BOUNDS FOR INITIAL MACLAURIN COEFFICIENTS OF A SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SUBORDINATION

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ABSTRACT. In this paper, we investigate the bounds of the coefficients for new subclasses of analytic and bi-univalent functions in the open unit disc defined by subordination. The coefficients bounds presented in this paper would generalize and improve those in related works of several earlier authors

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

The Koebe one-quarter theorem [8] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Recently some researchers have been devoted to study the bi-univalent functions class Σ and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [15]. In fact that this widely-cited work by Srivastava et al. [15] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al.[6, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], and others [10, 11, 14, 27]. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{2, 3\})$ for each $f \in \Sigma$, is still an open problem. In fact there is no direct way to get bound for coefficients greater than three. In special cases there are some papers in which the Faber polynomial methods were used for determining upper bounds for higher-order coefficients (for example see [20]).

More recently El-Ashwah [10] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses.

Definition 1. [10] *For $0 < \alpha \leq 1$; $\lambda \geq 1$, a function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \ |arg((1 - \lambda)\frac{(f * h)(z)}{z} + \lambda(f * h)'(z))| < \frac{\alpha\pi}{2}$$
$$(z \in \mathcal{U}),$$
(3)

and

$$|arg((1 - \lambda)\frac{(f * h)^{-1}(w)}{w} + \lambda((f * h)^{-1})'(w))| < \frac{\alpha\pi}{2}$$
$$(w \in \mathcal{U}),$$
(4)

where the functions $h(z)$ and $(f * h)^{-1}(w)$ are defined by:

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n \quad (h_n > 0),$$
(5)

and

$$(f * h)^{-1}(w) = w - a_2 h_2 w^2 + (2a_2^2 h_2^2 - a_3 h_3)w^3$$
$$- (5a_2^3 h_2^3 - 5a_2 h_2 a_3 h_3 + a_4 h_4)w^4 + \cdots .$$
(6)

theorem 1. [10] *Let $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{h_2\sqrt{(\lambda + 1)^2 + \alpha(1 + 2\lambda - \lambda^2)}}, \ |a_3| \leq \frac{1}{h_3}\left(\frac{4\alpha^2}{(\lambda + 1)^2} + \frac{2\alpha}{(2\lambda + 1)}\right).$$
(7)

Definition 2. [10] *For $0 \le \beta < 1$; $\lambda \ge 1$, a function $f(z)$ given by (1) is said to be in the class $\mathcal{B}_\Sigma(h, \beta, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \ Re\left((1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z)\right) > \beta \ (\ z \in \mathcal{U}), \qquad (8)$$

and

$$Re\left((1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w)\right) > \beta \ (\ w \in \mathcal{U}), \qquad (9)$$

where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by (5) and (6) respectively.

theorem 2. [10] *Let $f(z)$ given by (1) be in the class $\mathcal{B}_\Sigma(h, \beta, \lambda)$, $0 \le \beta < 1$ and $\lambda \ge 1$. Then*

$$|a_2| \le \frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(2\lambda+1)}}, \ |a_3| \le \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right). \qquad (10)$$

The object of the present paper is to introduce a new subclasses of the function class Σ and obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass. Our results would generalize and improve the Theorem 1 and Theorem 2.

2. Subclass $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$

An analytic function f is said to be subordinate to another analytic function g, written as

$$f(z) \prec g(z) \quad (z \in \mathcal{U}),$$

if there exists a Schwarz function w, which is analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathcal{U})$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathcal{U}$, then we have the following equivalence:
$f(z) \prec g(z) \iff f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let φ be an analytic function with positive real part in $\mathcal{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(\mathcal{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \qquad (11)$$

We now introduce the following class of bi-univalent functions.

Definition 3. Let $0 \le \gamma \le 1$ and $\tau \in \mathbb{C} - \{0\}$. A function $f \in \Sigma$ given by (1), is said to be in the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ if each of the following subordinate conditions holds true:

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] \prec \varphi(z) \quad (z \in \mathcal{U}), \qquad (12)$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) - 1\right] \prec \varphi(w) \quad (w \in \mathcal{U}), \quad (13)$$

where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by (5) and (6) respectively.

