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PARALLEL AND SEMIPARALLEL LIGHTLIKE HYPERSURFACES OF SEMI-RIEMANNIAN SPACE FORMS

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Abstract. In this paper, some properties of lightlike hypersurfaces with parallel and semiparallel second fundamental forms are investigated in semi-Riemannian space forms. Then some generalizations of these conditions are performed.

1. Introduction

The interest on submanifolds with parallel second fundamental forms increased in 1970s. The study on submanifolds with parallel second fundamental form of Euclidean spaces was started by J. Vilms [21] and similar case for hypersurfaces was studied by U. Simon and A. Weinstein [19]. A classification to the submanifolds with parallel second fundamental form of space forms was carried by Takeuchi [20] who makes the term parallel submanifolds more popular, especially from the local point of view. After then parallel submanifolds of Riemannian space forms and non-degenerate ones of semi-Riemannian space forms have been studied in many papers [1], [12, 13, 14], [16]. Later the condition for parallelity was generalized to higher orders and k-parallel submanifolds were introduced [4], [5], [15].

Parallel submanifolds were also extended to a more general class of submanifolds called semiparallel submanifolds. These wider class of submanifolds in Euclidean space was introduced and classified by J. Deprez [2], [3]. F. Dillen has given a classification of semiparallel hypersurfaces of a real space form [6]. Ü. Lumiste has written a book on this subject and its generalization including many of the old and recent studies [11].

Here some conditions related to parallel and semiparallel hypersurfaces are investigated for the degenerate case which is mostly ignored in the mentioned studies. We will use the screen distribution approach of a lightlike hypersurface explained as in the books [7],[9].

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2. Preliminaries

Let (M,g) be a hypersurface of an $(m+2)-$dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ of index $q\in 1,...,m+1$. As for any $p\in M, T_pM$ is a hyperplane of the semi-Euclidean space $(T_p\bar{M},\bar{g}_p)$, we consider

$$T_pM^{\perp}=\{V_p\in T_p\bar{M};\bar{g}_p(V_p,W_p)=0,\forall W_p\in T_pM\},$$

and

$$RadT_pM=T_pM\cap T_pM^{\perp}.$$

Then M is called a lightlike hypersurface of $\bar{M}$ if $RadT_pM\neq\{0\}$ at any $p\in M$. The semi-Riemannian metric $\bar{g}$ on $\bar{M}$ induces on M a symmetric tensor field g of type $(0,2)$, i.e., $g_p(X_p,Y_p)$, for any $p\in M$. Also we know that g has a constant rank m on M and $RadT_pM=TM^{\perp}$ [7].

The tangent bundle space TM of a lightlike hypersurface has the decomposition

$$TM=RadTM\perp S(TM) \tag{2.1}$$

where the complementary vector bundle $S(TM)$ is called the screen distribution on M. So, a lightlike hypersurface (M,g) of a semi-Riemannian manifold $\left(\bar{M},\bar{g}\right)$ is generally shown by $(M,g,S(TM))$. By [7, Theorem 1.1] there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero null section $\xi\in RadTM$ on a coordinate neighborhood $U\subset M$, there exists a unique null section N of $tr(TM)$ on U satisfying

$$\bar{g}\left(\xi,N\right)=1,\bar{g}\left(N,N\right)=\bar{g}\left(N,X\right)=0,\forall X\in\Gamma\left(S\left(TM\right)_{|U}\right)$$

where $tr\left(TM\right)$ and N are called the lightlike transversal vector bundle and the null transversal vector field of M with respect to $S\left(TM\right)$ respectively. Then we have the following decomposition of $T\bar{M}_{|M}$:

$$T\bar{M}|_M=S(TM)\perp\left(RadTM\oplus tr\left(TM\right)\right)=TM\oplus tr\left(TM\right).$$

