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OSTROWSKI TYPE INEQUALITIES FOR HARMONICALLY s-CONVEX FUNCTIONS

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Abstract. The author introduces the concept of harmonically s-convex functions and establishes some Ostrowski type inequalities and a variant of Hermite-Hadamard inequality for these classes of functions.

1. Introduction

Let $f : I\to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in I° (the interior of I) and let $a, b \in I^{\circ}$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

for all $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality (see [13]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$. For some results which generalize, improve and extend the inequalities(1.1) we refer the reader to the recent papers (see [2, 12]).

In [7], Hudzik and Maligranda considered the following class of functions:

Definition 1.1. A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and s fixed in $(0, 1]$. They denoted this by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

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In [5], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the s-convex functions.

Theorem 1.1. *Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is an s-convex function in the second sense, where $s\in[0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L[a,b]$, then the following inequalities hold*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

the constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

The above inequalities are sharp. For some recent results and generalizations concerning s-convex functions see [3, 4, 5, 6, 8, 10, 11].

In [9], the author gave harmonically convex and established Hermite-Hadamard's inequality for harmonically convex functions as follows:

Definition 1.2. Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{1.3}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 1.2. *Let $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

The above inequalities are sharp.

The goal of this paper is to introduce the concept of the harmonically s-convex functions, obtain the similar the inequalities (1.4) for harmonically s-convex functions and establish some new inequalities of Ostrowski type for harmonically s-convex functions.

2. Main Results

Definition 2.1. Let $I\subset(0,\infty)$ be an real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically s-convex (concave), if

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq(\geq)\,t^sf(y)+(1-t)^sf(x) \tag{2.1}$$

for all $x,y\in I$, $t\in[0,1]$ and for some fixed $s\in(0,1]$.

Proposition 2.1. *Let $I\subset(0,\infty)$ be an real interval and $f:I\to\mathbb{R}$ is a function, then ;*

1. *if f is s-convex and nondecreasing function then f is harmonically s-convex.*
2. *if f is harmonically s-convex and nonincreasing function then f is s-convex.*

Proof. Since $f:(0,\infty)\to\mathbb{R},\ f(x)=x$, harmonically convex function, we have

$$\frac{xy}{tx+(1-t)y}\le ty+(1-t)x \tag{2.2}$$

for all $x,y\in(0,\infty)$, $t\in[0,1]$. The proposition (1) and (2) is easily obtained from the inequality (2.2). $\square$

Example 2.1. Let $s\in(0,1]$ and $f:(0,1]\to(0,1]$, $f(x)=x^s$. Since f is s-convex (see [7]) and nondecreasing function, f is harmonically $s-$convex.

Proposition 2.2. *Let $s\in(0,1]$, $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a function and $g:[a,b]\to[a,b]$, $g(x)=\frac{ab}{a+b-x}$. Then f is harmonically s-convex on $[a,b]$ if and only if $f\circ g$ is s-convex on $[a,b]$.*

Proof. Since

$$(f\circ g)(ta+(1-t)b)=f\left(\frac{ab}{tb+(1-t)a}\right) \tag{2.3}$$

for all $t\in[0,1]$. The proof is obvious from equality (2.3). $\square$

The following result of the Hermite-Hadamard type holds.

Theorem 2.1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be an harmonically s-convex function, $s\in(0,1]$ and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold:*

$$2^{s-1}f\left(\frac{2ab}{a+b}\right)\le\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\le\frac{f(a)+f(b)}{s+1}. \tag{2.4}$$

Proof. Since $f:I\to\mathbb{R}$ is an harmonically s-convex function, we have, for all $x,y\in I$ (with $t=\frac{1}{2}$ in the inequality (2.1))

$$f\left(\frac{2xy}{x+y}\right)\le\frac{f(y)+f(x)}{2^s}$$

Choosing $x=\frac{ab}{ta+(1-t)b}$, $y=\frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2ab}{a+b}\right)\le\frac{f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)}{2^s}$$

Further, integrating for $t\in[0,1]$, we have

$$f\left(\frac{2ab}{a+b}\right)\le\frac{1}{2^s}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right] \tag{2.5}$$

Since each of the integrals is equal to $\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx$, we obtain the left-hand side of the inequality (2.4) from (2.5).

