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A DIFFERENT LOOK FOR PARANORMED RIESZ SEQUENCE SPACE DERIVED BY FIBONACCI MATRIX

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Abstract. This paper presents the generalized Riesz sequence space $r^q(\widehat{F}_u^p)$ which is formed all sequences whose $R_u^q\widehat{F}$-transforms are in the space $\ell(p)$, where $\widehat{F}$ is a Fibonacci matrix. α- β- and γ-duals of the newly described sequence space have been given in addition to some topological properties of its. Also, it has been established the basis of $r^q(\widehat{F}_u^p)$. Finally, we have been described a matrix class on the sequence space. Results obtained are more general and more comprehensive than presented up to now.

1. Preliminaries

The concept of sequence is widely considered to be one of the important concepts in summability theory, so let us begin by remembering the definition of it. A sequence is a function of which domain set is natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. In other words, an ordered list of numbers $x_0, x_1, ..., x_n, ...$ is a sequence. If it is an infinite sequence, it is illustrated with notation $\{x_n\}_{n=0}^{\infty}$, as a convenience, we write $\{x_n\}$ briefly. A sequence $\{x_n\}$ converges with limit a if each neighborhood of a contains almost all terms of the sequence, i.e., there must be at most only finitely many elements of $\{x_n\}$ outside any neighborhood of a. In this case, we say that $\{x_n\}$ converges to a as n goes to ∞. The set of all real or complex convergent sequences is indicated by c. Let $\{x_n\}$ be a sequence and define a new sequence $\{s_n\}$ called the sequence of partial sums of $\{x_n\}$with relation $s_n = \sum_{k=1}^{n} x_k$. When $\{s_n\}$ is convergent, we say that $\{x_n\}$ is summable and we point out the $\lim_n s_n$ by $\sum_{j=0}^{\infty} x_j$. A real or complex number sequence converges to zero is called null sequence. The set of all real or complex null sequences is denoted by c_0. A sequence is bounded, if all its terms remain between two numbers. The set of all bounded sequences is denoted by l_∞. We denote the family of all $\{x_n\}$ sequences by w, where

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x_n belongs to real or complex numbers set. Then w is a linear space under the usual pointwise addition and scalar multiplication over $\mathbb{C}$ and $\mathbb{R}$. Since any linear subspace of w is called a sequence space, also c, c_0 and ℓ_∞ are the subspaces of w, we concludes that they are sequence spaces. Further, we symbolizes the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series by bs, cs, ℓ_1, ℓ_p; respectively.

These spaces are Banach spaces with following norms:

$\|x\|_{\ell_\infty} = \|x\|_c = \|x\|_{c_0} = \sup_k |x_k|, \|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^n x_k|$, and $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{\frac{1}{p}}$.

For sake of brevity, here and after the summation without limits runs from 1 to ∞.

Now, let us look at historical information about Fibonacci sequence. Fibonacci sequence consist of $\{f_n\}$ numbers such that each its term is the sum of two terms preceding its. In this sequence, the first two terms are 1. If we write it clearly, it is a sequence of numbers $1, 1, 2, 3, 5, 8, 13, \cdots$. We can define it by the equation $f_n = f_{n-1} + f_{n-2}$, where $n \geq 2$ and $f_1 = f_0 = 1$. Fibonacci numbers were come out by Leonardo Pisano Bogollo (c-1170-c1250), he is known with his nickname Fibonacci. Numbers of the sequence is seen in the book "Liber Abaci "firstly written by Leonardo of Pisa. He helped to replace Roman numerical system with the numbers system used today consists of numbers from 0 to 9 in Europa. Fibonacci sequence has some well-known properties such as Golden Ratio and Cassini Formula. If we take ratio of two successive terms of Fibonacci sequences, limit of the this ratio is famous Golden Ratio which is 1.61803 and written by ϕ.

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi \quad \text{(Golden Ratio)}.$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \text{for each } n \in \mathbb{N}.$$

$$\sum_k \frac{1}{f_k} \text{ converges.}$$

$$f_{n-1}.f_{n+1} - f_n^2 = (-1)^{n+1} \qquad \text{for each n} \geq 1 \text{ (Cassini Formula).}$$

Let $A = (a_{nk})$ be a triangle matrix, that is $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. The equality $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Furthermore, a triangle matrix A has an inverse A^{-1}which is also a triangle matrix and unique such that for each $x \in \omega$, $x = A(A^{-1}x) = A^{-1}(Ax)$.

The domain X_A of an infinite matrix A which is a sequence space is defined as

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}, \tag{1.1}$$

in a sequence space X.

