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COSINE ENTROPY AND SIMILARITY MEASURES FOR FUZZY SETS

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Abstract. In the present paper, based on the cosine function, a new fuzzy entropy measure is defined. Some interesting properties of this measure are analyzed. Furthermore, a new fuzzy similarity measure has been proposed with its elegant properties. A relation between the proposed fuzzy entropy and fuzzy similarity measure has also been proved.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [19] in order to provide a scheme for handling non-statistical vague concepts. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines that include engineering, medical science, social science, artificial intelligence, signal processing, multi-agent systems, robotics, computer networks, and expert systems. Fuzzy entropy and similarity measures are as two important topics in fuzzy set theory, which have been investigated widely by many researchers from different points of view.

The first attempt to quantify fuzziness of a fuzzy set was made by Zadeh [20] in 1968, he proposed a probabilistic frame work and defined the entropy of a fuzzy set as weighted Shannon [10] entropy. In 1972, De Luca and Termini [3] first provided an axiomatic framework for the entropy of fuzzy sets based on the concept of Shannon's entropy. Kaufmann [5] introduced a fuzzy entropy measure based on a metric distance between a fuzzy set and its nearest crisp set. Yager [15] defined entropy of a fuzzy set in terms of a lack of distinction between the fuzzy set and its negation, a kind of 'norm'. Pal and Pal [7] proposed fuzzy entropy based on exponential function to measure the fuzziness called exponential fuzzy entropy. Bhandari and Pal [1] proposed generalized order-α fuzzy entropy to measure the fuzziness. In 2008, Parkash et al. [9] defined two new fuzzy entropy measures based on trigonometric functions and proved entropy maximization principle corresponding to these fuzzy entropies. Besides these, there exists quite a body of research work on applications of these theoretical studies [12, 16, 17 and 18].

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Similarity measures between two fuzzy sets, in particular, have found widespread applications in diverse fields like decision making, pattern recognition, machine learning, market prediction etc.

Talking of 'similarity measures', first, Wang [13] proposed a measure of similarity between two fuzzy sets. Salton and McGill [11] introduced a cosine similarity measure between fuzzy sets, which in essence is a kind of 'coefficient or a quotient' and applied it to information retrieval of words. Zwick et al. [21] used geometric distance and Huasdorff metrics for presenting similarity measures among fuzzy sets. Pappis and Karacapilidis [8] proposed three similarity measures for fuzzy sets based on union and intersection operations, the maximum difference, and the difference and sum of membership grades. Chen et al. [2] extended the work of Pappis and Karacapilidis [8], and defined some similarity measures on fuzzy sets based on the geometric model, the set theoretic approach, and matching function. Wang [14] proposed two similarity measures between fuzzy sets and between the elements of sets. Liu [6] as well as Fan and Xie [4] provided an axiomatic definition of similarity measure for fuzzy sets.

In the present paper two new measures called '*cosine fuzzy entropy*' and '*cosine fuzzy similarity*' are proposed. This paper is organized as follows:

In Section 2, some basic definitions related to probability theory and fuzzy sets are briefly discussed. In Section 3 cosine fuzzy entropy measure is proposed and there we verify its axiomatic requirement [3]. Some mathematical properties of the proposed entropy are also proved there. In Section 4 the cosine fuzzy similarity measure is introduced along with some of its properties. A relation between cosine fuzzy entropy and cosine fuzzy similarity is also established here.

2. Preliminaries

We start with probabilistic background. Let us denote the set of n-complete probability distributions by

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2. \tag{2.1}$$

For a probability distribution $P = (p_1, p_2, ..., p_n) \in \Gamma_n$, Shannon's entropy [14], is defined as

$$H(P) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i). \tag{2.2}$$

Definition 2.1. Fuzzy Set [19]: A fuzzy set A in a finite universe of discourse $X = \{x_1, x_2, ..., x_n\}$ is given by

$$A = \{ \langle x, \ \mu_A(x) \rangle \mid x \in X \}, \tag{2.3}$$

where $\mu_A(x) : X \to [0, 1]$ is the membership function of A. The number $\mu_A(x)$ describes the degree of membership of $x \in X$ in A.

