Konuralp Journal of Mathematics
Volume 3 No. 1 pp. 75–83 (2015) ©KJM

ON THE DETERMINATION OF A DEVELOPABLE SPHERICAL ORTHOTOMIC TIMELIKE RULED SURFACE

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Abstract. In this paper, a method for determination of developable spherical orthotomic ruled surfaces generated by a spacelike curve on dual hyperbolic unit sphere is given by using dual vector calculus in $\mathbb{R}^3_1$. We show that dual vectorial expression of a developable spherical orthotomic timelike ruled surface can be obtained from coordinates and the first derivatives of the base curve. The paper concludes with an example related to this method.

1. Introduction

In geometry, a surface is a called ruled surface if it is swept out by a moving line. The theory of ruled surfaces is a classical subject in differential geometry. Ruled surface, espicially developable ruled surface have been widely investigated in mathematics, engineering and architecture [13]. In today's manufacturing industries, the developable ruled surface desing and its application are extensively used in CAD, CAM and CNC. Also it has been popular in architecture such as saddle roofs, cooling towers, gridshell etc.

Dual numbers were first introduced by W.K. Clifford (1849–79) as a tool for his geometrical investigations. After him E. Study has done fundamental research with dual numbers and dual vectors on the geometry of lines and kinematics [2] which is so-called E. Study mapping. This mapping constitutes a one to one correpondence between the dual points of dual unit sphere S^2 and the directed lines of space of lines $\mathbb{R}^3$ [15]. If we consider the Minkowski 3-space $\mathbb{R}^3_1$ instead of $\mathbb{R}^3$ the E. Study mapping can be stated as follows. The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres $\mathbb{H}^2_0$ and $\mathbb{S}^2_1$ at the dual Lorentzian space $\mathbb{D}^3_1$ are in a one to one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines $\mathbb{R}^3_1$, respectively. Then a differentiable curve on $\mathbb{H}^2_0$ corresponds to a timelike ruled surface in $\mathbb{R}^3_1$. Similarly the timelike (resp.

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 53A35,53B30,53C50,53A04.
Key words and phrases. Ruled Surface, Developable Surface, Dual Hyperbolic Unit Sphere, Minkowski Space.
The author is supported by ...

spacelike) curve on $\mathbb{S}_1^2$ corresponds to any spacelike (resp. timelike) ruled surface at $\mathbb{R}^3$[19].

Let α be a regular curve and $\overrightarrow{T}$ be its tangent, and let u be a source. An orthotomic of α with respect to the source (u) is defined as a locus of reflection of u about tangents $\overrightarrow{T}$ [7]. Bruce and Giblin applied the unfolding theory to the study of the evolutes and orthotomics of plane and space curves [3], [4] and [5]. Georgiou, Hasanis and Koutroufiotis investigated the orthotomics in the Euclidean (n + 1)-space [6]. Alamo and Criado studied the antiorthotomics in the Euclidean (n+1)-space [1]. Xiong defined the spherical orthotomic and the spherical antiorthotomic [18]. Yıldız and Hacısalihoğlu examined the Study of spherical orthotomic of a circle [9]. Also, orthotomic concept can be apply to surface. For a given surface S and a fixed point (source) P, orthotomic surface of S relative to P is defined as a locus of reflection of P about all tangent planes of S [8].

Köse introduced a new method for determination of developable ruled surfaces [11]. Ekici and Özüsağlam [12] study this method in $\mathbb{R}_1^3$. And also, Yıldız et al. applied this method in $\mathbb{R}^3$ by using orthotomic concept [10]. For all these the following question is interesting: Can we obtain a remarkable method for determination of developable spherical orthotomic timelike ruled surface in $\mathbb{R}_1^3$. The answer is positive. In this article, we construct a method for determination of developable orthotomic timelike ruled surfaces by using dual vector calculus.

2. Basic concept

A dual number has the form $\widetilde{a} = a + \varepsilon a^*$ where a and a^* are real numbers and $\varepsilon = (0,1)$ stands for the dual unit which $\varepsilon^2 = 0$.

