Available online: February 27, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 136–148 (2019)
DOI: 10.31801/cfsuasmas.443685
ISSN 1303–5991 E-ISSN 2618-6470
http://communications.science.ankara.edu.tr/index.php?series=A1

GROWTH ESTIMATES OF ENTIRE FUNCTIONS WITH THE HELP OF THEIR RELATIVE L^*-TYPES AND RELATIVE L^*-WEAK TYPES

SANJIB KUMAR DATTA AND TANMAY BISWAS

Abstract. In this paper we attempt to prove some results related to the growth rates of entire functions on the basis of relative L^*-type and relative L^*-weak type of an entire function with respect to another entire function.

1. Introduction

Let $\mathbb{C}$ be the set of all finite complex numbers. For any entire function $f = \sum_{n=0}^{\infty} a_n z^n$ defined on $\mathbb{C}$, the function $M_f(r)$ is defined as

$$M_f(r) = \max_{|z|=r} |f(z)|$$

To start our paper we just recall the following definitions:

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log \log M_f(r)}{\log r} \text{ and } \lambda_f = \lim_{r\to\infty} \inf \frac{\log \log M_f(r)}{\log r}.$$

An entire function for which order and lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

Definition 2. *The type σ_f and lower type $\overline{\sigma}_f$ of an entire function f such that $0 < \sigma_f < 1$ are defined as*

$$\sigma_f = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{r^{\rho_f}}.$$

Received by the editors: June 19, 2017, Accepted: October 31, 2017.
2010 *Mathematics Subject Classification.* 30D35, 30D30, 30D20..
Key words and phrases. Entire function, slowly changing function, relative L^*-order (relative L^*-lower order), relative L^*-type (relative L^*- lower type), relative L^*-weak type.

Datta and Jha [6] introduced the definition of *weak type* of an entire function of finite positive *lower order* in the following way:

Definition 3. [6] *The weak type τ_f and the growth indicator $\overline{\tau}_f$ of an entire function f of finite positive lower order λ_f are defined by*

$$\overline{\tau}_f = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{r^{\lambda_f}}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a.

Somasundaram and Thamizharasi [9] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly. The more generalized concept for L-order and L-type for entire functions are L^*-order and L^*-type. Their definitions are as follows:

Definition 4. [9] *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \lim_{r\to\infty} \sup \frac{\log\log M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \lim_{r\to\infty} \inf \frac{\log\log M_f(r)}{\log\left[re^{L(r)}\right]}.$$

An entire function for which L^*-*order* and L^*-*lower order* are the same is said to be of *regular* L^*-*growth*. Functions which are not of *regular* L^*-*growth* are said to be of *irregular* L^*-*growth*.

Definition 5. [9] *The L^*-type $\sigma_f^{L^*}$ and L^*-lower type $\overline{\sigma}_f^{L^*}$ of an entire function f such that $0 < \rho_f^{L^*} < 1$ are defined as*

$$\sigma_f^{L^*} = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \text{ and } \overline{\sigma}_f^{L^*} = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

In order to determine the growth of two entire functions of same non zero finite L^*-lower order, one may define the L^*-weak type in the following way:

Definition 6. *The L^*-weak type $\tau_f^{L^*}$ of an entire function f such that $0 < \lambda_f^{L^*} < \infty$ are defined as*

$$\tau_f^{L^*} = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}.$$

Likewise the growth indicator $\overline{\tau}_f^{L^*}$ of an entire function f such that $0 < \lambda_f^{L^*} < \infty$ can be defined in the following manner :

$$\overline{\tau}_f^{L^*} = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : (|g(0)|, \to \infty$ exists and is such that

$$\lim_{s \to \infty} M_g^{-1} = \infty$$

In the line of Somasundaram and Thamizharasi [9] and Bernal [1] one may define the relative L^*-order of an entire function in the following manner :

Definition 7. *{ [5], [7]}The relative L^*-order $\rho_g^{L^*}(f)$ and relative L^*-lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\log \left[r e^{L(r)} \right]} \text{ and } \lambda_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\log \left[r e^{L(r)} \right]}.$$

In order to determine the relative growth of two entire functions having same non zero finite relative L^*-order with respect to another entire function, one may define the concept of relative L^*-type and relative L^*-lower type in the following manner:

Definition 8. *The relative L^*-type $\sigma_g^{L^*}(f)$ and relative L^*-lower type $\overline{\sigma}_g^{L^*}(f)$ of an entire function f with respect to g such that $0 < \rho_g^{L^*}(f) < 1$ are defined as follows:*

$$\sigma_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\rho_g^{L^*}(f)}} \text{ and } \overline{\sigma}_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\rho_g^{L^*}(f)}}.$$