Let ∇ be the induced connection on the lightlike hypersurface $(M,g,S(TM))$ and P be the projection morphism of TM on $S\left(TM\right)$ with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas are given by

$$\begin{aligned}
\bar{\nabla}_XY &= \nabla_XY+h\left(X,Y\right),\\
\bar{\nabla}_XN &= -A_NX+\nabla^t_XN,\\
\nabla_XPY &= \nabla^*_XPY+h^*\left(X,PY\right),\\
\nabla_X\xi &= -A^*_\xi X-\nabla^{*t}_X\xi,
\end{aligned} \tag{2.2}$$

for any $X,Y\in\Gamma(TM)$, where ∇^*,∇^t and ∇^{*t} are the linear connections on $S(TM), tr(TM)$ and $RadTM$, h and h^* are the second fundamental forms of M and $S(TM)$, A_N and A^*_ξ are the shape operators of M and $S(TM)$ respectively. Locally, suppose ξ,N is a pair of sections on $U\subset M$ satisfying (2). Then define a symmetric $slF(U)-$ bilinear form which is called the local second fundamental form of M and a 1-form τ on $U\subset M$ defined by

$$\begin{aligned}
B(X,Y) &= \bar{g}(h(X,Y),\xi),\\
\tau(X) &= \bar{g}(\nabla^t_XN,\xi)
\end{aligned}$$

for any $X, Y \in \Gamma(TM_{|U})$. It follows that

$$\begin{aligned} h(X,Y) &= B(X,Y)N, \\ \nabla^t_X N &= \tau(X)N, \\ \nabla^{*t}_X \xi &= \bar{g}(\nabla_X \xi, N) = -\bar{g}(\xi, \bar{\nabla}_X N) = -\tau(X)\xi. \end{aligned}$$

Also we define the local screen fundamental form of $S(TM)$ as

$$C(X, PY) = \bar{g}(h^*(X, PY), N).$$

Hence, on U the Gauss and Weingarten equations become

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X,Y)N, \\ \bar{\nabla}_X N &= -A_N X + \tau(X)N, \\ \nabla_X PY &= \nabla^*_X PY + C(X,PY)\xi, \\ \nabla_X \xi &= -A^*_\xi X + \tau(X)\xi, \end{aligned} \tag{2.3}$$

h is independent of the choice of $S(TM)$ and it satisfies the equation

$$h(X, \xi) = 0, \ \forall X \in \Gamma(TM). \tag{2.4}$$

The linear connection ∇ of M is not metric and satisfies the equation

$$(\nabla_X \ g)(Y,Z) = \bar{g}(h(X,Y), Z) + \bar{g}(h(X,Z), Y) \tag{2.5}$$

for any $X, Y, Z \in \Gamma(TM)$. But the connection ∇^* of $S(TM)$ is metric.

The second fundamental forms h and h^* are related to their shape operators with the equations

$$\bar{g}(h(X,Y), \xi) = B(X,Y) = g\left(A^*_\xi X, Y\right), \ \bar{g}\left(A^*_\xi X, N\right) = 0, \tag{2.6}$$

$$\bar{g}(h^*(X,PY), N) = C(X,PY) = g(A_N X, PY), \ \bar{g}(A_N X, N) = 0. \tag{2.7}$$

From (2.6), A^*_ξ is $S(TM)-$valued and self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.8}$$

Covariant derivatives of h and A_N with respect to the connection ∇ are defined as

$$(\nabla_X h)(Y,Z) = \nabla^t_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

$$\nabla_X (A_N Y) = (\nabla_X A_N) Y + A_N (\nabla_X Y). \tag{2.10}$$

The Riemann curvature tensor of a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ is given at [10] by

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &\quad + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z). \end{aligned} \tag{2.11}$$

Then for a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ we get the Gauss curvature equation as

$$R(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} - A_{h(X,Z)}Y + A_{h(Y,Z)}X \tag{2.12}$$

and the Codazzi equation as

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z).$$

For a lightlike hypersurface M of a semi-Euclidean space $\bar{M}$, using the equality $h(X,Y) = B(X,Y)N$ the equation (2.12) becomes

$$R(X,Y)Z = B(X,Z)A_N Y + B(Y,Z)A_N X. \tag{2.13}$$

The Ricci tensor of a lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$ is given at [8] by

$$R^{(0,2)}(X,Y) = mcg(X,Y) + B(X,Y)\, trA_N - B(Y, A_N X). \tag{2.14}$$

Let $(M, g, S(TM))$ be lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. M is totally umbilical, if and only if, locally, on each $U \subset M$ there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall X, Y \in \Gamma(TM|_U) \tag{2.15}$$

is satisfied [7].