Definition 2.2. A fuzzy set A^* is called a sharpened version of fuzzy set A if the following conditions are satisfied:

$$\begin{aligned} \mu_{A^*}(x_i) &\leq \mu_A(x_i) \qquad \text{if } \mu_A(x) \leq 0.5 \ \forall \ i, \\ \mu_{A^*}(x_i) &\geq \mu_A(x_i) \qquad \text{if } \mu_A(x) \geq 0.5 \ \forall \ i. \end{aligned}$$

Note: Throughout this paper, we shall denote the set of all fuzzy sets defined in X by$FS(X)$.

Definition 2.3. Set Operations on FSs [19]: Let $A, B \in FS(X)$ be given by

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$
$$B = \{\langle x, \mu_B(x)\rangle \mid x \in X\},$$

then usually set operations are defined as follows:

[(i)]

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x) \quad \forall x \in X$;
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
(3) $A^C = \{\langle x, 1 - \mu_A(x)\rangle \mid x \in X\}$;
(4) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x)\rangle \mid x \in X\}$;
(5) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x)\rangle \mid x \in X\}$;

where $\vee$, $\wedge$ stand for max. and min. operators, respectively.

In fuzzy set theory, a measure of fuzziness is the 'fuzzy entropy' which expresses the amount of aggregated ambiguity of a fuzzy set A. The first attempt to quantify the fuzziness was made in 1968 by Zadeh [20], who defined the entropy of a fuzzy set A with respect to (X, P) as

$$H(A, P) = -\sum_{i=1}^{n} \mu_A(x_i)\, p(x_i) \log_2 p(x_i). \tag{2.4}$$

De Luca and Termini [3] defined fuzzy entropy for a fuzzy set A corresponding (2.2) as

(2.5)
$$H_{DT}(A) = -\frac{1}{n}\sum_{i=1}^{n} \left[\mu_A(x_i)\ \log_2\ (\mu_A(x_i)) + (1 - \mu_A(x_i)) \log_2\ (1 - \mu_A(x_i))\right].$$

Based on exponential function, Pal and Pal [7] introduced exponential fuzzy entropy for fuzzy set A as

$$_eH(A) = \frac{1}{n(\sqrt{e} - 1)} \sum_{i=1}^{n} \left[\mu_A(x_i)\ e^{1-\mu_A(x_i)} + (1 - \mu_A(x_i))\ e^{\mu_A(x_i)} - 1\right]. \tag{2.6}$$

Later, Bhandari and Pal [1]made a survey on entropy measures on fuzzy sets and introduced the following parametric fuzzy entropy for fuzzy set A as

$$H_\alpha(A) = \frac{1}{n(1-\alpha)} \sum_{i=1}^{n} \log\left[\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha\right]. \tag{2.7}$$

Parkash et al. [9] defined two fuzzy entropy measures for fuzzy set A based on trigonometric functions (sine and cosine) given by

(2.8)
$$H_{OPR1}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\sin\frac{\pi\mu_A(x_i)}{2} + \sin\frac{\pi(1-\mu_A(x_i))}{2} - 1\right\} \times \frac{1}{(\sqrt{2}-1)}\right],$$

(2.9)
$$H_{OPR2}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left\{\cos\frac{\pi\mu_A(x_i)}{2} + \cos\frac{\pi(1-\mu_A(x_i))}{2} - 1\right\} \times \frac{1}{(\sqrt{2}-1)}\right].$$

Definition 2.4. Similarity Measure of FSs [6]: A real function $S : FS(X) \times FS(X) \to [0,1]$ is called the similarity measure of the fuzzy sets, if S satisfies the following properties:

[S1.]

(1) $0 \le S(A,B) \le 1 \;\forall\; A,B \in FS(X)$.
(2) $S(A,B) = S(B,A) \;\forall\; A,B \in FS(X)$.
(3) $S(A,B) = 1$ if and only if $A = B$, i.e. $\mu_A(x_i) = \mu_B(x_i)$ for all $i = 1, 2, ..., n$.
(4) For all $A,B,C \in FS(X)$, if $A \subseteq B \subseteq C$, then $S(A,C) \le S(A,B)$, $S(A,C) \le S(B,C)$.

In the next section, we introduce a new entropy measure on fuzzy sets called '*cosine fuzzy entropy*' and verify its axiomatic validity.

