symmetric dialgebra is Hom-associative, then from (3.23), we get (3.5). □

The next statement is one of the main results of this paper ; it states that the commutator bracket of any left-Hom-symmetric dialgebra gives rise to a Hom-Leibniz algebra.

Theorem 3.3. *Let $(S, \dashv, \vdash, \alpha)$ be a left-Hom-symmetric dialgebra. Then the Loday commutator defined by*

$$[x, y] := x \dashv y - y \vdash x, \tag{3.25}$$

defines a structure of Hom-Leibniz algebra on S.

Proof. The proof follows by a straighforward computation in which the identities (3.21) and (3.22) are used once. In fact, for any $x, y, z \in S$, we have

$$\begin{aligned} Leib_\alpha(x, y, z) &= [\alpha(x), [y, z]] - [[x, y], \alpha(z)] + [[x, z], \alpha(y)] \\ &= \alpha(x) \dashv (y \dashv z) - \alpha(x) \dashv (z \vdash y) - (y \dashv z) \vdash \alpha(x) + (z \vdash y) \vdash \alpha(x) \\ &\quad -(x \dashv y) \dashv \alpha(z) + (y \vdash x) \dashv \alpha(z) + \alpha(z) \vdash (x \dashv y) - \alpha(z) \vdash (y \vdash x) \\ &\quad +(x \dashv z) \dashv \alpha(y) - (z \vdash x) \dashv \alpha(y) - \alpha(y) \vdash (x \dashv z) + \alpha(y) \vdash (z \vdash x) \\ &= 0. \end{aligned}$$

Now, by (3.23) and (3.24) it follows that $Leib_\alpha(x, y, z) = 0$. This completes the proof. $\square$

We need the below definition in the next theorem.

Definition 3.7. Let $(S, \dashv, \vdash, \alpha)$ and $(S', \dashv', \vdash', \alpha')$ be two left-Hom-symmetric dialgebras. A map $f : S \to S'$ is said to be a morphism of left-Hom-symmetric dialgebras if

$$\alpha' \circ f = f \circ \alpha, \quad f(x \dashv y) = f(x) \dashv' f(y) \ \text{ and } \ f(x \vdash y) = f(x) \vdash' f(y),$$

for any $x, y \in S$.

Twisting a left-symmetric dialgebra by a left-symmetric dialgebras endomorphism, we get a left-Hom-symmetric dialgebra ; this is stated in the following theorem.

Theorem 3.4. *Let $(S, \dashv, \vdash)$ be a left-symmetric dialgebra and $\alpha : S \to S$ be a morphism of left-symmetric dialgebras. Then $(S, \dashv_\alpha, \vdash_\alpha, \alpha)$ is a multiplicative left-Hom-symmetric dialgebra with*

$$\begin{aligned} x \vdash_\alpha y &= \alpha(x \vdash y), \\ x \dashv_\alpha y &= \alpha(x \dashv y). \end{aligned}$$

Proof. The proof is similar to that of Proposition 3.1. $\square$

In the rest of this section, we introduce affine Hom-Leibniz structures on Hom-Leibniz algebras.

Definition 3.8. Let $(L, [-, -], \alpha)$ be a Hom-Leibniz algebra. A pair $(\bigtriangledown_1, \bigtriangledown_2)$ of bilinear maps

$$\bigtriangledown_1 : L \times L \to L$$

and

$$\bigtriangledown_2 : L \times L \to L.$$

is called an affine Hom-Leibniz structure if

$$\bigtriangledown_2(x,y) - \bigtriangledown_1(y,x) = [x,y], \tag{3.26}$$

$$\bigtriangledown_1(\bigtriangledown_1(x,y),\alpha(z)) = \bigtriangledown_1(\bigtriangledown_2(x,y),\alpha(z)), \tag{3.27}$$

$$\bigtriangledown_2(\alpha(x),\bigtriangledown_2(y,z))) = \bigtriangledown_2(\alpha(x),\bigtriangledown_1(y,z))), \tag{3.28}$$

$$\bigtriangledown_2(\alpha(x),\bigtriangledown_1(y,z)) - \bigtriangledown_1(\alpha(y),\bigtriangledown_2(x,z)) = \bigtriangledown_2([x,y],\alpha(z)), \tag{3.29}$$

and

$$\bigtriangledown_1(\alpha(x),\bigtriangledown_1(y,z)) - \bigtriangledown_1(\alpha(y),\bigtriangledown_1(x,z)) = \bigtriangledown_1([x,y],\alpha(z)), \tag{3.30}$$

for all $x, y, z \in L$.

The next result is the Hom-type of ([10], Theorem 11).