Generally X_A constructed by the limitation matrix A is either the expansion or the contraction of the space X itself, where X is a sequence space. Sometimes they are overlap. The inclusion $X_S \subset X$ is provided strictly for $X \in \{\ell_\infty, c, c_0\}$. From this property, it can be concluded that the inclusion $X \subset X_{\Delta^{(1)}}$ is also provided firmly for $X \in \{\ell_\infty, c, c_0, \ell_p\}$. But, if X is taken as $X := c_0 \oplus span\{z\}$ for each $x \in X$, there exist an $s \in c_0$ and an $\alpha \in \mathbb{C}$ such that $x := s + \alpha z$, where $z = ((-1)^k)$ and it is considered the matrix A with the rows A_n defined by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, then we obtain $Ae = z \in \lambda$ when $Az = e \notin \lambda$ resulting in the sequences

$z \in X \setminus X_A$ and $e \in X_A \setminus \lambda$, here $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ represents a sequence of which n^{th} term is 1 for each $n \in \mathbb{N}$ and the others are 0. Namely, the sequence spaces X_A and X are overlap when none of them contains the other one [10].

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and all $x \in X$, where θ is the zero vector in the linear space X.

Let us suppose that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$ and $1/p_k + 1/p_k^{'} = 1$ provided $1 < inf p_k \leq H < \infty$. The linear spaces $\ell_\infty(p)$ and $\ell(p)$ were defined by Maddox in [56, 57] (see also Simons [68] and Nakano [63]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\},$$

and

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$h_1(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M} \quad \text{and} \quad h_2(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M} \quad \text{iff} \quad \inf p_k > 0,$$

respectively. In addition to this, by notation $\mathcal{F}$, we denote the collection consisting of all nonempty and finite subsets of $\mathbb{N}$.

Constructing a new sequence space by means of the matrix domain of a particular triangle has been used in literature as the sequence spaces $X_p = (\ell_p)_{C_1}$ [64], $r^t(p) = (\ell(p))_{R_t}$ [2], $e^r_p = (\ell_p)_{E^r}$ and $e^r(p) = (\ell(p))_{E^r}$ [7, 48, 61]. $Z(u, v, \ell_p) = (\ell_p)_{G(u,v)}$ and $\ell(u, v, p) = (\ell(p))_{G(u,v)}$ [4, 60], $a^r(p) = (\ell_p)_{A^r}$ and $a^r(u, p) = (\ell(p))_{A^r_u}$ [8, 9], $bv_p = (\ell_p)_\Delta$ and $bv(u, p) = (\ell(p))_{A_u}$ [3, 11, 59], $\overline{\ell(p)} = (\ell(p))_S$ [37], $\ell^\lambda_p = (\ell_p)_\Lambda$ in [62], $\lambda_{B(r,s)}$ in [53] $\lambda_{B(\tilde{r},\tilde{s})}$ in [25], $f_0(B)$ and $f(B)$ in [12], $f_0(\widetilde{B})$ and $f(\widetilde{B})$ in [26], where $C_1 = \{c_{nk}\}$, $R^t = \{r^t_{nk}\}$, $E^r = \{e^r_{nk}\}$, $S = \{s_{nk}\}$, $\Delta = \{\delta_{nk}\}$, $G(u, v) = \{g_{nk}\}$, $\Delta^{(m)} = \{\Delta^{(m)}_{nk}\}$, $A^r = \{a^r_{nk}\}$, $A^r_u = \{a_{nk}(r)\}$, $A^u = \{a^u_{nk}\}$, $B(r, s) = \{b_{nk}(r, s)\}$, $B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\}$, $\Lambda = \{\lambda_{nk}\}^\infty_{n,k=0}$ and $A(\lambda) = \{a_{nk}(\lambda)\}$ denote the Cesàro, Riesz, Euler, generalized weighted means or factorable matrix, summation matrix, difference matrix, generalized difference matrix and sequential band matrix, respectively [6, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 27, 28, 29, 40, 41, 42, 50, 51, 52, 54, 55, 66]. Let us note here, there are many different ways to construct new sequence spaces from old ones. To get more detailed information, one can look at the articles [24, 30, 35, 36, 69].

Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk}, where $n, k \in \mathbb{N}$ and let X, Y be sequence spaces. For any sequence x, A-transform of x is written as $Ax = \big((Ax)_n\big)$. If it is A-transform of x, it means that $(Ax)_n = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$ then A is called a matrix mapping from X into Y and is denoted by $A : X \to Y$. We illustrate the class of all infinite matrices such that $A : X \to Y$ by $(X : Y)$.

The new sequence spaces derived by Riesz mean (R, q_n) and Fibonacci matrix $\widehat{F} = \{\widehat{f}_{nk}\}$ are given in this study.