The set of all dual numbers is denoted by $\mathbb{D}$ which is a commutative ring over $\mathbb{R}$.

$\mathbb{D}^3$ is the set of all triples of dual numbers. $\mathbb{D}^3$ can be written as

$$\mathbb{D}^3 = \left\{ \overrightarrow{\widetilde{a}} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3) \,|\widetilde{a}_i \in \mathbb{D}, 1 \leq i \leq 3 \right\}.$$

A dual vector has the form $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$, where $\overrightarrow{a}$ and $\overrightarrow{a}^*$ are real vectors in $\mathbb{R}^3$. The set $\mathbb{D}^3$ becomes a modul under addition and scalar multiplication on the set $\mathbb{D}$ [17].

For any dual Lorentzian vector $\overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^*$ and $\overrightarrow{\widetilde{b}} = \overrightarrow{b} + \varepsilon \overrightarrow{b}^*$, inner product is defined by

$$\left\langle \overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \varepsilon \left(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle \right)$$

where $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle$ is the Lorentzian inner product with signature $(+,+,-)$ of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in the Minkowski 3-Space $\mathbb{R}_1^3$.

A dual vector $\overrightarrow{\widetilde{a}}$ is said to be time-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle < 0$, space-like if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle > 0$ and light-like (or null) if $\langle \overrightarrow{a}, \overrightarrow{a} \rangle = 0$ and $\overrightarrow{a} \neq 0$. The set of all dual Lorentzian vector is called dual Lorentzian space and is denoted by

$$\mathbb{D}_1^3 = \left\{ \overrightarrow{\widetilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a}^* \,|\, \overrightarrow{a}, \overrightarrow{a}^* \in \mathbb{R}_1^3 \right\}$$

For any vector $\vec{\tilde{a}} = \vec{a} + \varepsilon \vec{a}^*$ and $\vec{\tilde{b}} = \vec{b} + \varepsilon \vec{b}^*$, vector product is defined by

$$\vec{\tilde{a}} \wedge \vec{\tilde{b}} = \vec{a} \wedge \vec{b} + \varepsilon \left(\vec{a} \wedge \vec{b}^* + \vec{a}^* \wedge \vec{b} \right),$$

where $\vec{a} \wedge \vec{b}$ is the Lorentzian vector product.

The norm $\left\| \vec{\tilde{a}} \right\|$ of $\vec{\tilde{a}} = \vec{a} + \varepsilon \vec{a}^*$ is defined as

$$\left\| \vec{\tilde{a}} \right\| = \| \vec{a} \| + \varepsilon \frac{\langle \vec{a}, \vec{a}^* \rangle}{\| \vec{a} \|}, \quad \vec{a} \neq 0.$$

The dual vector $\vec{\tilde{a}}$ with norm 1 is called a dual unit vector.

The dual Lorentzian unit sphere and the dual hyperbolic unit sphere are

$$\mathbb{S}_1^2 = \left\{ \vec{\tilde{x}} = x + \varepsilon x^* \in \mathbb{D}_1^3 \mid < \tilde{x}, \tilde{x} >= 1; x, x^* \in \mathbb{R}_1^3 \right\}$$

and

$$\mathbb{H}_0^2 = \left\{ \vec{\tilde{x}} = x + \varepsilon x^* \in \mathbb{D}_1^3 \mid < \tilde{x}, \tilde{x} >= -1; x, x^* \in \mathbb{R}_1^3 \right\}$$

respectively. The dual spacelike unit vectors of dual Lorentzian sphere $\mathbb{S}_1^2$ represent oriented spacelike lines is $\mathbb{R}_1^3$. The dual timelike unit vectors of dual hyperbolic unit sphere $\mathbb{H}_0^2$ represent oriented timelike lines in $\mathbb{R}_1^3$.

For $\mathbb{R}_1^3$, the Study Mapping is defined as follows: "There are one-to-one correspondence between the directed timelike (resp. spacelike) lines in three dimensional Minkowski space and the dual point on the surface of a dual hyperbolic (resp. dual Lorentzian) unit sphere (resp.) in three dimensional dual Lorentzian space " [16].