The proof of the second inequality follows by using (2.1) with $x=a$ and $y=b$ and integrating with respect to t over $[0,1]$. $\square$

In order to prove our main theorems, we need the following lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\begin{aligned}
& f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \\
= & \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2}f'\left(\frac{ax}{ta+(1-t)x}\right)dt\right. \\
& \left. -(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2}f'\left(\frac{bx}{tb+(1-t)x}\right)dt\right\}
\end{aligned}$$

Proof. Integrating by part and changing variables of integration yields

$$\begin{aligned}
& \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2}f'\left(\frac{ax}{ta+(1-t)x}\right)dt\right. \\
& \left. -(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2}f'\left(\frac{bx}{tb+(1-t)x}\right)dt\right\}
\end{aligned}$$

$$\begin{aligned}
= & \frac{1}{x(b-a)}\left[b(x-a)\int_0^1 tdf\left(\frac{ax}{ta+(1-t)x}\right)+a(b-x)\int_0^1 tdf\left(\frac{bx}{tb+(1-t)x}\right)\right] \\
= & \frac{1}{x(b-a)}\left[b(x-a)\left\{tf\left(\frac{ax}{ta+(1-t)x}\right)\Big|_0^1-\int_0^1 f\left(\frac{ax}{ta+(1-t)x}\right)dt\right\}\right] \\
& +\frac{1}{x(b-a)}\left[a(b-x)\left\{tf\left(\frac{bx}{tb+(1-t)x}\right)\Big|_0^1-\int_0^1 f\left(\frac{bx}{tb+(1-t)x}\right)dt\right\}\right] \\
= & f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du.
\end{aligned}$$

□

Theorem 2.2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically s- convex on $[a,b]$ for $q \geq 1$, then for all $x \in [a,b]$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \tag{2.6}$$

$$\begin{aligned}
\leq & \frac{ab}{b-a}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,q)\left|f'(x)\right|^q+\lambda_2((a,x,s,q,q)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\
& \left.+(b-x)^2\left(\lambda_3(b,x,s,q,q)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,q)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\lambda_1(a,x,s,\vartheta,\rho)=\frac{\beta(\rho+s+1,1)}{x^{2\vartheta}}.{}_2F_1\left(2\vartheta,\rho+s+1;\rho+s+2;1-\frac{a}{x}\right),$$
$$\lambda_2(a,x,s,\vartheta,\rho)=\frac{\beta(\rho+1,1)}{x^{2\vartheta}}.{}_2F_1\left(2\vartheta,\rho+1;\rho+s+2;1-\frac{a}{x}\right),$$
$$\lambda_3(b,x,s,\vartheta,\rho)=\frac{\beta(1,\rho+s+1)}{b^{2\vartheta}}.{}_2F_1\left(2\vartheta,1;\rho+s+2;1-\frac{x}{b}\right),$$
$$\lambda_4(b,x,s,\vartheta,\rho)=\frac{\beta(s+1,\rho+1)}{b^{2\vartheta}}.{}_2F_1\left(2\vartheta,s+1;\rho+s+2;1-\frac{x}{b}\right),$$

β is Euler Beta function defined by

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x,y>0,$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\ c>b>0,\ |z|<1 \text{ (see [1])}.$$

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\begin{aligned}
&\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|\\
\leq\ &\frac{ab}{b-a}\left\{(x-a)^2\int_0^1\frac{t}{(ta+(1-t)x)^2}\left|f'\left(\frac{ax}{ta+(1-t)x}\right)\right|dt\right.\\
&\left.+(b-x)^2\int_0^1\frac{t}{(tb+(1-t)x)^2}\left|f'\left(\frac{bx}{tb+(1-t)x}\right)\right|dt\right\}\\
\leq\ &\frac{ab(x-a)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\frac{t^q}{(ta+(1-t)x)^{2q}}\left[t^s|f'(x)|^q+(1-t)^s|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{ab(b-x)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\frac{t^q}{(tb+(1-t)x)^{2q}}\left[t^s|f'(x)|^q+(1-t)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}},
\end{aligned}\tag{2.7}$$

where an easy calculation gives

(2.8)
$$\int_0^1 \frac{t^{q+s}}{(ta+(1-t)x)^{2q}}dt = \frac{\beta(q+s+1,1)}{x^{2q}}.{}_2F_1\left(2q,q+s+1;q+s+2;1-\frac{a}{x}\right),$$

$$\int_0^1 \frac{t^{q+s}}{(tb+(1-t)x)^{2q}}dt = \frac{\beta(1,q+s+1)}{b^{2q}}.{}_2F_1\left(2q,1;q+s+2;1-\frac{x}{b}\right),$$