Theorem 3.5. *Let $(L, [-,-], \alpha)$ be a Hom-Leibniz algebra and let $(\bigtriangledown_1, \bigtriangledown_2)$ be an affine Hom-Leibniz structure. Then L is a left-Hom-symmetric dialgebra with $\vdash$ and $\dashv$ defined as*

$$x \vdash y = \bigtriangledown_1(x,y), \quad x \dashv y = \bigtriangledown_2(x,y). \tag{3.31}$$

Proof. Relations (3.27) and (3.28) imply (3.21) and (3.22) respectively. Next, (3.23) follows from (3.26) and (3.29). Finally, (3.24) is established by applying (3.21), (3.26) and (3.30). □

Corollary 3.1. *Let $(\bigtriangledown, \bigtriangledown)$ be an affine structure on the Hom-Leibniz algebra $(L, [-,-], \alpha)$. Then $(L, \bigtriangledown, \alpha)$ is a left-Hom-symmetric algebra.*

4. Hom-dendriform algebras

This section in devoted to modules over Hom-dendriform algebras.

Definition 4.1. [2] A Hom-dendriform algebra is a vector space D together with bilinear maps $\dashv : D \otimes D \to D$, $\vdash : D \otimes D \to D$ and linear map $\alpha : S \to S$ such that

$$\alpha(x) \vdash (y \dashv z) = (x \vdash y) \dashv \alpha(z), \tag{4.1}$$

$$(x \dashv y) \dashv \alpha(z) = \alpha(x) \dashv (y \dashv z) + \alpha(x) \dashv (y \vdash z), \tag{4.2}$$

$$\alpha(x) \vdash (y \vdash z) = (x \dashv y) \vdash \alpha(z) + (x \vdash y) \vdash \alpha(z). \tag{4.3}$$

Lemma 4.1. [2] *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dendriform algebra. Defining $x \circ y = x \vdash y - y \dashv x$, one obtains a left-Hom-symmetric algebra structure on D.*

The following result is the Hom-analogue of Proposition 5.3 in [7].

Proposition 4.1. *Let $(D, \dashv, \vdash, \alpha_D)$ and $(\mathcal{D}, \prec, \succ, \alpha_{\mathcal{D}})$ be a Hom-dialgebra and a Hom-dendriform algebra respectively. Then, on the tensor product $D \otimes \mathcal{D}$, the bracket*

$$\begin{aligned} [x \otimes a, y \otimes b] \quad := \quad & (x \dashv y) \otimes (a \prec b) - (y \vdash x) \otimes (b \succ a) \\ & -(y \dashv x) \otimes (b \prec a) + (x \vdash y) \otimes (a \succ b), \end{aligned}$$

where $x, y \in D, a, b \in \mathcal{D}$, defines a structure of Hom-Lie algebra on $D \otimes \mathcal{D}$, with $\alpha_{D \otimes \mathcal{D}} = \alpha_D \otimes \alpha_{\mathcal{D}}$.

Proof. The bracket is skew-symmetric by definition. Hence, it suffices to show that the Hom-Jacobi identity is fulfilled.

The Hom-Jacobi identity for $x \otimes a$, $y \otimes b$, $z \otimes c$ gives a total of 48 terms, in fact $8 \times 3!$ terms. There are 8 terms for which x, y, z (and also a, b, c) stay in the same order. The other set of 8 terms are permutations of this set which reads :

$$\begin{aligned}
&\alpha(x) \dashv (y \dashv z) \otimes \alpha(a) \prec (b \prec c) - (x \dashv y) \dashv \alpha(z) \otimes (a \prec b) \prec \alpha(c),\\
&\alpha(x) \vdash (y \dashv z) \otimes \alpha(a) \succ (b \prec c) - (x \vdash y) \dashv \alpha(z) \otimes (a \succ b) \prec \alpha(c),\\
&\alpha(x) \dashv (y \vdash z) \otimes \alpha(a) \prec (b \succ c) - (x \dashv y) \vdash \alpha(z) \otimes (a \prec b) \succ \alpha(c),\\
&\alpha(x) \vdash (y \vdash z) \otimes \alpha(a) \succ (b \succ c) - (x \vdash y) \vdash \alpha(z) \otimes (a \succ b) \succ \alpha(c).
\end{aligned}$$

The terms 1 and 3 in column 1 together with the term 1 in column 2 cancel due to Definition 3.2 and (4.2). Similarly, the terms 41, 32 and 42 cancel due to Definition 3.2 and (4.3). Finally the terms 21 and 22 cancel due to Definition 3.2 and (4.1). ☐

Corollary 4.1. *If D and $\mathcal{D}$ are multiplicative, then $D \otimes \mathcal{D}$ is also a multiplicative Hom-Lie algebra.*