Remark 1. *There are many choices of φ which would provide interesting subclasses of class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$.*

- *If we take $\tau = 1$ and $\varphi = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \le 1)$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to Definition 1.*
- *If we take $\tau = 1$, $\varphi = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \le 1)$ and $h(z) = \frac{z}{1-z}$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [11].*
- *If we take $\tau = \lambda = 1$, $\varphi = \left(\frac{1+z}{1-z}\right)^\alpha$ $(0 < \alpha \le 1)$ and $h(z) = \frac{z}{1-z}$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to the class $\mathcal{H}_\Sigma^\alpha$ introduced and studied by Srivastava et al. [15].*
- *If we take $\tau = 1$ and $\varphi = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to Definition 2.*
- *If we take $\tau = 1$ and $\varphi = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to the class $\mathcal{B}_\Sigma(\beta, \lambda)$ introduced and studied by Frasin and Aouf [11].*
- *If we take $\tau = \lambda = 1$ and $\varphi = \frac{1+(1-2\beta)z}{1-z}$ $(0 \le \beta < 1)$, then the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ reduce to the class $\mathcal{H}_\Sigma(\beta)$ introduced and studied by Srivastava et al. [15].*

3. Coefficient bounds for the class $\mathcal{B}_\Sigma(\varphi, \tau, \lambda)$

In order to derive our main results, we have to recall here the following lemma.

Lemma 1. [13] *Let $p \in \mathcal{P}$ the family of all functions p analytic in $\mathcal{U}$ for which $\Re p(z) > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ $(z \in \mathcal{U})$ for $z \in \mathcal{U}$. Then $|p_n| \le 2$, for each n.*

theorem 3. *Let $f(z) \in \mathcal{B}_\Sigma(\varphi, \tau, \lambda)$ be of the form (1). Then*

$$|a_2| \le \min\left[\frac{|\tau|B_1}{h_2(1+\lambda)}, \frac{1}{h_2}\sqrt{\frac{|\tau|(B_1 + |B_1 - B_2|)}{(1+2\lambda)}},\right.$$

$$\left.\frac{|\tau|B_1\sqrt{B_1}}{h_2\sqrt{|(B_1 - B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda)|}}\right]. \quad (14)$$

$$|a_3| \le \min\left[\frac{|\tau|(B_1 + |B_1 - B_2|)}{h_3(1+2\lambda)}, \frac{|\tau|^2 B_1^2}{h_3(1+\lambda)^2} + \frac{|\tau|B_1}{h_3(1+2\lambda)}\right]. \quad (15)$$

where the coefficients B_1 and B_2 are given as in (11).

Proof. For $f \in \mathcal{B}_\Sigma(\varphi, \tau, \lambda)$, there are analytic functions $u, v : \mathcal{U} \longrightarrow \mathcal{U}$, with $u(0) = v(0) = 0$, satisfying the following conditions:

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] = \varphi(u(z)) \quad (z \in \mathcal{U}) \qquad (16)$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) - 1\right] = \varphi(v(w)) \quad (w \in \mathcal{U}). \qquad (17)$$

Now we define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (18)$$

and

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \qquad (19)$$

Then p_1 and p_2 are analytic in $\mathcal{U}$ with positive real parts and $p_1(0) = 1 = p_2(0)$. Therefore, in view of the Lemma 1, we have

$$|b_n| \leq 2 \quad and \quad |c_n| \leq 2 \quad (n \in \mathbb{N}). \qquad (20)$$

Solving (18) and (19) for $u(z)$ and $v(z)$, we get

$$u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right] \quad (z \in \mathbb{U}) \qquad (21)$$

and

$$v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right)z^2 + \cdots\right] \quad (z \in \mathbb{U}). \qquad (22)$$

Clearly, upon substituting from (21) and (22) into (16) and (17), respectively, if we make use of (11), we find that

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) - 1\right] = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right)$$

$$= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right]z^2 + \cdots \qquad (23)$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) - 1\right] = \varphi\left(\frac{p_2(w) - 1}{p_2(w) + 1}\right)$$

$$= 1 + \frac{1}{2}B_1 b_1 w + \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right]w^2 + \cdots . \qquad (24)$$

Clearly it follow from (23) and (24), we have

$$\frac{(1+\lambda)a_2 h_2}{\tau} = \frac{1}{2}B_1 c_1, \tag{25}$$

$$\frac{(1+2\lambda)a_3 h_3}{\tau} = \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right], \tag{26}$$

$$\frac{-(1+\lambda)a_2 h_2}{\tau} = \frac{1}{2}B_1 b_1 \tag{27}$$

and

$$\frac{(1+2\lambda)(2a_2^2 h_2^2 - a_3 h_3)}{\tau} = \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right]. \tag{28}$$