In this paper, section 2 is dedicated for the spaces of difference sequences and given some historical developments about this subject. In addition, the definition of Fibonacci Matrix and the paranormed sequence space $r^q(\widehat{F}_u^p)$ of non-absolute type which is the set of all sequences whose $R_u^q\widehat{F}$-transforms are in the space $\ell(p)$ are presented. In section 3, alpha-, beta- and gamma-duals of the sequence space $r^q(\widehat{F}_u^p)$ are found. Moreover, the basis of the space $r^q(\widehat{F}_u^p)$ is attained. In the final section, we characterize a matrix class on the sequence space.

2. Difference operator and the Riesz Sequence Space $r^q(\widehat{F}_u^p)$ of Non-absolute Type

Before following non-absolute type the Riesz sequence space $r^q(\widehat{F}_u^p)$, firstly, let us recall some definitions. We remember the idea of difference operator. The difference sequence spaces have been introduced by Kızmaz [49]. For $\lambda \in \{\ell_\infty, c, c_0\}$, $\lambda(\Delta)$ consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k+1}) \in \lambda$ is called the difference sequence spaces [49]. The difference spaces bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [5], and in the case $1 \leq p < \infty$ by Başar and Altay [11], and Çolak, et.al. [38].

The concept of difference sequences was generalized by Çolak and Et [39]. They defined and analyzed some property of these sequence spaces

$$\Delta^m \lambda = \Big\{ x = (x_k) \in \omega : \Delta^m x \in \lambda \Big\},$$

where $\Delta^1 x = (x_k - x_{k+1})$ and $\Delta^m x = \Delta(\Delta^{m-1}x)$ for $m \in \{1, 2, 3, \ldots\}$. Malkowsky and Parashar [58] introduced the sequence spaces as follows

$$\Delta^{(m)} \lambda = \Big\{ x = (x_k) \in \omega : \Delta^{(m)} x \in \lambda \Big\},$$

where $m \in \mathbb{N}$, $\Delta^{(1)}x = (x_k - x_{k-1})$ and $\Delta^{(m)}x = \Delta^{(1)}(\Delta^{(m-1)}x)$. Polat and Başar [65] introduced the spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e_0^r, e_c^r and e_∞^r, respectively. Altay [1] studied the space $\ell_p(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are $p-$absolutely summable which is a generalization of the spaces bv_p [11, 38].

The transformation given by

$$q_n = \frac{q_1 s_1 + \cdots + q_n s_n}{Q_n}$$

is called the Riesz mean (R, q_n) or simply the (R, q_n) mean, where (q_k) is a sequence of positive numbers and $Q_n = q_1 + q_2 + \cdots + q_n$.

The (R, q_n) matrix method is given by

$$r_{nk}^t := \begin{cases} \frac{q_k}{Q_n} & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n). \end{cases}$$

The Riesz sequence spaces $r^q(u, p)$ and $r^q(\Delta_u^p)$ of non-absolute type had been studied by Ganie and Sheikh [43, 67]. After then, Candan and Güneş [32] had examined the sequence space $r^q(B_u^p)$.

Many mathematician used Fibonacci numbers to construct new sequence space. Some of them are here. Kara [46] defined $\ell_p(\widehat{F})$ sequence space. After Kara et al. [47] characterized some class of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1 \leq p \leq \infty$. Also, Başarır et al. [15] introduced the sequence space $\lambda(\widehat{F})$ and $\lambda(\widehat{F}, p)$. Later, Candan [31] presented the sequence spaces $c_0(\widehat{F}\,(r,s))$ and $c(\widehat{F}\,(r,s))$. After then, Candan and Kayaduman [34] introduced the sequence space $\widehat{c}^{f(r,s)}$ derived by generalized difference Fibonacci matrix. Finally, Candan and Kara [33] studied the space $\ell_p(\widehat{F}(r,s))$, where $1 \leq p \leq \infty$.

Let f_n be the $n-$th Fibonacci number for every $n \in \mathbb{N}$. Then we define the Fibonacci matrix $\widehat{F} = \{\widehat{f}_{nk}\}$ by

$$\widehat{f}_{nk} := \begin{cases} \frac{f_n}{f_{n+1}} & , \quad k = n, \\ -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \quad \text{or} \quad k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

For $0 < p_k \leq H < \infty$, let us define the set $r^q(\widehat{F}^p_u)$ as the set of all sequences whose $R^q_u\widehat{F}$-transforms are in the sequence space $\ell(p)$, that is

$$r^q(\widehat{F}^p_u) = \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j \widehat{F} x_j \right|^{p_k} < \infty \right\}.$$