For a $(r,s)-$ tensor field T we define the second covariant derivative $(\nabla^2 T)$ as the $(r, s+2)-$tensor field [17]

$$\begin{aligned}(\nabla^2 T)(W_1, ..., W_s; U, V) &= (\nabla^2_{U,V} T)(W_1, ..., W_s) \\ &= \nabla_U((\nabla_V T)(W_1, ..., W_s)) \\ &- (\nabla_{\nabla_U V} T)(W_1, ..., W_s) \\ &- (\nabla_V T)(\nabla_U W_1, ..., W_s) \\ &- ... - (\nabla_V T)(W_1, ..., \nabla_U W_s).\end{aligned} \tag{2.16}$$

3. Parallel and 2-Parallel Lightlike Hypersurfaces

A tensor field is said to be parallel if its covariant derivative vanishes. A hypersurface whose second fundamental form h is parallel, that is $\nabla h = 0$, is called a parallel hypersurface. In general if the second fundamental form h of a hypersurface satisfies the condition

$$\nabla^k h = 0, \quad \nabla^s h \neq 0 \ (s < k),$$

then the hypersurface is said to be $k-$parallel [11]. Thus, a $0-$parallel hypersurface is simply a totally geodesic one and a $1-$paralel hypersurface is parallel that is not totally geodesic.

We already have the following theorem for parallel lightlike hypersurfaces:

Theorem 3.1. [18] *Let M be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then the second fundamental form of M is parallel if and only if M is totally geodesic.*

For the general case the following theorem can be proved.

Theorem 3.2. *There exists no proper totally umbilical $2-$parallel lightlike hypersurface of a semi-Riemannian space form.*

Proof. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. Using the identity (3.1) the second order covariant derivative of the second fundamental form h of M can be found as

$$\begin{aligned}\left(\nabla^2_{V,W} h\right)(X,Y) &= \nabla^t_V\left((\nabla_W h)(X,Y)\right) - (\nabla_W h)(\nabla_V X, Y) \\ &\quad - (\nabla_W h)(X, \nabla_V Y) - (\nabla_{\nabla_V W} h)(X,Y)\end{aligned}$$

for any $X, Y, V, W \in \Gamma(TM)$. If we assume that M is $2-$ parallel, setting $W = X = \xi$, we get

$$\begin{aligned}0 &= \nabla^t_V\left((\nabla_\xi h)(\xi, Y)\right) - (\nabla_\xi h)(\nabla_V \xi, Y) \\ &\quad - (\nabla_\xi h)(\xi, \nabla_V Y) - (\nabla_{\nabla_V \xi} h)(\xi, Y).\end{aligned} \tag{3.1}$$

Substituting (2.4) into (2.9) and using the last equation of (2.2) with (2.8) we obtain $(\nabla_\xi h)(\xi, Y) = 0$ and $(\nabla_\xi h)(\xi, \nabla_V Y) = 0$. Since h is symmetric, from the equation (2.9) we see that ∇h is also symmetric. Then by the Codazzi equation we can write

$$(\nabla_\xi h)(\nabla_V \xi, Y) = (\nabla_{\nabla_V \xi} h)(\xi, Y) = (\nabla_Y h)(\xi, \nabla_V \xi).$$

So the equation (3.1) becomes

$$0 = -2(\nabla_Y h)(\xi, \nabla_V \xi).$$

Again by the equations (2.4),(2.6),(2.8),(2.9),(2.15), the last equation of (2.2) and since the lightlike hypersurface is totally umbilical we obtain the result

$$0 = h(A_\xi^* Y, A_\xi^* V) = B(A_\xi^* Y, A_\xi^* V)N = \rho^2 h(Y, V).$$

Since the second fundamental form of a $2-$parallel lightlike hypersurface can not vanish and $\rho \neq 0$, we get a contradiction and the theorem is proved. □

