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ON THE BINOMIAL SUMS OF HORADAM SEQUENCE

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Abstract. The main purpose of this paper is to establish some new properties of Horadam numbers in terms of binomial sums. By that, we can obtain these special numbers in a new and direct way. Moreover, some connections between Horadam and generalized Lucas numbers are revealed to get a more strong result.

1. Introduction

For $a, b, p, q \in \mathbb{Z}$, Horadam [1] considered the sequence $W_n(a, b\ ;\ p, q)$, shortly W_n, which was defined by the recursive equation

$$W_n(a, b\ ;\ p, q) = pW_{n-1} + qW_{n-2} \qquad (n \geq 2), \tag{1.1}$$

where initial conditions are $W_0 = a,\ W_1 = b$ and $n \in \mathbb{N}$.

In equation (1.1), for special choices of $a,\ b,\ p$ and q, the following recurrence relations can be obtained.

- For $a = 0,\ b = 1$, it is obtained generalized Fibonacci numbers:

$$U_n = pU_{n-1} + qU_{n-2}. \tag{1.2}$$

- For $a = 2,\ b = p$, it is obtained generalized Lucas numbers:

$$V_n = pV_{n-1} + qV_{n-2}. \tag{1.3}$$

- Finally, we should note that choosing suitable values on $p,\ q,\ a$ and b in equation (1.1), it is actually obtained others second order sequences such as Fibonacci, Pell, Jacobsthal, Horadam and etc. (for example, see [16] and references therein).

Considering [1] (or [4]), one can clearly obtain the characteristic equation of (1.1) as the form $t^2 - pt - q = 0$ with the roots

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{1.4}$$

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Hence the Binet formula

$$W_n = W_n (a, b\ ;\ p, q) = A\alpha^n + B\beta^n, \tag{1.5}$$

where $A = \frac{b-a\beta}{\alpha-\beta},\ B = \frac{a\alpha-b}{\alpha-\beta}$, can be thought as a solution of the recursive equation in (1.1).

The number sequences have been interested by the researchers for a long time. Recently, there have been so many studies in the literature that concern about subsequences of Horadam numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers. They were widely used in many research areas as Physics, Engineering, Architecture, Nature and Art (see [1-16]). For example, in [7], Taskara et al. examined the properties of Lucas numbers with binomial coefficients.

In [3], they also computed the sums of products of the terms of the Lucas sequence $\{V_{kn}\}$. In addition in [2], the authors established identities involving sums of products of binomial coefficients.

And, in [8], we obtained Horadam numbers with positive and negative indices by using determinants of some special tridiagonal matrices.

In this study, we are mainly interested in some new properties of the binomial sums of Horadam numbers.

2. Main Results

Let us first consider the following lemma which will be needed later in this section. In fact, this lemma enables us to construct a relation between Horadam numbers and generalized Lucas numbers by using their subscripts.

Lemma 2.1. *[3]For $n \geq 1$, we have*

$$W_{ni+i} = V_i W_{ni} - (-q)^i W_{ni-i}. \tag{2.1}$$

Theorem 2.1. *For $n \geq 2$, the following equalities are hold:*

$$W_{ni+i} = V_i^{n-1} W_{2i} - (-q)^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij}.$$

Proof. Let us show this by induction, for $n = 2$, we can write

$$W_{3i} = V_i W_{2i} - (-q)^i W_i,$$

which coincides with equation (2.1). Now, assume that, it is true for all positive integers m, i.e.

$$W_{mi+i} = V_i^{m-1} W_{2i} - (-q)^i \sum_{j=1}^{m-1} V_i^{m-j-1} W_{ij}. \tag{2.2}$$

Then, we need to show that above equality holds for $n = m + 1$, that is,

$$W_{(m+1)i+1} = V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij}. \tag{2.3}$$

By considering the right hand side of equation (2.3), we can expand the summation as

$$\begin{aligned} V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} &= V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m-1} V_i^{m-j} W_{ij} - (-q)^i W_{mi} \\ &= V_i \left(V_i^{m-1} W_{2i} - (-q)^i \sum_{j=1}^{m-1} V_i^{m-j-1} W_{ij} \right) - (-q)^i W_{mi}. \end{aligned}$$

Then, using equation (2.2), we have

$$V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} = V_i W_{mi+i} - (-q)^i W_{mi}$$

Finally, by considering (2.1), we obtain

$$V_i^m W_{2i} - (-q)^i \sum_{j=1}^{m} V_i^{m-j} W_{ij} = W_{(m+1)i+i}$$

which ends up the induction. □

Choosing some suitable values on $a,\ b,\ p$ and q, one can also obtain the sums of the well known Fibonacci, Lucas and etc. in terms of the sum in Theorem 2.1.

Corollary 2.1. *In Theorem 2.1, for special choices of $a,\ b,\ p$ and q, the following results can be obtained for well-known number sequences in literature.*

- For $a = 0,\ b = 1$, it is obtained generalized Fibonacci numbers:
$$U_{ni+i} = V_i^{n-1} U_{2i} - (-q)^i \sum_{j=1}^{n-1} V_i^{n-1-j} U_{ij}.$$
- For $a = 2,\ b = p$, it is obtained generalized Lucas numbers:
$$V_{ni+i} = V_i^{n-1} V_{2i} - (-q)^i \sum_{j=1}^{n-1} V_i^{n-1-j} V_{ij}.$$
- By choosing other suitable values on $a,\ b,\ p$ and q, almost all other special numbers can also be obtained in terms of the sum in Theorem 2.1.