3. Cosine Fuzzy Entropy

We submit following formal definition of a new measure of 'fuzzy entropy':

Definition 3.1. Cosine Fuzzy Entropy: Let A be a fuzzy set defined on $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_i),\, i = 1, 2, ..., n$. We define the cosine fuzzy entropy for fuzzy set A, $H_{\cos}(A)$ as:

$$H_{\cos}(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\left(\frac{(2\mu_A(x_i)-1)}{2}\pi\right)\right]. \tag{3.1}$$

As a first step, in the next theorem, we establish properties that according to De Luca and Termini [3] justify the above proposed measure to be a valid 'fuzzy entropy'.

Theorem 3.1. *The $H_{\cos}(A)$ measure in (3.1) of the cosine fuzzy entropy satisfies the following propositions:*

*[P1.]**(Sharpness)**: $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x_i) = 0$ or $1 \;\forall\; x_i \in X$.* ***(Maximality)**: $H(A)$ is maximum if and only if A is a most fuzzy set, i.e. $\mu_A(x_i) = 0.5 \;\forall\; x_i \in X$.* ***(Resolution)**: $H(A^*) \le H(A)$, where A^* is a sharpened version of the set A.* ***(Symmetry)**: $H(A) = H\left(A^C\right)$, where A^C is the complement set of the fuzzy set A.*

(4) *Proof.* Let $\Delta_A = \left(\frac{(2\mu_A(x_i)-1)}{2}\pi\right)$ and then from $0 \le \mu_A(x_i) \le 1$, we note that

$$-\frac{\pi}{2} \le \Delta_A \le \frac{\pi}{2} \Rightarrow 0 \le \cos\frac{(2\mu_A(x_i)-1)}{2}\pi \le 1 \Rightarrow 0 \le H_{\cos}(A) \le 1.$$

P1. ***(Sharpness)*:** First, let A be a crisp set with membership values either 0 or 1 for all $x_i \in X$. Then from (3.1) we simply obtain

$$H_{\cos}(A) = 0. \tag{3.2}$$

This proves 'if' part of the statement. Next let us suppose that $H_{\cos}(A) = 0$, i.e.

$$\sum_{i=1}^{n}\left[\cos\frac{(2\mu_A(x_i)-1)}{2}\pi\right] = 0. \tag{3.3}$$

Then, this being the sum of n terms and each term in the summation is non negative, then for all i,

$$\cos\frac{(2\mu_A(x_i)-1)}{2}\pi = 0. \tag{3.4}$$

From (3.4), it is easy to deduce that $\mu_A(x_i) = 0$ or 1 for all $x_i \in X$, that is A is crisp.

P2. ***(Maximality)***: Let $\mu_A(x_i) = 0.5$ for all $x_i \in X$. From (3.1) we obtain $H_{\cos}(A) = 1$.

Now, let $H_{\cos}(A) = 1$, and then also from (3.1), we have

$$\cos\Delta_A = 1 \Rightarrow \Delta_A = 0 \Rightarrow \mu_A(x_i) = 0.5\,\forall\, x_i \in X.$$

P3. ***(Resolution):*** Let

$$f(\mu_A(x_i)) = \cos\frac{(2\mu_A(x_i)-1)}{2}\pi \qquad \forall\, x_i \in X. \tag{3.5}$$

Since $f(\mu_A(x_i))$ is an increasing function of $\mu_A(x_i)$ in the range $[0, 0.5)$ and is a decreasing function of $\mu_A(x_i)$ in the range $(0.5, 1]$, therefore

$$\begin{aligned}\mu_{A^*}(x_i) \le \mu_A(x_i) &\Rightarrow \frac{(2\mu_{A^*}(x_i)-1)}{2}\pi \le \frac{(2\mu_A(x_i)-1)}{2}\pi\\ &\Rightarrow f(\mu_{A^*}(x_i)) \le f(\mu_A(x_i))\,\forall\, x_i \in [0\ ,\ 0.5)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\mu_{A^*}(x_i) \ge \mu_A(x_i) &\Rightarrow \frac{(2\mu_{A^*}(x_i)-1)}{2}\pi \ge \frac{(2\mu_A(x_i)-1)}{2}\pi\\ &\Rightarrow f(\mu_{A^*}(x_i)) \ge f(\mu_A(x_i))\,\forall\, x_i \in (0.5\ ,\ 1]\,.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we have