We can rewrite the set $r^q(\widehat{F}^p_u)$ by means of the notation of (1.1) as follow

$$r^q(\widehat{F}^p_u) = \{\ell(p)\}_{R^q_u \widehat{F}},$$

where $R^q_u F = (r^{q^u_F}_{nk})$ is a matrix defined as follows:

$$r^{q^u_{\widehat{F}}}_{nk} = \begin{cases} \frac{1}{Q_n}\left(\frac{f_k}{f_{k+1}} u_k q_k - \frac{f_{k+2}}{f_{k+1}} u_{k+1} q_{k+1}\right) & , \quad 0 \leq k \leq n-1, \\ \frac{f_n}{f_{n+1}} \frac{q_n u_n}{Q_n} & , \quad k = n, \\ 0 & , \quad k > n. \end{cases}$$

If $y = (y_k)$ is a $R^q_u\widehat{F}$- transform of any given sequence $x = (x_k)$, then it is written as

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j \widehat{F} x_j. \tag{2.1}$$

Hereafter, when we talk about the sequences $x = (x_k)$ and $y = (y_k)$, we will mean that they are connected with the relation (2.1).

For the sake of simplicity, here and what follows, we shall write

$$\pi_i := \frac{f_{i+1}}{f_i u_i q_i} - \frac{f_{i+1}}{f_{i+2} u_{i+1} q_{i+1}}, \ \varphi_i := \frac{f_i}{f_{i+1}} u_i q_i - \frac{f_{i+2}}{f_{i+1}} u_{i+1} q_{i+1}$$

for every $i \in \mathbb{N}$.

Now, it is time to give the following theorem.

Theorem 2.1. *The set $r^q(\widehat{F}^p_u)$ is a linear space together with coordinatewise addition and scalar multiplication, that is, $r^q(\widehat{F}^p_u)$ is a sequence space.*

Proof. The proof of this theorem is obtained by using elementary calculations of linear algebra. □

Theorem 2.2. *Let $0 < p_k \leq H < \infty$. Then, $r^q(\widehat{F}_u^p)$ is the complete linear metric space with h paronorm defined by the following equality*

$$h_{\widehat{F}}(x) = \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j x_j + \frac{f_k}{f_{k+1}}\frac{u_k q_k}{Q_k}x_k\right|^{p_k}\right]^{\frac{1}{M}}.$$

Proof. According to the definition of paranorm reminded in introduction, it is sufficient to show that the conditions of the paranorm are satisfied. It is easy to see that $h_{\widehat{F}}(\theta) = 0$ for the null element of $r^q(\widehat{F}_u^p)$ and $h_{\widehat{F}}(x) = h_{\widehat{F}}(-x)$ for all $x \in r^q(\widehat{F}_u^p)$.

Now, we shall show the subadditivity of h. By taking $z, x \in r^q(\widehat{F}_u^p)$, we have

$$\begin{aligned}
(2.2)\quad h_{\widehat{F}}(x+z) &= \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j (x_j+z_j) + \frac{f_k}{f_{k+1}}\frac{u_k q_k}{Q_k}(x_k+z_k)\right|^{p_k}\right]^{\frac{1}{M}} \\
&\leq \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j x_j + \frac{f_k}{f_{k+1}}\frac{u_k q_k}{Q_k}x_k\right|^{p_k}\right]^{\frac{1}{M}} \\
&\quad + \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j z_j + \frac{f_k}{f_{k+1}}\frac{u_k q_k}{Q_k}z_k\right|^{p_k}\right]^{\frac{1}{M}} \\
&= h_{\widehat{F}}(x) + h_{\widehat{F}}(z).
\end{aligned}$$

For an arbitrary $\alpha \in \mathbb{R}$ (see [57, p. 30])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{2.3}$$

Again, the inequalities (2.2) and (2.3) are come out by the subadditivity of h and the following inequality clearly holds

$$h_{\widehat{F}}(\alpha x) \leq \max\{1, |\alpha|^M\} h_{\widehat{F}}(x).$$

Finally, we show that the scalar multiplication is continuous. Let α be any complex number and (x^n) be any sequence in $r^q(\widehat{F}_u^p)$ such that $h_{\widehat{F}}(x^n - x) \to 0$. Additionally, let (α_n) be an arbitrary sequence of scalars such that $\alpha_n \to \alpha$, we get

$$\begin{aligned}
h_{\widehat{F}}(\alpha_n x^n - \alpha x) &= \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j (\alpha_n x_j^n - \alpha x_j)\right|^{p_k}\right]^{\frac{1}{M}} \\
&\leq |\alpha_n - \alpha|^{\frac{1}{M}} h_{\widehat{F}}(x^n) + |\alpha|^{\frac{1}{M}} h_{\widehat{F}}(x^n - x),
\end{aligned}$$

tending to zero, for $n \to \infty$, since $\left\{h_{\widehat{F}}(x^n)\right\}$ is bounded due to the inequality

$$h_{\widehat{F}}(x^n) \leq h_{\widehat{F}}(x) + h_{\widehat{F}}(x^n - x).$$

Because of subadditive of $h_{\widehat{F}}$, it is valid. It means that the scalar multiplication is continuous and $h_{\widehat{F}}$ is a paranorm on the space $r^q(\widehat{F}_u^p)$.