Now, we will show the relation between Horadam numbers and generalized Lucas numbers using binomial sums as follows.

Theorem 2.2. *For $n \geq 2$, the following equalities are satisfied:*

$$W_{ni+i} = \begin{cases} \sum\limits_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum\limits_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}, & i \text{ is odd} \\ \sum\limits_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum\limits_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}, & i \text{ is even.} \end{cases}$$

Proof. There are two cases of subscript i.

Case 1: Let be i is odd. Then, by Theorem 2.1, we can write

$$\begin{aligned} W_{ni+i} &= V_i^{n-1} W_{2i} + q^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij} \\ &= V_i^{n-1} W_{2i} + q^i V_i^{n-2} W_i + q^i V_i^{n-3} W_{2i} + \cdots + q^i W_{(n-1)i}. \end{aligned}$$

We must note that the proof should be investigated for both cases of n.

If n is odd, then we have

$$\begin{aligned} W_{ni+i} &= V_i^{n-2}\left(V_i W_{2i} + q^i W_i\right) + q^i V_i^{n-4}\left(V_i W_{2i} + W_{3i}\right) \\ &\quad + \cdots + q^i V_i \left(V_i W_{(n-3)i} + W_{(n-2)i}\right) + q^i W_{(n-1)i}. \end{aligned} \tag{2.4}$$

Hence, it is given the binomial summation, when the recursive substitutions equation (2.4) by using (2.1),

$$W_{ni+i} = \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}. \tag{2.5}$$

If n is even, then similar approach can be applied to obtain

$$\begin{aligned} W_{ni+i} &= V_i^{n-2}\left(V_i W_{2i} + q^i W_i\right) + q^i V_i^{n-4}\left(V_i W_{2i} + W_{3i}\right) \\ &\quad + \cdots + q^i V_i^0 \left(V_i W_{(n-2)i} + W_{(n-1)i}\right). \end{aligned}$$

and

$$W_{ni+i} = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\frac{n-2}{2}} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij}. \tag{2.6}$$

For the final step, we combine (2.5) and (2.6) to see the equality

$$W_{ni+i} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} V_i^{n-2j} q^{ij} W_i + q^i a \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} V_i^{n-2j-1} q^{ij},$$

as required. Now, for the next case, consider

Case 2: Let be i is even. Then, by Theorem 2.1, we know

$$\begin{aligned} W_{ni+i} &= V_i^{n-1} W_{2i} - q^i \sum_{j=1}^{n-1} V_i^{n-1-j} W_{ij} \\ &= V_i^{n-1} W_{2i} - q^i V_i^{n-2} W_i - q^i V_i^{n-3} W_{2i} - \cdots - q^i W_{(n-1)i}. \end{aligned}$$

and therefore, we write

(2.7)

$$W_{ni+i} = \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum_{j=0}^{\frac{n-1}{2}} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}$$

if n is odd. And we get

(2.8)

$$W_{ni+i} = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum_{j=0}^{\frac{n-2}{2}} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}$$

if n is even. Thus, by combining (2.7) and (2.8), we obtain

$$W_{ni+i} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-1)^j V_i^{n-2j} q^{ij} W_i - q^i a \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (-1)^j V_i^{n-2j-1} q^{ij}.$$

Hence the result follows. □

Choosing some suitable values on $i,\ a,\ b,\ p$ and q, one can also obtain the binomial sums of the well known Fibonacci, Lucas, Pell, Jacobsthal numbers, etc. in terms of binomial sums in Theorem 2.2.

Corollary 2.2. *In Theorem 2.2, for special choices of $i,\ a,\ b,\ p,\ q$, the following result can be obtained.*

- For $i = 1$,
  - For $a = 0$ and $b, p, q = 1$, Fibonacci number
$$F_{n+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j},$$
  - For $a = 2$ and $b, p, q = 1$, Lucas number
$$L_{n+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} + 2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}.$$
  - For $a = 0,\ b = 1,\ p = 2$ and $q = 1$, Pell number
$$P_{n+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} 2^{n-2j}.$$
  - For $a = 0,\ b = 1,\ p = 1$ and $q = 2$, Jacobsthal number
$$J_{n+1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} 2^j.$$
- For $i = 2$,
  - For $a = 0$ and $b, p, q = 1$, Fibonacci number
$$F_{2n+2} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-1)^j 3^{n-2j}.$$
  - For $a = 2$ and $b, p, q = 1$, Lucas number
$$L_{2n+2} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-1)^j 3^{n+1-2j} - 2 \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (-1)^j 3^{n-1-2j}.$$
  - For $a = 0,\ b = 1,\ p = 2$ and $q = 1$, Pell number
$$P_{2n+2} = 2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-j}{j} (-1)^j 6^{n-2j}.$$

* For $a=0$, $b=1$, $p=1$ and $q=2$, Jacobsthal number

$$J_{2n+2}=\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-j}{j}(-1)^{j}\,2^{j}.$$

- By choosing other suitable values on $i,\ a,\ b,\ p$ and q, almost all other special numbers can also be obtained in terms of the binomial sum in Theorem 2.2.

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