$$f(\mu_{A^*}(x_i)) \le f(\mu_A(x_i))\,. \tag{3.8}$$

Since $H_{\cos}(A) = \frac{1}{n}\sum_{i=1}^{n}(f(\mu_A(x_i)))$ and $H_{\cos}(A^*) = \frac{1}{n}\sum_{i=1}^{n}(f(\mu_{A^*}(x_i)))$, then we obtain

$$H_{\cos}(A^*) \le H_{\cos}(A)\,. \tag{3.9}$$

P4. ***(Symmetry):*** It is clear from definition of $H_{\cos}(A)$ and with $\mu_{A^C}(x_i) = 1-\mu_A(x_i)$, we conclude that

$$H(A) = H\left(A^C\right). \tag{3.10}$$

Hence $H_{\cos}(A)$ is an axiomatically valid measure of fuzzy entropy.

This proves the theorem. □

We now turn to study of properties of $H_{\cos}(A)$. The proposed cosine fuzzy entropy $H_{\cos}(A)$, satisfies the following interesting properties.

Theorem 3.2. *Let $A, B \in FS(X)$ be given by*

$$A = \{\langle x,\, \mu_A(x)\rangle\,|\, x \in X\}\,,$$
$$B = \{\langle x,\, \mu_B(x)\rangle\,|\, x \in X\}\,,$$

such that they satisfy for any x_i either $A \subseteq B$ or$A \supset B$, then we have

$$H_{\cos}(A \cup B) + H_{\cos}(A \cap B) = H_{\cos}(A) + H_{\cos}(B)\,.$$

Proof. Let us separate X into two parts X_1 and X_2, where

$$X_1 = \{x_i \in X : A \subseteq B\},$$

and

$$X_2 = \{x_i \in X : A \supset B\}.$$

That is, for all $x_i \in X_1$

$$\mu_A(x_i) \leq \mu_B(x_i), \tag{3.11}$$

and for all $x_i \in X_2$

$$\mu_A(x_i) > \mu_B(x_i). \tag{3.12}$$

From definition in (3.1), we have

$$H_{\cos}(A \cup B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(2\mu_{A\cup B}(x_i) - 1)}{2}\pi\right]$$

$$= \frac{1}{n}\left[\left\{\sum_{x_i \in X_1}\cos\frac{(2\mu_B(x_i) - 1)}{2}\pi\right\} + \left\{\sum_{x_i \in X_2}\cos\frac{(2\mu_A(x_i) - 1)}{2}\pi\right\}\right]. \tag{3.13}$$

Again from definition in (3.1), we have

$$H_{\cos}(A \cap B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(2\mu_{A\cap B}(x_i) - 1)}{2}\pi\right]$$

$$= \frac{1}{n}\left[\left\{\sum_{x_i \in X_1}\cos\frac{(2\mu_A(x_i) - 1)}{2}\pi\right\} + \left\{\sum_{x_i \in X_2}\cos\frac{(2\mu_B(x_i) - 1)}{2}\pi\right\}\right]. \tag{3.14}$$

Now adding (3.13) and (3.14), we get

$$H_{\cos}(A \cup B) + H_{\cos}(A \cap B) = H_{\cos}(A) + H_{\cos}(B).$$

This proves the theorem. □

Corollary 3.1. *For any $A \in FS(X)$, and A^C the complement of fuzzy setA, then*

$$H_{\cos}(A) = H_{\cos}\left(A^C\right) = H_{\cos}\left(A \cup A^C\right) = H_{\cos}\left(A \cap A^C\right). \tag{3.15}$$

Proof. This follows from the result $H(A) = H\left(A^C\right)$ and the above theorem. □

In the next section, we propose a new similarity measure between fuzzy sets called '*cosine fuzzy similarity*' and study their properties. We have also given a relation between cosine fuzzy entropy and cosine fuzzy similarity here.