Let $\mathbb{S}_1^2$ (resp. $\mathbb{H}_0^2$), O and $\left\{ O; \vec{\tilde{e}}_1, \vec{\tilde{e}}_2, \vec{\tilde{e}}_3 \right\}$ denote the dual hyperbolic (resp. Lorentzian) unit sphere , the center of $\mathbb{S}_1^2$ (resp. $\mathbb{H}_0^2$) and dual orthonormal system at O, respectively, where

$$\vec{\tilde{e}}_i = \vec{e}_i + \varepsilon \vec{e}_i^*; \quad 1 \leq i \leq 3.$$

Let S_3 be the group of all the permutations of the set $\{1, 2, 3\}$, then it can be written as

$$\left. \begin{array}{c} \vec{\tilde{e}}_{\sigma(1)} = sgn(\sigma) \vec{\tilde{e}}_{\sigma(2)} \wedge \vec{\tilde{e}}_{\sigma(3)}, sgn(\sigma) = \pm 1, \\ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix} \end{array} \right\}.$$

In the case that the orthonormal system

$$\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

is the system of $\mathbb{R}_1^3$.

By using the Study Mapping, we can conclude that there exists a one to one correspondence between the dual orthonormal system and the real orthonormal system.

Now, define of spherical normal, spherical tangent and spherical orthotomic of a spherical curve α. Let $\left\{ \vec{T}, \vec{N}, \vec{B} \right\}$ be the Frenet frame of α. The spherical normal of α is the great circle, passing through $\alpha(s)$, that is normal to α at $\alpha(s)$ and given by

$$\begin{cases} \langle \overrightarrow{x}, \overrightarrow{x} \rangle = 1, \\ \left\langle \overrightarrow{x}, \overrightarrow{T} \right\rangle = 0, \end{cases}$$

where x is an arbitrary point of the spherical normal. The spherical tangent of α is the great circle which tangent to α at $\alpha(s)$ and given by

$$\begin{cases} \langle \overrightarrow{y}, \overrightarrow{y} \rangle = 1, \\ \left\langle \overrightarrow{y}, (\overrightarrow{\alpha} \wedge \overrightarrow{T}) \right\rangle = 0, \end{cases} \tag{2.1}$$

where y is an arbitrary point of the spherical tangent.

Let u be a source on a sphere. Then, Xiong defined the spherical orthotomic of α relative to u as to be the set of reflections of u about the planes, lying on the above great circles (2.1) for all $s \in I$ and given by

$$\overrightarrow{\widetilde{u}} = 2 \left\langle (\overrightarrow{\alpha} - \overrightarrow{u}), \overrightarrow{v} \right\rangle \overrightarrow{v} + \overrightarrow{u} \tag{2.2}$$

where $\overrightarrow{v} = \frac{\overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha}}{\left\| \overrightarrow{B} - \left\langle \overrightarrow{B}, \overrightarrow{\alpha} \right\rangle \overrightarrow{\alpha} \right\|}$ [18].

3. The dual vector formulation

Let L be a line and x denotes the direction and p be the position vector of any point on L. Dual vector representation allows us the Plucker vectors x and $p \wedge x$. Thus, dual Lorentzian vector $\widetilde{x}(t)$ can be written as

$$\widetilde{x}(t) = x + \varepsilon(p \wedge x) = x + \varepsilon x^*,$$

where ε is the dual unit and $\varepsilon^2 = 0$.