4. Semiparallel and 2-Semiparallel Lightlike Hypersurfaces

The integrability condition of the differential system $\nabla h = 0$ is given by the equation

$$R(X, Y) \cdot h = 0$$

where $R(X, Y)$ is the curvature operator and h is the second fundamental form. This equation characterizes the semiparallel hypersurfaces. Equivalently, for $X, Y, Z, W \in \Gamma(TM)$ any hypersurface satisfying the equation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0 \tag{4.1}$$

is called a semiparallel hypersurface [18]. As a generalization of this, we consider the following integrability condition of the system $\nabla^k h = 0$:

$$R(X, Y) \cdot \nabla^{k-1} h = 0. \tag{4.2}$$

Hypersurfaces with this condition are said to be $k-$semiparallel. $1-$semiparallel is simply a semiparallel one. We know that non-degenerate parallel hypersurfaces of semi-Riemannian spaces are semiparallel [11]. It is clear that the converse of this is not true. We know the following theorem for the lightlike hypersurfaces of semi-Euclidean spaces:

Theorem 4.1. *Let $(M, g, S(TM))$ be a semiparallel lightlike hypersurface of semi-Euclidean $(n+2)-$space. Then either M is totally geodesic or $C\left(\xi, A_\xi^* U\right) = 0$ for any $U \in (S(TM))$ and $\xi \in \Gamma\left(TM^\perp\right)$, where C and A_ξ^* are the second fundamental form and the shape operator of the screen distribution $S(TM)$, respectively* [18].

This theorem can be extended as to be valid also for Lorentzian space forms:

Theorem 4.2. *Let $(M, g, S(TM))$ be a semiparallel lightlike hypersurface of a Lorentzian space form $(\bar{M}(c), \bar{g})$. Then, for any $Z \in \Gamma(TM)$, either M is totally geodesic or the equation $R^{(0,2)}(\xi, Z) = 0$ is satisfied.*

Proof. Substituting (2.12) in (4.1) we get

$$\begin{aligned} & h\left(R\left(X,Y\right)Z,W\right)+h\left(Z,R\left(X,Y\right)W\right) \\ = \; & c\{g\left(Y,Z\right)B\left(X,W\right)-g\left(X,Z\right)B\left(Y,W\right) \\ & +g\left(Y,W\right)B\left(X,Z\right)-g\left(X,W\right)B\left(Y,Z\right)\} \\ & -B\left(X,Z\right)h\left(A_N Y,W\right)+B\left(Y,Z\right)h\left(A_N X,W\right) \\ & -B\left(X,W\right)h\left(A_N Y,Z\right)+B\left(Y,W\right)h\left(A_N X,Z\right). \end{aligned} \tag{4.3}$$

Since the lightlike hypersurface is semiparallel, setting $X=\xi$ and $Z=W$ in the equation above, with (2.4) and (2.6) we find

$$0=B\left(Y,Z\right)g\left(A_N\xi,A_\xi^* Z\right).$$

Then by the definition of Ricci tensor (2.14) we obtain $g\left(A_N\xi,A_\xi^* Z\right)=R^{(0,2)}\left(\xi,Z\right)$. Hence, either $B=0$, that is M is totally geodesic, or $R^{(0,2)}\left(\xi,Z\right)=0$. □

Corollary 4.1. *Let $(M,g,S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c),\bar{g})$. M is semiparallel if and only if M is semiparallel as a lightlike hypersurface of the ambient semi-Euclidean space.*

Proof. Since M is totally umbilical, substituting (2.15) in (4.3) we get

$$\begin{aligned} & h\left(R\left(X,Y\right)Z,W\right)+h\left(Z,R\left(X,Y\right)W\right)= \\ = \; & c\rho\{g\left(Y,Z\right)g\left(X,W\right)-g\left(X,Z\right)g\left(Y,W\right) \\ & +g\left(Y,W\right)g\left(X,Z\right)-g\left(X,W\right)g\left(Y,Z\right)\} \\ & -B\left(X,Z\right)h\left(A_N Y,W\right)+B\left(Y,Z\right)h\left(A_N X,W\right) \\ & -B\left(X,W\right)h\left(A_N Y,Z\right)+B\left(Y,W\right)h\left(A_N X,Z\right) \\ = \; & -h\left(B\left(X,Z\right)A_N Y-B\left(Y,Z\right)A_N X,W\right) \\ & -h\left(Z,B\left(X,W\right)A_N Y-B\left(Y,W\right)A_N X\right) \end{aligned} \tag{4.4}$$