$$\int_0^1 \frac{t^{q}(1-t)^s}{(ta+(1-t)x)^{2q}}dt = \frac{\beta(q+1,s+1)}{x^{2q}}.{}_2F_1\left(2q,q+1;s+q+2;1-\frac{a}{x}\right),$$

$$\int_0^1 \frac{t^{q}(1-t)^s}{(tb+(1-t)x)^{2q}}dt = \frac{\beta(s+1,q+1)}{b^{2q}}.{}_2F_1\left(2q,s+1;s+q+2;1-\frac{x}{b}\right). \tag{2.9}$$

Hence, If we use (2.8)-(2.9) in (2.7), we obtain the desired result. This completes the proof. □

Corollary 2.1. *In Theorem 2.2, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\begin{aligned}&\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|\\ \leq\ & \frac{ab}{b-a}M\left\{(x-a)^2\left(\lambda_1(a,x,s,q,q)+\lambda_2((a,x,s,q,q)\right)^{\frac{1}{q}}\right.\\ &\left.+(b-x)^2\left(\lambda_3(b,x,s,q,q)+\lambda_4(b,x,s,q,q)\right)^{\frac{1}{q}}\right\}\end{aligned}$$

holds.

Theorem 2.3. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a,b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically s- convex on* $[a,b]$ *for* $q \geq 1$, *then for all* $x \in [a,b]$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \tag{2.10}$$

$$\begin{aligned}\leq\ & \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)\left|f'(x)\right|^q+\lambda_2(a,x,s,q,1)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+(b-x)^2\left(\lambda_3(b,x,s,q,1)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}\end{aligned}$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \tag{2.11}$$

$$\leq \frac{ab\left(x-a\right)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{t}{\left(ta+(1-t)x\right)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(a\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\left(b-x\right)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \frac{t}{\left(tb+(1-t)x\right)^{2q}}\left[t^s\left|f'\left(x\right)\right|^q+(1-t)^s\left|f'\left(b\right)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)\left|f'\left(x\right)\right|^q+\lambda_2(a,x,s,q,1)\left|f'\left(a\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(b-x\right)^2\left(\lambda_3(b,x,s,q,1)\left|f'\left(x\right)\right|^q+\lambda_4(b,x,s,q,1)\left|f'\left(b\right)\right|^q\right)^{\frac{1}{q}}\right\}$$

This completes the proof. □

Corollary 2.2. *In Theorem 2.3, additionally, if* $|f'(x)|\leq M$, $x\in[a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$

$$\leq \frac{ab}{b-a}M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,1)+\lambda_2((a,x,s,q,1)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(b-x\right)^2\left(\lambda_3(b,x,s,q,1)+\lambda_4(b,x,s,q,1)\right)^{\frac{1}{q}}\right\}$$

holds.

Theorem 2.4. *Let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I°, $a,b\in I^\circ$ *with* $a<b$, *and* $f'\in L[a,b]$. *If* $|f'|^q$ *is harmonically s- convex on* $[a,b]$ *for* $q\geq 1$, *then for all* $x\in[a,b]$, *we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \tag{2.12}$$

$$\leq \frac{ab}{b-a}\left\{\lambda_5^{1-\frac{1}{q}}(a,x)(x-a)^2\left(\lambda_1(a,x,s,1,1)\left|f'(x)\right|^q+\lambda_2(a,x,s,1,1)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\lambda_5^{1-\frac{1}{q}}(b,x)(b-x)^2\left(\lambda_3(b,x,s,1,1)\left|f'(x)\right|^q+\lambda_4(b,x,s,1,1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}$$

where

$$\lambda_5(\theta,x)=\frac{1}{x-\theta}\left\{\frac{1}{\theta}-\frac{\ln x-\ln\theta}{x-\theta}\right\},$$

and λ_1, λ_2, λ_3 and λ_4 are defined as in Theorem 2.2.