From (25) and (27), it follows that

$$c_1 = -b_1 \tag{29}$$

and

$$\frac{2(1+\lambda)^2 a_2^2 h_2^2}{\tau^2} = \frac{1}{4}B_1^2(c_1^2 + b_1^2). \tag{30}$$

By using lemma 1, we obtain

$$|a_2| \le \frac{|\tau|B_1}{h_2(1+\lambda)}. \tag{31}$$

By adding (26) and (28) we have

$$\frac{(1+2\lambda)2a_2^2 h_2^2}{\tau} = \left[\frac{1}{2}B_1(b_2 + c_2) - \frac{1}{4}B_1(b_1^2 + c_1^2) + \frac{1}{4}B_2(b_1^2 + c_1^2)\right]. \tag{32}$$

i.e.,

$$a_2^2 = \frac{\tau\left[2B_1(b_2 + c_2) - B_1(b_1^2 + c_1^2) + B_2(b_1^2 + c_1^2)\right]}{8(1+2\lambda)h_2^2}.$$

Since $B_1 > 0$, $h_2 > 0$, $0 \le \lambda \le 1$ and by using lemma 1, we obtain

$$|a_2|^2 \le \frac{|\tau|(B_1 + |B_1 - B_2|)}{(1+2\lambda)h_2^2},$$

$$|a_2| \le \frac{1}{h_2}\sqrt{\frac{|\tau|(B_1 + |B_1 - B_2|)}{(1+2\lambda)}}. \tag{33}$$

On the other by using (30) in (32) we obtain

$$\frac{2(1+2\lambda)a_2^2 h_2^2}{\tau} = \left[\frac{1}{2}B_1(b_2 + c_2) - \frac{2(B_1 - B_2)(1+\lambda)^2 a_2^2 h_2^2}{\tau^2 B_1^2}\right]. \tag{34}$$

Then

$$\frac{2a_2^2 h_2^2}{\tau^2 B_1^2}\left((B_1 - B_2)(1+\lambda)^2 + \tau B_1^2(1+2\lambda)\right) = \frac{1}{2}B_1(b_2 + c_2). \tag{35}$$

i.e.,

$$a_2^2 = \frac{\tau^2 B_1^3 (b_2 + c_2)}{4h_2^2((B_1 - B_2)(1 + \lambda)^2 + \tau B_1^2(1 + 2\lambda))}.$$

By applying lemma 1, we obtain

$$|a_2|^2 \leq \frac{4\tau^2 B_1^3}{4h_2^2|(B_1 - B_2)(1 + \lambda)^2 + \tau B_1^2(1 + 2\lambda)|},$$

$$|a_2| \leq \frac{|\tau| B_1 \sqrt{B_1}}{h_2 \sqrt{|(B_1 - B_2)(1 + \lambda)^2 + \tau B_1^2(1 + 2\lambda)|}}. \tag{36}$$

Now from (31), (34) and (36), we can find the bound for $|a_2|$.

Similarly, upon subtracting (28) from (26) and using (29) we get

$$\frac{2(1 + 2\lambda)a_3 h_3}{\tau} - \frac{2(1 + 2\lambda)a_2^2 h_2^2}{\tau} = \frac{1}{2} B_1 (c_2 - b_2). \tag{37}$$

Now if we use (30) in (37) we obtain

$$\frac{2(1 + 2\lambda)a_3 h_3}{\tau} = \frac{\tau B_1^2 (1 + 2\lambda)(c_1^2 + b_1^2)}{4(1 + \lambda)^2} + \frac{1}{2} B_1 (c_2 - b_2). \tag{38}$$

$$a_3 = \frac{\tau^2 B_1^2 (1 + 2\lambda)(c_1^2 + b_1^2)}{8(1 + \lambda)^2(1 + 2\lambda)h_3} + \frac{\tau B_1 (c_2 - b_2)}{4(1 + 2\lambda)h_3},$$

By applying lemma 1, we have

$$|a_3| \leq \frac{|\tau^2| B_1^2}{h_3(1 + \lambda)^2} + \frac{|\tau| B_1}{h_3(1 + 2\lambda)}. \tag{39}$$