By using the dual Lorentzian vector function $\widetilde{x}(t) = x(t) + \varepsilon(p(t) \wedge x(t)) = x(t) + \varepsilon x^*(t)$, a ruled surface can be given as

$$m(u, t) = p(t) + ux(t).$$

It is known that the dual unit Lorentzian vector $\widetilde{x}(t)$ is a differentiable curve on the dual hyperbolic unit sphere and also having unit magnitude [14].

$$\begin{aligned} \langle \widetilde{x}, \widetilde{x} \rangle &= \langle x + \varepsilon p \wedge x, x + \varepsilon p \wedge x \rangle \\ &= \langle x, x \rangle + \langle 2\varepsilon x, p \wedge x \rangle + \varepsilon^2 \langle p \wedge x, p \wedge x \rangle \\ &= \langle x, x \rangle \\ &= -1. \end{aligned}$$

The dual arc-length of the dual Lorentzian curve $\widetilde{x}(t)$ is defined as

$$\hat{s}(t) = \int_0^t \left\| \frac{d\hat{x}}{dt} \right\| dt. \tag{3.1}$$

The integrant of (3.1) is the dual speed, $\widetilde{\delta}$ of $\widetilde{x}(t)$ and is

$$\widetilde{\delta} = \left\| \frac{d\widehat{x}}{dt} \right\| = \left\| \frac{dx}{dt} \right\| \left(1 + \varepsilon \frac{\left\langle \frac{dx}{dt}, \frac{dp}{dt} \wedge x \right\rangle}{\left\| \frac{dx}{dt} \right\|^2} \right) = \left\| \frac{dx}{dt} \right\| (1 + \varepsilon \Delta).$$

The curvature function

$$\Delta = \frac{\left\langle \frac{dx}{dt}, \frac{dp}{dt} \wedge x \right\rangle}{\left\| \frac{dx}{dt} \right\|^2} = \frac{\left\langle \frac{dx}{dt}, \frac{dx^*}{dt} \right\rangle}{\left\| \frac{dx}{dt} \right\|^2}$$

is the well-known distribution parameter (drall) of the ruled surface.

4. The Determination of Timelike Developable Spherical Orthotomic Ruled Surface

Let $\widetilde{x}(t)$ be a point on hyperbolic unit sphere, centered at the origin. The dual coordinates of $\widetilde{x}(t) = x_i + \varepsilon x_i^*$ can be expressed as

$$\begin{aligned} \widetilde{x_1} &= x_1 + \varepsilon x_1^* = \sinh \widetilde{\varphi} \cos \widetilde{\psi}, \\ \widetilde{x_2} &= x_2 + \varepsilon x_2^* = \sinh \widetilde{\varphi} \sin \widetilde{\psi}, \\ \widetilde{x_3} &= x_3 + \varepsilon x_3^* = \cosh \widetilde{\varphi}. \end{aligned} \tag{4.1}$$

where $\widetilde{\varphi} = \varphi + \varepsilon\varphi^*$and $\widetilde{\psi} = \psi + \varepsilon\psi^*$ are dual hyperbolic angle and dual angle respectively. Since $\varepsilon^2 = \varepsilon^3 = ... = 0$ according to the Taylor series expansion from (4.1), we obtain the real parts of $\widetilde{x}(t)$ as

$$\begin{aligned} x_1 &= \sinh \varphi \cos \psi, \\ x_2 &= \sinh \varphi \sin \psi, \\ x_3 &= \cosh \varphi, \end{aligned}$$

and the dual parts of $\widetilde{x}(t)$ as

$$\begin{aligned} x_1^* &= \varphi^* \cosh \varphi \cos \psi - \psi^* \sinh \varphi \sin \psi, \\ x_2^* &= \varphi^* \cosh \varphi \sin \psi + \psi^* \sinh \varphi \cos \psi, \\ x_3^* &= \varphi^* \sinh \varphi. \end{aligned}$$

Hence, the dual Lorentzian curve $\widetilde{x}(t) = x(t) + \varepsilon x^*(t)$ may be represented by

$$\begin{aligned} \widetilde{x}(t) &= (\sinh \varphi(t) \cos \psi(t), \sinh \varphi(t) \sin \psi(t), \cosh \varphi(t)) \\ &\quad + \varepsilon \begin{pmatrix} \varphi^*(t) \cosh \varphi(t) \cos \psi(t) - \psi^*(t) \sinh \varphi(t) \sin \psi(t), \\ \varphi^*(t) \cosh \varphi(t) \sin \psi(t) + \psi^*(t) \sinh \varphi(t) \cos \psi(t), \\ \varphi^*(t) \sinh \varphi(t) \end{pmatrix}. \end{aligned}$$