Let us suppose that $\{x^i\}$ is an arbitrary Cauchy sequence in the space $r^q(\widehat{F}_u^p)$, where $x^i = \{x_0^i, x_1^i, ...\}$. In that case, there exists a positive integer $n_0(\epsilon)$

$$h_{\widehat{F}}(x^i - x^j) < \infty, \tag{2.4}$$

for all $i, j \geq n_0(\epsilon)$ for a given $\epsilon > 0$. By using definition of $h_{\widehat{F}}$, for each fixed $k \in \mathbb{N}$

$$\left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right| \leq \left[\sum_k \left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right|^{p_k}\right]^{\frac{1}{M}} < \infty,$$

for $i, j \geq n_0(\epsilon)$, and $\left\{(R_u^q \widehat{F} x^0)_k, (R_u^q \widehat{F} x^1)_k, ...\right\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges. Therefore, we can write $(R_u^q \widehat{F} x^i)_k \to (R_u^q \widehat{F} x)_k$, for $i \to \infty$. Using these infinitely limits $(R_u^q \widehat{F} x)_0, (R_u^q \widehat{F} x)_1, ...$, we can constitute the sequence $\left\{(R_u^q \widehat{F} x)_0, (R_u^q \widehat{F} x)_1, ...\right\}$. From inequality (2.4) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$, we have

$$\sum_{k=0}^{m} \left|(R_u^q \widehat{F} x^i)_k - (R_u^q \widehat{F} x^j)_k\right|^{p_k} \leq h_{\widehat{F}}(x^i - x^j)^M < \epsilon^M. \tag{2.5}$$

For j and $m \to \infty$ inequality (2.5) becomes

$$h_{\widehat{F}}(x^i - x) < \infty.$$

Taking $\epsilon = 1$, $i \geq n_0(1)$ in inequality (2.5) and using Minkowsky's inequality, for each $m \in \mathbb{N}$, we get

$$\left[\sum_{k=0}^{m} \left|(R_u^q \widehat{F} x)_k\right|^{p_k}\right]^{\frac{1}{M}} \leq h_{\widehat{F}}(x^i - x) + h_{\widehat{F}}(x^i) \leq 1 + h_{\widehat{F}}(x^i),$$

i.e., $x \in r^q(\widehat{F}_u^p)$. Because $h_{\widehat{F}}(x^i - x) \leq \infty$ for all $i \geq n_0(\epsilon)$, $x^i \to x$ as $i \to \infty$, thus it is proved that $r^q(\widehat{F}_u^p)$ is complete. $\square$

It is seen that the absolute property is invalid on the space $r^q(\widehat{F}_u^p)$, in other words $h_{\widehat{F}}(x) \neq h_{\widehat{F}}(\mid x \mid)$ holds for at least one sequence in the space $r^q(\widehat{F}_u^p)$ i.e., $r^q(\widehat{F}_u^p)$ is a sequence space of non-absolute type.

Theorem 2.3. *Let $0 < p_k \leq H < \infty$. Then the sequence space $r^q(\widehat{F}_u^p)$ is linearly isomorphic to the space $\ell(p)$.*

Proof. To prove this theorem's assertion, we firstly have to make sure that there exists a transformation T between the spaces $r^q(\widehat{F}_u^p)$ and $\ell(p)$. Let us take into account the transformation T from $r^q(\widehat{F}_u^p)$ to $\ell(p)$ by $x \to y = Tx$. Since it is obvious to show that T is linear, we omit the details. Now, it is necessary to prove that both T is injective and surjective. If we take $x = \theta$, we obtain that $Tx = \theta$ and this shows that T is injective.