Proof. From Lemma 2.1, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \tag{2.13}$$
$$\leq \frac{ab(x-a)^2}{b-a}\left(\int_0^1\frac{t}{(ta+(1-t)x)^2}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(ta+(1-t)x)^2}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab(b-x)^2}{b-a}\left(\int_0^1\frac{t}{(tb+(1-t)x)^2}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(tb+(1-t)x)^2}\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}.$$

It is easily check that

$$\int_0^1\frac{t}{(ta+(1-t)x)^2}dt=\frac{1}{x-a}\left\{\frac{1}{a}-\frac{\ln x-\ln a}{x-a}\right\}, \tag{2.14}$$

$$\int_0^1\frac{t}{(tb+(1-t)x)^2}dt=\frac{1}{b-x}\left\{\frac{\ln b-\ln x}{b-x}-\frac{1}{b}\right\},$$

Hence, If we use (2.8)-(2.9) for $q=1$ and (2.14) in (2.13), we obtain the desired result. This completes the proof. □

Corollary 2.3. *In Theorem 2.4, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\begin{aligned} &\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\ \leq\ & \frac{ab}{b-a} M \left\{ \lambda_5^{1-\frac{1}{q}}(a,x)\,(x-a)^2 \left(\lambda_1(a,x,s,1,1) + \lambda_2(a,x,s,1,1)\right)^{\frac{1}{q}} \right. \\ &\left. + \lambda_5^{1-\frac{1}{q}}(b,x)\,(b-x)^2 \left(\lambda_3(b,x,s,1,1) + \lambda_4(b,x,s,1,1)\right)^{\frac{1}{q}} \right\} \end{aligned}$$

holds.

Theorem 2.5. *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a,b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically* s*-convex on* $[a,b]$ *for* $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \tag{2.15}$$

$$\begin{aligned} \leq\ & \frac{ab}{b-a} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left\{ (x-a)^2 \left(\lambda_1(a,x,s,q,0)\,|f'(x)|^q + \lambda_2(a,x,s,q,0)\,|f'(a)|^q\right)^{\frac{1}{q}} \right. \\ &\left. + (b-x)^2 \left(\lambda_3(b,x,s,q,0)\,|f'(x)|^q + \lambda_4(b,x,s,q,0)\,|f'(b)|^q\right)^{\frac{1}{q}} \right\}. \end{aligned}$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have

$$\begin{aligned} &\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \\ \leq\ & \frac{ab\,(x-a)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 \frac{1}{(ta+(1-t)x)^{2q}} \left[t^s\,|f'(x)|^q + (1-t)^s\,|f'(a)|^q \right] dt \right)^{\frac{1}{q}} \\ &+ \frac{ab\,(b-x)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 \frac{1}{(tb+(1-t)x)^{2q}} \left[t^s\,|f'(x)|^q + (1-t)^s\,|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,0)\left|f'(x)\right|^q+\lambda_2(a,x,s,q,0)\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(\lambda_3(b,x,s,q,0)\left|f'(x)\right|^q+\lambda_4(b,x,s,q,0)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.4. *In Theorem 2.5, additionally, if* $|f'(x)| \leq M$, $x \in [a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab}{b-a}M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\left(\lambda_1(a,x,s,q,0)+\lambda_2(a,x,s,q,0)\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(\lambda_3(b,x,s,q,0)+\lambda_4(b,x,s,q,0)\right)^{\frac{1}{q}}\right\}$$

holds.

Theorem 2.6. *Let* $f: I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I°, $a,b \in I^\circ$ *with* $a < b$, *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically s-convex on* $[a,b]$ *for* $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$

$$\leq \frac{ab}{b-a}\left\{\left(\lambda_1(a,x,0,p,p)\right)^{\frac{1}{p}}(x-a)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\lambda_3(b,x,0,p,p)\right)^{\frac{1}{p}}(b-x)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

where λ_1, λ_2, λ_3 *and* λ_4 *are defined as in Theorem 2.2.*

Proof. From Lemma 2.1, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab(x-a)^2}{b-a}\left(\int_0^1 \frac{t^p}{(ta+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 \frac{t^p}{(tb+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left[t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\left\{(\lambda_1(a,x,0,p,p))^{\frac{1}{p}}(x-a)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+(\lambda_3(b,x,0,p,p))^{\frac{1}{p}}(b-x)^2\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.5. *In Theorem 2.6, additionally, if* $|f'(x)|\leq M$, $x\in[a,b]$, *then inequality*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right|$$

$$\leq \frac{ab}{b-a}M\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\left\{(\lambda_1(a,x,0,p,p))^{\frac{1}{p}}(x-a)^2\right.$$

$$\left.+(\lambda_3(b,x,0,p,p))^{\frac{1}{p}}(b-x)^2\right\}$$

holds.

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