Definition 4.2. Let $(S, \circ, \alpha)$ be a left-Hom-symmetric algebra. An S-bimodule is a vector space M endowed with a linear map $\beta : M \to M$, two bilinear maps $S \otimes M \to M, x \otimes m \mapsto x \prec m$ and $M \otimes S \to M, m \otimes x \mapsto m \succ x$, such that

$$\alpha(x) \prec (y \prec m) - (x \circ y) \prec \beta(m) - \alpha(y) \prec (x \prec m) + (y \circ x) \prec \beta(m) = 0,$$

and,

$$\alpha(x) \prec (m \succ y) - (x \prec m) \succ \alpha(y) - \beta(m) \succ (x \circ y) + (m \succ x) \succ \alpha(y) = 0.$$

Example 4.1. Any left-Hom-symmetric algebra is a bimodule over itself.

The following theorem gives a kind of connection between left-Hom-symmetric algebras and left-Hom-symmetric dialgebras.

Proposition 4.2. *Let $(S, \cdot, \alpha)$ be a left-Hom-symmetric algebra and I be a bimodule over S. Assume that, for all $i, j \in I$ and $a, b, c, d \in S$,*

$$\begin{aligned}
\alpha(i) \cdot (a \cdot b) - (i \cdot a) \cdot \alpha(b) &= \alpha(a) \cdot (i \cdot b) - (a \cdot i) \cdot \alpha(b),\\
\alpha(c) \cdot (d \cdot j) - (c \cdot d) \cdot \alpha(j) &= \alpha(d) \cdot (c \cdot j) - (d \cdot c) \cdot \alpha(j).
\end{aligned}$$

Then $(S \oplus I, \dashv, \vdash, \alpha_{S \oplus I})$ is a left-Hom-symmetric dialgebra with

$$\begin{aligned}
\alpha_{S \oplus I} &= \alpha_S \oplus \alpha_I,\\
(i_1 + a_1) \dashv (i_2 + a_2) &= i_1 a_2 + a_1 a_2,\\
(i_1 + a_1) \vdash (i_2 + a_2) &= a_1 i_2 + a_1 a_2.
\end{aligned}$$

Proof. It is straighforward by calculation. ☐

Corollary 4.2. *Let $(S, \cdot, \alpha)$ be a left-Hom-symmetric algebra and I be an ideal of S. Then $(S \oplus I, \dashv, \vdash, \alpha_{S \oplus I})$ is a left-Hom-symmetric dialgebra.*

Now, we define bimodules over Hom-dendriform algebras which are Hom-analogue of ([9], Definition 5.5).

Definition 4.3. Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dendriform algebra. A D-bimodule is a Hom-module (M, β) together with four bilinear maps

$$\begin{aligned}
&D \otimes M \to M, x \otimes m \mapsto x \succ m; \qquad D \otimes M \to M, x \otimes m \mapsto x \prec m;\\
&M \otimes D \to M, m \otimes x \mapsto m \succ x; \qquad M \otimes D \to M, m \otimes x \mapsto m \prec x
\end{aligned}$$

such that

$$\alpha(x) \succ (y \prec m) = (x \vdash y) \prec \beta(m), \tag{4.4}$$
$$(x \dashv y) \prec \beta(m) = \alpha(x) \prec (y \prec m) + \alpha(x) \prec (y \succ m), \tag{4.5}$$
$$\alpha(x) \succ (y \succ m) = (x \dashv y) \succ \beta(m) + (x \vdash y) \succ \beta(m), \tag{4.6}$$
$$\alpha(x) \succ (m \prec y) = (x \succ m) \prec \alpha(y), \tag{4.7}$$
$$(x \prec m) \prec \alpha(y) = \alpha(x) \prec (m \prec y) + \alpha(x) \prec (m \succ y), \tag{4.8}$$
$$\alpha(x) \succ (m \succ y) = (x \prec m) \succ \alpha(y) + (x \succ m) \succ \alpha(y), \tag{4.9}$$
$$\beta(m) \succ (x \dashv y) = (m \succ x) \prec \alpha(y), \tag{4.10}$$
$$(m \prec x) \prec \alpha(y) = \beta(m) \prec (x \dashv y) + \beta(m) \prec (x \vdash y), \tag{4.11}$$
$$\beta(m) \succ (x \vdash y) = (m \prec x) \succ \alpha(y) + (m \succ x) \succ \alpha(y). \tag{4.12}$$