We consider an arbitrary sequence $y \in \ell(p)$ and later define the sequence $x = (x_k)$

$$x_k = \sum_{n=0}^{k-1} \prod_{j=n}^{k-1} \left(\frac{f_{j+2}}{f_{j+1}}\right)^2 \pi_n Q_n y_n + \frac{f_{k+1}}{f_k} \frac{Q_k}{u_k q_k} y_k,$$

for $k \in \mathbb{N}$. If we use the newly defined sequence $x = (x_k)$, then we have

$$\begin{aligned} h_{\widehat{F}}(x) &= \left[\sum_k \left|\frac{1}{Q_k}\sum_{j=0}^{k-1}\varphi_j x_j + \frac{f_k}{f_{k+1}}\frac{u_k q_k}{Q_k}x_k\right|^{p_k}\right]^{\frac{1}{M}} \\ &= \left[\sum_k \left|\sum_{j=0}^{k}\delta_{kj}y_j\right|^{p_k}\right]^{\frac{1}{M}} \\ &= \left[\sum_k |y_k|^{p_k}\right]^{\frac{1}{M}} \\ &= h_1(y) < \infty \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1 & , \quad k = j, \\ 0 & , \quad k \neq j. \end{cases}$$

This shows that $x \in r^q(\widehat{F}_u^p)$. In other words, T is surjective and paranorm preserving. Thus, the transformation T is a linear bijection which means that $r^q(\widehat{F}_u^p)$ and $\ell(p)$ are linearly isomorphic. This completes the proof. □

3. Schauder Basis and $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\widehat{F}_u^p)$

In the present section, firstly, let us recall the definitions of alpha-, beta-, and gamma- dual concepts.

If $\lambda, \mu \subset w$ and z is an arbitrary sequence, we write

$$z^{-1} * \lambda = \{x = (x_k) \in w : xz \in \lambda\},$$

and

$$M(\lambda, \mu) = \cap_{x \in \lambda} x^{-1} * \mu.$$

If we choose $\mu = \ell_1$, cs and bs, then we obtain the $\alpha-, \beta-$ and $\gamma-$ duals of the space λ, respectively as

$$\lambda^{\alpha} = M(\lambda, \ell_1) = \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1 \text{ for all } x \in \lambda\},$$

$$\lambda^{\beta} = M(\lambda, cs) = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x \in \lambda\},$$

$$\lambda^{\gamma} = M(\lambda, bs) = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \text{ for all } x \in \lambda\}.$$

Now, we are going to give the following lemmas necessary to prove the theorems related to the $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\widehat{F}_u^p)$.

Lemma 3.1. [44]

(i) *Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{K \in \mathcal{F}} \sum_k \left|\sum_{n \in K} a_{nk} B^{-1}\right|^{p_k'} < \infty.$$

(ii) *Let $0 < p_k \leq 1$. Then $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \sup_k \left|\sum_{n \in K} a_{nk}\right|^{p_k} < \infty.$$

Lemma 3.2. [45]

(i) *Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_n \sum_k \left|a_{nk} B^{-1}\right|^{p_k'} < \infty. \tag{3.1}$$

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

Lemma 3.3. [45] *$A \in (\ell(p) : c)$ if and only if there exists an integer $B > 1$ provided that (3.1) and (3.2) hold,*

$$\lim_n a_{nk} = \beta_k \text{ for } k \in \mathbb{N}, \tag{3.3}$$

also holds, where $0 < p_k \leq H < \infty$ for every given $k \in \mathbb{N}$.

Theorem 3.1. *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. The sets $D_1(u,p), D_2(u,p)$ and $D_3(u,p)$ are defined by following equations:*

$$D_1(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_k \left| \sum_{n \in K} \left[\prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k + \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n \right] B^{-1} \right|^{p_k} < \infty \right\},$$

$$D_2(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sum_k \left| \left[\left(\frac{f_{k+1}}{f_k} \frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] B^{-1} \right|^{p_k'} < \infty \right\},$$

$$D_3(u,p) = \left\{ a = (a_k) \in w : \sum_{i=k+1}^{\infty} a_i \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \text{ exists} \right\}.$$

In this case,

$$[r^q(B_u^p)]^\alpha = D_1(u,p), \qquad [r^q(B_u^p)]^\beta = D_2(u,p) \cap D_3(u,p), \qquad [r^q(B_u^p)]^\gamma = D_2(u,p).$$

Proof. Let us take any $a = (a_k) \in w$. Then, we obtain

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n-1} \prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k y_k + \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n y_n \\ &= (Dy)_n, \end{aligned} \tag{3.4}$$

where the matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \prod_{j=k}^{n-1} \left(\frac{f_{j+2}}{f_{j+1}} \right)^2 \pi_k a_n Q_k & , \quad 0 \leq k \leq n-1, \\ \frac{f_{n+1}}{f_n} \frac{a_n}{u_n q_n} Q_n & , \quad k = n, \\ 0 & , \quad k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus from Eq.(3.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in r^q(F_u^p)$ if and only if $Dy \in \ell_1$ whenever $y \in \ell(p)$. This means that $D \in (\ell(p), \ell_1)$, and Lemma 3.1(ii) gives that $\left[r^q(\widehat{F}_u^p) \right]^\alpha = D_1(u,p)$.