If we use (32) in (37), we get

$$\frac{2(1 + 2\lambda)a_3 h_3}{\tau} = \left[\frac{1}{2} B_1 (c_2 + c_2) - \frac{1}{4} B_1 (b_1^2 + c_1^2) + \frac{1}{4} B_2 (b_1^2 + c_1^2) \right], \tag{40}$$

$$a_3 = \frac{\tau \left(2B_1 (c_2 + c_2) - B_1 (b_1^2 + c_1^2) + B_2 (b_1^2 + c_1^2) \right)}{8(1 + 2\lambda)h_3}.$$

Since $B_1 > 0$, $h_3 > 0$, $0 \leq \lambda \leq 1$ and by using lemma 1, we get

$$|a_3| \leq \frac{|\tau|(B_1 + |B_1 - B_2|)}{h_3(1 + 2\lambda)}. \tag{41}$$

Now from (39) and (41), we can find the bound for $|a_3|$.

$\square$

4. Conclusions

If we take $\tau = 1$ and

$$\varphi = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \ (0 < \alpha \leq 1),$$

in Theorem 3, we conclude the following result which is an improvement of Theorem 1.

Corollary 1. *Let $f(z) \in \mathcal{B}_{\Sigma}(h, \alpha, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2\alpha}{h_2(1+\lambda)}, \frac{1}{h_2}\sqrt{\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}}, \frac{2\alpha}{h_2\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda - \lambda^2)}}\right]. \tag{42}$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{h_3(1+2\lambda)}, \frac{4\alpha^2}{h_3(1+\lambda)^2} + \frac{2\alpha}{h_3(1+2\lambda)}\right]. \tag{43}$$

If we take $h(z) = \frac{z}{1-z}$ in Corollary 1 we obtain the following result which is an improvement of theorem obtained by Frasin and Aouf [11].

Corollary 2. *Let $f(z) \in \mathcal{B}_{\Sigma}(\alpha, \lambda)$ be of the form (1). Then*

$$|a_2| \leq \min\left[\frac{2\alpha}{(1+\lambda)}, \sqrt{\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}}, \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda - \lambda^2)}}\right]. \tag{44}$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{(1+2\lambda)}, \frac{4\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{(1+2\lambda)}\right]. \tag{45}$$

If we take $\lambda = 1$ in Corollary 2, then we have the following result which is an improvement of result obtained by Srivastava et al. [15].

Corollary 3. *Let $f(z) \in \mathcal{H}_{\Sigma}^{\alpha}$ be of the form (1). Then*

$$|a_2| \leq \min\left[\sqrt{\frac{4\alpha - 2\alpha^2}{3}}, \frac{2\alpha}{\sqrt{4 + 2\alpha}}\right]. \tag{46}$$

$$|a_3| \leq \min\left[\frac{4\alpha - 2\alpha^2}{3}, \alpha^2 + \frac{2\alpha}{3}\right]. \tag{47}$$

If we take $\tau = 1$ and

$$\varphi = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \ (0 \leq \beta < 1)$$

in Theorem 3, we conclude the following result which is an improvement of Theorem 2.

Corollary 4. *Let $f(z) \in \mathcal{B}_\Sigma(h, \beta, \lambda)$ be of the form* (1). *Then*

$$|a_2| \leq \min\left[\frac{2(1-\beta)}{h_2(1+\lambda)}, \frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(1+2\lambda)}}\right]. \tag{48}$$

$$|a_3| \leq \frac{2(1-\beta)}{h_3(1+2\lambda)}. \tag{49}$$

If we take $h(z) = \frac{z}{1-z}$ in Corollary 4 we obtain the following result which is an improvement of theorem obtained by Frasin and Aouf [11].

Corollary 5. *Let $f(z) \in \mathcal{B}_\Sigma(\beta, \lambda)$ be of the form* (1). *Then*

$$|a_2| \leq \min\left[\frac{2(1-\beta)}{(1+\lambda)}, \sqrt{\frac{2(1-\beta)}{(1+2\lambda)}}\right]. \tag{50}$$

$$|a_3| \leq \frac{2(1-\beta)}{(1+2\lambda)}. \tag{51}$$

If we take $\lambda = 1$ in Corollary 5 we obtain the following result which is an improvement of theorem obtained and studied by Srivastava et al. [15].

Corollary 6. *Let $f(z) \in \mathcal{H}_\Sigma(\beta)$ be of the*

$$|a_2| \leq \min\left[(1-\beta), \sqrt{\frac{2(1-\beta)}{3}}\right]. \tag{52}$$

$$|a_3| \leq \frac{2(1-\beta)}{3}. \tag{53}$$

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