4. Cosine Fuzzy Similarity Measure

In this section, we propose a new similarity measure for FSs. The formal definition is as follows:

Definition 4.1. Cosine Fuzzy Similarity Measure: Given two fuzzy sets A and B defined in $X = \{x_1, x_2, ..., x_n\}$ having the membership values $\mu_A(x_i)$, $i = 1, 2, ..., n$ and $\mu_B(x_i)$, $i = 1, 2, ..., n$ respectively, we define the measure of cosine fuzzy similarity, $S_{FS}(A, B)$, between FSs A and B, as

$$S_{FS}(A, B) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right]. \tag{4.1}$$

In the next theorem, we establish properties that according to Liu [6], justify our proposed measure to be a valid 'fuzzy similarity':

Theorem 4.1. *The $S_{FS}(A,B)$ measure in* (4.1) *of the fuzzy similarity satisfies the following properties:*

[S1.] $0 \le S_{FS}(A,B) \le 1$*;* $S_{FS}(A,B) = S_{FS}(B,A)$*;* $S_{FS}(A,B) = 1$ *if and only if* $A = B$*, i.e.* $\mu_A(x_i) = \mu_B(x_i)$ *for all* $i = 1, 2, ..., n$*. For all* $A, B, C \in FS(X)$*, if* $A \subseteq B \subseteq C$*, then* $S_{FS}(A,C) \le S_{FS}(A,B)$*,* $S_{FS}(A,C) \le S_{FS}(B,C)$*.*

(3) *Proof. S1.* Let $\Delta_{(A,B)} = \frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi$, then from $0 \le \mu_A(x_i), \mu_B(x_i) \le 1$, we have

$$-\frac{\pi}{2} \le \Delta_{(A,B)} \le \frac{\pi}{2} \Rightarrow 0 \le \cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi \le 1 \Rightarrow 0 \le S_{FS}(A,B) \le 1. \tag{4.2}$$

S2. This simply follows from symmetric expression of $S_{FS}(A,B)$.

S3. Let $A = B$, i.e. $\mu_A(x_i) = \mu_B(x_i)$ for all $i = 1, 2, ..., n$. Then from (4.1) we obtain that

$$S_{FS}(A,B) = 1. \tag{4.3}$$

This proves 'if' part of the statement. Next suppose that $S_{FS}(A,B) = 1$, i.e.

$$\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right] = n. \tag{4.4}$$

Then, this being the sum of n terms, each term in the summation being less than or equal to 1, then for all i,

$$\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi = 1 \tag{4.5}$$

or

$$\mu_A(x_i) - \mu_B(x_i) = 0. \tag{4.6}$$

From (4.6), it immediately follows that $\mu_A(x_i) = \mu_B(x_i)$ for any $x_i \in X$, i.e. $A = B$.

S4. Since

$$A \subseteq B \subseteq C \Rightarrow \mu_A(x_i) \le \mu_B(x_i) \le \mu_C(x_i), \tag{4.7}$$

then

$$\left.\begin{array}{l} \frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi \ge \frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi \\ \frac{(\mu_B(x_i) - \mu_C(x_i))}{2}\pi \ge \frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi \end{array}\right\}. \tag{4.8}$$

From (4.8) and the nature of cosine function, we get

$$\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi \ge \cos\frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi \Rightarrow S_{FS}(A,C) \le S_{FS}(A,B), \tag{4.9}$$

$$\cos\frac{(\mu_B(x_i) - \mu_C(x_i))}{2}\pi \ge \cos\frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi \Rightarrow S_{FS}(A,C) \le S_{FS}(B,C). \tag{4.10}$$

This proves the theorem. □

The importance and strength of this measure lies in its properties that we study in the following theorems.
For proofs of the properties, we will consider separation of X into two parts X_1 and X_2, such that

$$X_1 = \{x_i \in X : A \subseteq B\},$$

and

$$X_2 = \{x_i \in X : A \supset B\}.$$

That is, for all $x_i \in X_1$

$$\mu_A(x_i) \leq \mu_B(x_i), \tag{4.11}$$

and for all $x_i \in X_2$

$$\mu_A(x_i) > \mu_B(x_i). \tag{4.12}$$

Theorem 4.2. *For $A, B \in FS(X)$, and if they satisfy that for any $x_i \in X$, either $A \subseteq B$ or $A \supset B$, then*

$$S_{FS}(A \cup B, A \cap B) = S_{FS}(A, B).$$

Proof. Using Definition 4.1, we have

$$\begin{aligned}
&S_{FS}(A \cup B, A \cap B)\\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i) - \mu_{A\cap B}(x_i))}{2}\pi\right]\\
&= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_B(x_i) - \mu_A(x_i))}{2}\pi\right\} + \sum_{x_2\in X_2}\left\{\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right\}\right]\\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right]\\
&= S_{FS}(A, B).
\end{aligned}$$