For $\beta-$ dual of space $r^q(F_u^p)$, let us consider following equation,

$$\begin{aligned}(3.5)\quad \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\left(\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2\right) Q_k\right] y_k \\ &= (Ey)_n,\end{aligned}$$

where, $E = (e_{nk})$ is defined as

$$e_{nk} = \begin{cases} \left[\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2\right] Q_k & , \quad 0 \le k \le n, \\ 0 & , \quad k > n. \end{cases}$$

From Eq.(3.5), $ax = (a_k x_k) \in cs$ whenever $x \in r^q(\widehat{F}_u^p)$ if and only if $Ey \in c$ whenever $y \in \ell(p)$. In other words, $E \in (\ell(p), c)$. We obtain $\left[r^q(\widehat{F}_u^p)\right]^\beta = D_2(u,p) \cap D_3(u,p)$, using Lemma 3.3.

For $\gamma-$ dual of space $r^q(\widehat{F}_u^p)$, using Eq.(3.5) $ax = (a_k x_k) \in bs$ whenever $x \in r^q(\widehat{F}_u^p)$ iff $Ey \in \ell_\infty$ whenever $y \in \ell(p)$. In other words, $a = (a_k) \in [r^q(\widehat{F}_u^p)]^\gamma$ iff $E \in (\ell(p), \ell_\infty)$. Then from Lemma 3.2 (ii) we obtain $[r^q(\widehat{F}_u^p)]^\gamma = D_2(u,p)$. □

Theorem 3.2. *Let $1 < p_k \le H < \infty$ for every $k \in \mathbb{N}$ and define the sets $D_4(u,p)$ and $D_5(u,p)$ with the following equations:*

$$D_4(u,p) = \left\{ a = (a_k) \in w : \sup_{K \in F} \sup_k \left| \sum_{n \in K} \left[\prod_{j=n+1}^{k} \left(\frac{f_{j+2}}{f_{j+1}}\right)^2 \pi_n a_n Q_n + \frac{f_{k+1}}{f_k}\frac{a_n}{u_k q_k} Q_k \right] \right|^{p_k} < \infty \right\},$$

$$D_5(u,p) = \left\{ a = (a_k) \in w : \sup_k \left| \left(\frac{f_{k+1}}{f_k}\frac{a_k}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_i \prod_{j=k+1}^{i}\left(\frac{f_{j+1}}{f_j}\right)^2 \right) Q_k \right|^{p_k} < \infty \right\}.$$

Then

$$[r^q(F_u^p)]^\alpha = D_4(u,p), \qquad \left[r^q(\widehat{F}_u^p)\right]^\beta = D_3(u,p) \cap D_5(u,p), \qquad \left[r^q(\widehat{F}_u^p)\right]^\gamma = D_5(u,p).$$

Proof. It can be done as that of Theorem 3.1. □

Theorem 3.3. *Let $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(q) = \left\{ b_n^{(k)}(q) \right\}$ of the elements of the space $r^q(\widehat{F}_u^p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \frac{f_{k+1}}{f_k}\frac{Q_k}{u_k q_k} & , \quad n = k, \\ \prod_{j=k+1}^{n}\left(\frac{f_{j+1}}{f_j}\right)^2 \pi_k Q_k & , \quad n > k, \\ 0 & , \quad n < k. \end{cases}$$

Then, the sequence $b^{(k)}(q)$ is a basis for the space $r^q(\widehat{F}_u^p)$ and any $x \in r^q(\widehat{F}_u^p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^k(q), \tag{3.6}$$

where $\lambda_k(q) = (R_u^q \widehat{F} x)_k$ for all $k \in \mathbb{N}$.

Proof. Let $0 < p_k \leq H < \infty$, and for $k \in \mathbb{N}$

$$R_u^q \widehat{F} b^{(k)}(q) = e^{(k)} \in \ell(p), \tag{3.7}$$

where $e^{(k)}$ is a sequence of which k^{th} term is 1 and the others are 0 for each $k \in \mathbb{N}$. Moreover, let $x \in r^q(\widehat{F}_u^p)$. For all non-negative integer m, we get

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q). \tag{3.8}$$

Putting $R_u^q \widehat{F}$ to Eq.(3.8), for $i, m \in \mathbb{N}$, we have

$$R_u^q \widehat{F} x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) R_u^q \widehat{F} b^{(k)}(q) = \sum_{k=0}^{m} (R_u^q \widehat{F} x)_k e^{(k)},$$

and hence

$$\left(R_u^q \widehat{F}(x - x^{[m]})\right)_i = \begin{cases} 0 & , \quad 0 \leq i \leq m, \\ (R_u^q \widehat{F} x)_i & , \quad i > m. \end{cases}$$