The result is obvious by the equation above. □

Example 4.1. In Minkowski space $\mathbb{R}_1^{m+2}$ the lightlike cone $\wedge_0^{m+1}$ is given by the equations

$$-\left(x^0\right)^2+\sum_{a=1}^{m+1}\left(x^a\right)^2=0,\ x=\sum_{A=0}^{m+1}x^A\frac{\partial}{\partial x^A}\neq 0.$$

The radical space and the lightlike transversal vector bundle of M are spanned by the lightlike vector fields

$$\xi=\sum_{A=0}^{m+1}x^A\frac{\partial}{\partial x^A}$$

and

$$N=\frac{1}{2\left(x^0\right)^2}\left\{-x^0\frac{\partial}{\partial x^0}+\sum_{a=1}^{m+1}x^a\frac{\partial}{\partial x^a}\right\}$$

respectively, for any $X\in S\left(T\wedge_0^{m+1}\right)$, $X=\sum_{a=1}^{m+1}X^a\frac{\partial}{\partial x^a}$. The lightcone and its screen ditribution $S\left(T\Lambda_0^{m+1}\right)$ are totally umbilical as the equations

$$B\left(X,Y\right)=-g\left(X,Y\right)$$

ve

$$C(X,Y) = -\frac{1}{2(x^0)^2} g(X,Y)$$

are satisfied for any $X, Y \in S\left(T\wedge_0^{m+1}\right)$. Also the Riemann curvature tensor of Λ_0^{m+1} is calculated as

$$R(X,Y)Z = -\frac{1}{2(x^0)^2}\{g(Y,Z)X - g(X,Z)Y\}$$

similar to the given in [7]. Since Λ_0^{m+1} is not totally geodesic, it is not parallel. But using the definition of semiparallelity for any $X, Y, Z, W \in \Gamma(TM)$ we get

$$\begin{aligned}(R(X,Y)\cdot h)(Z,W) &= -h(R(X,Y)Z,W) - h(Z,R(X,Y)W)\\ &= \frac{1}{2(x^0)^2}\{g(Y,Z)h(X,W) - g(X,Z)h(Y,W)\}\\ &\quad +\frac{1}{2(x^0)^2}\{g(Y,W)h(Z,X) - g(X,W)h(Z,Y)\}\end{aligned}$$

and with $B(X,Y) = -g(X,Y)$ we have

$$(R(X,Y)\cdot h)(Z,W) = 0.$$

So Λ_0^{m+1} is semiparallel.

Theorem 4.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian space form $(\bar{M}(c), \bar{g})$. If M is $2-$semiparallel, then either M is totally geodesic or it satisfies the equation $R^{(0,2)}\left(A_\xi^* W, \xi\right) = 0$.*

Proof. From (4.2), if M is $2-$semiparallel, then we get

$$\begin{aligned}0 &= (R(X,Y)\cdot\nabla h)(U,V,W) = (R(X,Y)\cdot\nabla_W h)(U,V)\\ &= -(\nabla_W h)(R(X,Y)U,V) - (\nabla_W h)(U,R(X,Y)V)\\ &\quad -\left(\nabla_{R(X,Y)W}h\right)(U,V)\\ &= -B(Y,U)(\nabla_W h)(A_N X,V) + B(X,U)(\nabla_W h)(A_N Y,V)\\ &\quad -B(Y,V)(\nabla_W h)(U,A_N X) + B(X,V)(\nabla_W h)(U,A_N Y)\\ &\quad -B(Y,W)(\nabla_{A_N X}h)(U,V) + B(X,W)(\nabla_{A_N Y}h)(U,V).\end{aligned}$$

Setting $U = X = \xi$ we have

$$\begin{aligned}0 &= B(Y,V)h(\nabla_W\xi, A_N\xi) + B(Y,W)h(\nabla_{A_N\xi}\xi, V)\\ &= -B(Y,V)h\left(A_\xi^* W, A_N\xi\right) - B(Y,W)h\left(A_\xi^*(A_N\xi), V\right)\end{aligned}$$

and taking $V = W$ it becomes

$$0 = -2B(Y,W)h\left(A_\xi^* W, A_N\xi\right).$$

Hence, either $B = 0$, that is M is totally geodesic, or using (2.14) we see that it satisfies $g\left(A_\xi^* A_\xi^* W, A_N\xi\right) = R^{(0,2)}\left(A_\xi^* W, \xi\right) = 0$. □

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