Theorem 4.1. *Let $(D, \dashv, \vdash, \alpha)$ be a Hom-dendriform algebra and $(M, \prec, \succ, \beta)$ be a dendriform bimodule over D. Then $(M, \triangleleft, \triangleright, \beta)$ is a left-symmetric bimodule over the left-Hom-symmetric algebra associated to $(D, \dashv, \vdash, \alpha)$ (i.e. $(D, \circ, \alpha)$, where $x \circ y = x \vdash y - y \dashv x$) by means of*

$$x \triangleleft m = x \succ m - m \prec x \quad \text{and} \quad m \triangleright x = m \succ x - x \prec m.$$

Proof. The first condition in Definition 4.2 is proved by expanded

$$\alpha(x) \triangleleft (y \triangleleft m) - (x \circ y) \triangleleft \beta(m) - \alpha(y) \triangleleft (x \triangleleft m) + (y \circ x) \triangleleft \beta(m)$$

by means of $\dashv$, $\vdash$, $\prec$ and $\succ$, and using (4.6), (4.7) and (4.11). The second condition is proved similarly by using the rest of relations. □

5. Hom-Poisson dialgebras

In this section, we introduce Hom-Poisson dialgebras and we give some examples and some construction theorems.

Definition 5.1. A Hom-Poisson dialgebra is a quintuple $(P, \dashv, \vdash, [-,-], \alpha)$ in which P is a vector space, $\dashv, \vdash, [-,-] : P \otimes P \to P$ are three bilinear maps and $\alpha : P \to P$ is a linear map such that

$$[x \dashv y, \alpha(z)] = \alpha(x) \dashv [y, z] + [x, z] \dashv \alpha(y), \tag{5.1}$$

$$[x \vdash y, \alpha(z)] = \alpha(x) \vdash [y, z] + [x, z] \vdash \alpha(y), \tag{5.2}$$

$$[\alpha(x), y \dashv z] = \alpha(y) \vdash [x, z] + [x, y] \dashv \alpha(z) = [\alpha(x), y \vdash z]. \tag{5.3}$$

for all $x, y, z \in P$.

Example 5.1. Any Poisson dialgebra is a Hom-Poisson dialgebra with $\alpha = Id$.

Example 5.2. If $(A, \cdot, \alpha)$ is a symmetric Hom-Leibniz algebra [11] i.e. both left and right Hom-Leibniz algebra, then $(A, \dashv, \vdash, [-,-], \alpha)$ is a Hom-Poisson dialgebra, with $[-,-] = \cdot = \dashv = \vdash$.

Example 5.3. Let $(P, \dashv, \vdash, [-,-], \alpha)$ and $(P', \dashv', \vdash', [-,-]', \alpha')$ be two Hom-Poisson dialgebras. Then the direct product $P \times P'$ is also a Hom-Poisson dialgebra with componentwise operation. In particular, for any non-negative integer n, $P^n = P \times P \times \cdots \times P$ (n times) is a Hom-Poisson dialgebra.

The below theorem generalizes Proposition 2.6 in [3].

Theorem 5.1. *Let* $(D, \dashv, \vdash, \alpha)$ *be a Hom-dialgebra. Then* $(D, \dashv, \vdash, [-,-], \alpha)$ *is a Hom-Poisson dialgebra, where*

$$[x, y] = x \dashv y - y \vdash x,$$

for any $x, y \in D$.

Proof. It follows from axioms in Definition 3.2. □

Observe that by setting $\dashv = \mu$ and $\vdash = \mu^{op}$, we recover ([3], Proposition 2.6).

Definition 5.2. Let $(P, \dashv, \vdash, [-,-], \alpha)$ and $(P', \dashv', \vdash', [-,-]', \alpha')$ be two Hom-Poisson dialgebras. A linear map $f : P \to P'$ is said to be a morphism of Hom-Poisson dialgebras, if $\alpha' \circ f = f \circ \alpha$ and for any $x, y \in P$,

$$f(x \dashv y) = f(x) \dashv' f(y), \quad f(x \vdash y) = f(x) \vdash' f(y), \quad f([x, y]) = [f(x), f(y)]'.$$

The following theorem allows to obtain a Hom-Poisson dialgebra from Poisson dialgebra and an endomorphism.

Theorem 5.2. *Let* $(P, \dashv, \vdash, [-,-])$ *be a Poisson dialgebra and* $\alpha : P \to P$ *an endomorphism of Poisson dialgebras. Then* $(P, \dashv_\alpha, \vdash_\alpha, [-,-]_\alpha, \alpha)$ *is a Hom-Poisson dialgebra, with*

$$x \dashv_\alpha y = \alpha(x \dashv y), \quad x \vdash_\alpha y = \alpha(x \vdash y), \quad [x, y]_\alpha = \alpha([x, y]),$$

for all $x, y \in P$.

Proof. The proof is analogue to the one of Theorem 3.1. □

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