Also, for any given $\epsilon > 0$, there exists an integer m_0 such that for every $m \geq m_0$

$$\left(\sum_{i=m_0}^{\infty} \left|(R_u^q \widehat{F} x)_i\right|^{p_k}\right)^{\frac{1}{M}} < \frac{\epsilon}{2}.$$

Hence, it is obtained that for all $m \geq m_0$

$$\begin{aligned} h_{\widehat{F}}(x - x^m) &= \left(\sum_{i=m}^{\infty} \left|(R_u^q \widehat{F} x)_i\right|^{p_k}\right)^{\frac{1}{M}} \\ &\leq \left(\sum_{i=m_0}^{\infty} \left|(R_u^q \widehat{F} x)_i\right|^{p_k}\right)^{\frac{1}{M}} \\ &< \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

By using limit properties, $lim_{m\to\infty} h_{\widehat{F}}(x - x^m) = 0$ is obtained. Thus x is represented as Eq.(3.6).

Let us suppose that it has two representation as $x = \sum_k \mu_k(q) b^{(k)}$ and $x = \sum_k \lambda_k(q) b^{(k)}$. Since the linear transformation from $r^q(\widehat{F}_u^p)$ to $\ell(p)$ is continuous, we get

$$\begin{aligned} (R_u^q \widehat{F} x)_n &= \sum_k \mu_k(q) \left(R_u^q F b^{(k)}(q)\right)_n \\ &= \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q), \end{aligned}$$

for $n \in \mathbb{N}$. Taking $(R_u^q \widehat{F} x)_n = \lambda_n$ for all $n \in \mathbb{N}$, it is obtained $\lambda_n(q) = \mu_n(q)$ thus we get Eq. (3.6). □

4. Matrix Mapping on the Space $r^q(\widehat{F}_u^p)$

In this section, we characterize the matrix class $\left(r^q(\widehat{F}_u^p), \ell_\infty\right)$.

Theorem 4.1.

(i) $A \in \left(r^q(\widehat{F}_u^p), \ell_\infty\right)$ *if and only if there exists an integer $B > 0$ such that*

(4.1)
$$C(B) = \sup_n \sum_k \left| \left[\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right] Q_k B^{-1} \right|^{p_k'} < \infty,$$

and
$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs \qquad (n \in \mathbb{N}),$$
where $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$.

(ii) $A \in \left(r^q(\widehat{F}_u^p), \ell_\infty\right)$ *if and only if*
$$\sup_{n,k} \left| \left[\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{n} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right] Q_k \right|^{p_k} < \infty, \tag{4.2}$$

and
$$\{a_{nk}\}_{k\in\mathbb{N}} \in cs \qquad (n \in \mathbb{N}),$$
where $0 < p_k \leq 1 < \infty$ for every $k \in \mathbb{N}$.

Proof.

(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$ and $A \in \left(r^q(\widehat{F}_u^p), \ell_\infty\right)$. Then Ax exists for $x \in r^q(\widehat{F}_u^p)$, $\quad \{a_{nk}\}_{k\in\mathbb{N}} \in \left[r^q(\widehat{F}_u^p)\right]^\beta$ for each $n \in \mathbb{N}$. Further, let us consider the following equality obtained by using the relation (3.4) that
$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \left[\frac{f_{k+1}}{f_k} \left(\frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{m} a_{nj} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] y_k. \tag{4.3}$$

From Lemma 3.1 and Eq.(4.3), we obtain the expression.

Conversely, $\{a_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$, $x \in r^q(\widehat{F}_u^p)$. Since $\{a_{nk}\}_{k\in\mathbb{N}} \in \left[r^q(\widehat{F}_u^p)\right]^\beta$ for every fixed $n \in \mathbb{N}$ A-transform of x exists. We derive from Eq.(4.3) as $m \to \infty$ that
$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\left(\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{\infty} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] y_k. \tag{4.4}$$

Now, by combining Eq.(4.4) and inequality holding for an arbitrary $B > 0$ and complex numbers a, b
$$|ab| \leq B \left\{ |aB^{-1}|^{p'} + |b|^p \right\},$$
where $p > 1$ and $1/p + 1/p' = 1$. We obtain
$$\begin{aligned} \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| &\leq \sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} \left| \left[\left(\frac{f_{k+1}}{f_k} \frac{a_{nk}}{u_k q_k} + \pi_k \sum_{i=k+1}^{\infty} a_{ni} \prod_{j=k+1}^{i} \left(\frac{f_{j+1}}{f_j} \right)^2 \right) Q_k \right] \right| |y_k| \\ &\leq B[C(B) + h_1^M(y)] < \infty. \end{aligned}$$

This mean that $Ax \in \ell_\infty$ whenever $x \in r^q(\widehat{F}_u^p)$.

(ii) The proof of (ii) can be obtained same way. $\square$

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