Let $\widetilde{\sigma}(t) = \sigma(t) + \varepsilon\sigma^*(t)$ be spherical orthotomic of the great circle, which lies on the $\overrightarrow{e}_2 \overrightarrow{e}_3$ plane, relative to the dual curve $\widetilde{x}(t)$. By (2.2), we get $\widetilde{\sigma}(t) = (-\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ where $\widetilde{x}_i$'s are the coordinates of $\widetilde{x}(t)$ for $i = 1, 2, 3$. By considering the spherical orthotomic dual curve, we have;

$$\begin{aligned} \widetilde{\sigma}(t) &= (-\sinh \varphi(t) \cos \psi(t), \sinh \varphi(t) \sin \psi(t), \cosh \varphi(t)) \\ &\quad + \varepsilon \begin{pmatrix} -\varphi^*(t) \cosh \varphi(t) \cos \psi(t) + \psi^*(t) \sinh \varphi(t) \sin \psi(t), \\ \varphi^*(t) \cosh \varphi(t) \sin \psi(t) + \psi^*(t) \sinh \varphi(t) \cos \psi(t), \\ \varphi^*(t) \sinh \varphi(t) \end{pmatrix} \end{aligned} \tag{4.2}$$

on the hyperbolic unit sphere corresponding to a timelike developable spherical orthotomic ruled surface $m(t, u) = p(t) + u\sigma(t)$. Because of two timelike vectors are never ortogonal, then a base curve, $p(t)$, must be a spacelike.

Since $\sigma^* = p \wedge \sigma$, we have the following system of linear equations in variables p_1, p_2, p_3;

$$\begin{aligned}
-\varphi^* \cosh\varphi \cos\psi + \psi^* \sinh\varphi \sin\psi &= p_2 \cosh\varphi - p_3 \sinh\varphi \sin\psi \\
\varphi^* \cosh\varphi \sin\psi + \psi^* \sinh\varphi \cos\psi &= -p_1 \cosh\varphi - p_3 \sinh\varphi \cos\psi \\
\varphi^* \sinh\varphi &= -p_1 \sinh\varphi \sin\psi - p_2 \sinh\varphi \cos\psi.
\end{aligned}$$

where p_i's are the coordinates of $p(t)$ for $i = 1, 2, 3$.

The matrix of coefficients of unknowns p_1, p_2 and p_3 is

$$\begin{bmatrix} 0 & \cosh\varphi & -\sinh\varphi \sin\psi \\ -\cosh\varphi & 0 & -\sinh\varphi\cos\psi \\ -\sinh\varphi\sin\psi & -\sinh\varphi\cos\psi & 0 \end{bmatrix}$$

and therefore its rank is 2 .

$$\begin{aligned}
p_1 &= -(p_3 + \psi^*) \cos\psi \tanh\varphi - \varphi^* \sin\psi, \\
p_2 &= (p_3 + \psi^*) \sin\psi \tanh\varphi - \varphi^* \cos\psi, \qquad (4.3) \\
p_3 &= p_3.
\end{aligned}$$

Since $p_3(t)$ can be chosen arbitrarily, then we may take $p_3(t) = -\psi^*(t)$. In this case, (4.3) reduces to

$$\begin{aligned}
p_1 &= -\varphi^* \sin\psi, \\
p_2 &= -\varphi^* \cos\psi, \qquad (4.4) \\
p_3 &= -\psi^*.
\end{aligned}$$

The distribution parameter of the timelike spherical orthotomic ruled surface given by (4.2) is obtained as follows

$$\begin{aligned}
\Delta &= \frac{< \frac{dx}{dt}, \frac{dx^*}{dt} >}{\| \frac{dx}{dt} \|^2} \\
&= \frac{\frac{d\psi}{dt}\frac{d\psi^*}{dt}\sinh^2\varphi(t) + \varphi^*\left(\frac{d\psi}{dt}\right)^2 \cosh\varphi(t)\sinh\varphi(t) + \frac{d\varphi^*}{dt}\frac{d\varphi}{dt}}{\left(\frac{d\psi}{dt}\right)^2 \sinh^2\varphi(t) + \left(\frac{d\varphi}{dt}\right)^2}. \qquad (4.5)
\end{aligned}$$