This proves the theorem. □

Theorem 4.3. *For $A, B, C \in FS(X)$,*
[(i).]$S_{FS}(A \cup B, C) \leq S_{FS}(A, C) + S_{FS}(B, C)$, $S_{FS}(A \cap B, C) \leq S_{FS}(A, C) + S_{FS}(B, C)$.

(2) *Proof.* We prove (i) only, (ii) can be proved analogously.
(i) Let us consider the expressions for

$$S_{FS}(A, C) + S_{FS}(B, C) - S_{FS}(A \cup B, C) \tag{4.13}$$

$$\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - \mu_C(x_i))}{2}\pi\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_B(x_i) - \mu_C(x_i))}{2}\pi\right]\\
&\qquad - \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i) - \mu_C(x_i))}{2}\pi\right]
\end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right]$$
$$- \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\}\right]$$

$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\}\right]$$
$$\geq 0.$$

This proves the theorem. □

Theorem 4.4. *For $A, B, C \in FS(X)$,*

$$S_{FS}(A\cup B, C) + S_{FS}(A\cap B, C) = S_{FS}(A, C) + S_{FS}(B, C).$$

Proof. From Definition 4.1, we first have:

$$S_{FS}(A\cup B, C)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cup B}(x_i)-\mu_C(x_i))}{2}\pi\right]$$

(4.14)
$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\}\right].$$

Next, again from Definition 4.1, we have

$$S_{FS}(A\cap B, C)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_{A\cap B}(x_i)-\mu_C(x_i))}{2}\pi\right]$$

(4.15)
$$= \frac{1}{n}\left[\sum_{x_i\in X_1}\left\{\cos\frac{(\mu_A(x_i)-\mu_C(x_i))}{2}\pi\right\} + \sum_{x_i\in X_2}\left\{\cos\frac{(\mu_B(x_i)-\mu_C(x_i))}{2}\pi\right\}\right].$$

After adding (4.14) and (4.14), we get the result.
This proves the theorem. □

Theorem 4.5. *For $A, B \in FS(X)$,*
[(i).] $S_{FS}(A, B) = S_{FS}(A^C, B^C)$; $S_{FS}(A, B^C) = S_{FS}(A^C, B)$; $S_{FS}(A, B) + S_{FS}(A^C, B) = S_{FS}(A^C, B^C) + S_{FS}(A, B^C)$;

where A^C and B^C represent complements of the fuzzy sets A and B, respectively.

(3) *Proof. (i).* It simply follows from the relation that membership of an element in a set has with its complement.
(ii). Let us consider the expressions for

(4.16) $$S_{FS}(A, B^C) - S_{FS}(A^C, B)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(\mu_A(x_i) - (1-\mu_B(x_i)))}{2}\pi\right] - \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{((1-\mu_A(x_i)) - \mu_B(x_i))}{2}\pi\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(1-\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right] - \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{(1-\mu_A(x_i) - \mu_B(x_i))}{2}\pi\right]$$
$$= 0.$$

(iii). It is obvious from *(i)* and *(ii)*.

This proves the theorem. □

Interestingly, the cosine fuzzy similarity measure given in (4.1) leads to interesting situations when it is consider between a set and its complement. The measure (4.1) reduces to cosine fuzzy entropy (3.1), as shown in the next theorem.

Theorem 4.6. *For each* $A \in FS(X)$,

$$S_{FS}\left(A, A^C\right) = H_{\cos}(A). \tag{4.17}$$

Proof. The proof follows directly from the Definitions 2.3, 3.1 and 4.1. □

5. Conclusions

We have introduced two measures using cosine function. These measures having elegant properties, present a new vista for applications and further considerations.

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