If this timelike spherical orthotomic ruled surface is a developable, then $\Delta = 0$ and by (4.5) becomes

$$\frac{d\varphi^*}{dt}\frac{d}{dt}(\coth\varphi(t)) - \varphi^*\left(\frac{d\psi}{dt}\right)^2 \coth\varphi(t) - \frac{d\psi}{dt}\frac{d\psi^*}{dt} = 0.$$

Setting

$$y(t) = \coth\varphi(t), A(t) = -\frac{\varphi^*\left(\frac{d\psi}{dt}\right)^2}{\frac{d\varphi^*}{dt}}, B(t) = -\frac{\frac{d\psi}{dt}\frac{d\psi^*}{dt}}{\frac{d\varphi^*}{dt}}$$

we are lead to a linear differential equation of first degree

$$\frac{dy}{dt} + A(t)y + B(t) = 0. \qquad (4.6)$$

Let $p(t)$ be a curve. Then we can find a developable spherical orthotomic ruled surface such that its base curve is the curve $p(t)$ and by (4.4), we have

$$\begin{aligned} \tan\psi &= \frac{p_1}{p_2}, \\ \varphi^* &= \sqrt{p_1^2 + p_2^2}, \\ \psi^* &= -p_3. \end{aligned}$$

Now only $\varphi(t)$ remains to be determined. The solution of the linear differential (4.6) gives $\coth\varphi(t)$. This solution includes an integral constant therefore we have infinitely many timelike developable spherical orthotomic ruled surface such that its base curve is $p(t)$.

Moreover, it is to be noted that $\varphi^*(t)$ has two values; by using the minus sign we obtain the reciprocal of the timelike developable spherical orthotomic ruled surface $\widetilde{x}(t)$ obtained by using the plus sign for a given integral constant.

Example 4.1. Consider $p(t) = \left(t, t, 2t^3 + 1\right)$. If $-\frac{1}{\sqrt[4]{18}} < t < \frac{1}{\sqrt[4]{18}}$, then the ruled surface is timelike. Then we have,

$$\tan\psi = 1, \varphi^* = \sqrt{2}t,\ \frac{d\psi^*}{dt} = -6t^2,\ \frac{d\psi}{dt} = 0 \text{ and } \frac{d\varphi^*}{dt} = \sqrt{2}.$$

Substituting these values into (4.6) we obtain the linear differential equation of first degree

$$\frac{dy}{dt} = 0.$$

The solution of this differential equation gives

$$\coth\varphi(t) = c.$$

Hence, the family of the developable timelike ruled surface is given by

$$m(t,u) = p(t) + u\sigma(t)$$

where $\sigma(t) = \left(\frac{p_2}{\varphi^*}\sinh\varphi, -\frac{p_1}{\varphi^*}\sinh\varphi, \cosh\varphi\right)$.

The graph of the developable timelike ruled surface given by this equation for $c = 2$ in domain

$$D : \begin{cases} -\frac{1}{\sqrt[4]{18}} < t < \frac{1}{\sqrt[4]{18}} \\ -1 < u < 1 \end{cases}$$

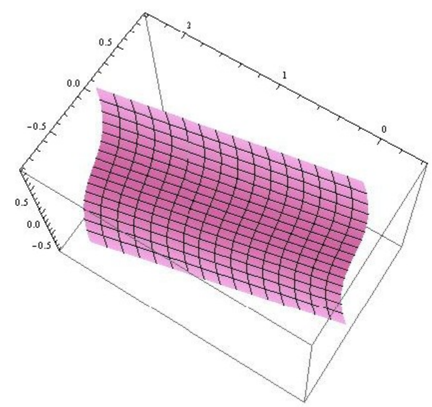

Figure 1. Spherical Orthotomic Timelike Ruled Surface

is given in Fig. 1.

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