Analogously, in order to determine the relative growth of two entire functions having same non zero finite relative L^*-lower order with respect to another entire function, one can define the relative L^*-weak type in the following way:

Definition 9. *The relative L^*-weak type $\tau_g^{L^*}(f)$ of an entire function f with respect to g of finite positive relative L^*-lower order $\lambda_g^{L^*}(f)$ is defined as:*

$$\tau_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\lambda_g^{L^*}(f)}}.$$

Similarly the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function f with respect to another entire function g $0 < \lambda_g^{L^*}(f) < 1$ can be defined in the following manner:

$$\overline{\tau}_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\lambda_g^{L^*}(f)}}.$$

In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis of relative L^*-type and relative L^*-weak type. In fact some works on different relative growth indicators have also been explored by Datta et al {[3], [4]}. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [10].

2. Some Examples

In this section we present some examples in connection with definitions given in the previous section.

Example 1. ***(Order and L^* Order)*** *Given any natural number n, let $f(z) = exp(nz)$. Then $M_f(r) = exp(nr)$. Therefore*

$$\rho_f = \lim_{r\to\infty} \sup \frac{\log\log M_f(r)}{\log r} = 1 \text{ and } \lambda_f = \lim_{r\to\infty} \inf \frac{\log\log M_f(r)}{\log r} = 1.$$

Further we take $L(r) = \log r$; then

$$\rho_f^{L^*} = \lambda_f^{L^*} = \frac{1}{2}.$$

Example 2. ***(Type, Weak type, L* Type and L* weak Type)*** *Let us consider $f(z) = exp(nz)$ for any natural number n. Then*

$$\sigma_f = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{r^{\rho_f}} = \frac{nr}{r} = n \text{ and } \overline{\sigma}_f = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{r^{\rho_f}} = \frac{nr}{r} = n,$$

since $\rho_f = 1$. Similarly

$$\tau_f = \lim_{r\to\infty} \inf \frac{\log M_f(r)}{r^{\lambda_f}} = \frac{nr}{r} = n \text{ and } \overline{\tau}_f = \lim_{r\to\infty} \sup \frac{\log M_f(r)}{r^{\lambda_f}} = \frac{nr}{r} = n.$$

as $\lambda_f = 1$. Further if we take $L(r) = logr$, then $\rho_f^{L^*} = \lambda_f^{L^*} = \frac{1}{2}$ and therefore we get that

$$\sigma_f^{L^*} = \overline{\sigma}_f^{L^*} = \tau_f^{L^*} = \overline{\tau}_f^{L^*} = \infty.$$

Example 3. ***(Relative Order and relative L^* Order)*** *Suppose $f = g = expz.$Therefore*

$$\rho_g(f) = \lambda_g(f) = 1.$$

Further if we take $L(r) = logr$, then

$$\rho_g^{L^*}(f) = \lambda_g^{L^*}(f) = \frac{1}{2}.$$

Example 4. ***(Relative Type, relative weak type etc.)*** *Suppose $f = g = expz.$Therefore*

$$\sigma_g(f) = \lim_{r\to\infty} \sup \frac{\log M_g^{-1} M_f(r)}{r^{\rho_g(f)}} = 1 \text{ and } \overline{\sigma}_g(f) = \lim_{r\to\infty} \inf \frac{\log M_g^{-1} M_f(r)}{r^{\rho_g(f)}} = 1,$$

since $\rho_g(f) = 1$. Likewise

$$\tau_g(f) = \lim_{r\to\infty} \sup \frac{\log M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} = 1 \text{ and } \overline{\tau}_g(f) = \lim_{r\to\infty} \inf \frac{\log M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} = 1,$$

as $\lambda_g(f) = 1$.Further if we take $L(r) = logr$, then $\rho_g^{L^*}(f) = \lambda_g^{L^*}(f) = \frac{1}{2}$ and therefore we obtain that

$$\sigma_g^{L^*}(f) = \overline{\sigma}_g^{L^*}(f) = \tau_g^{L^*}(f) = \overline{\tau}_g^{L^*}(f) = 1.$$

3. Lemmas

First of all let us recall the following theorem due to Datta et al. [2] :

Theorem A Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and $0 \leq \lambda_g \leq \rho_g \leq \infty$. Then

$$\frac{\lambda_f^{L^*}}{\rho_g} \leq \lambda_g^{L^*}(f) \leq \min\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \leq \max\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \leq \rho_g^{L^*}(f) \leq \frac{\rho_f^{L^*}}{\lambda_g}.$$

Now From the conclusion of the above theorem, we present the following two lemmas which will be needed in the sequel.

Lemma 1. *([2]) Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then*

$$\rho_g^{L^*}(f) = \frac{\rho_f^{L^*}}{\rho_g} \text{ and } \lambda_g^{L^*}(f) = \frac{\lambda_f^{L^*}}{\lambda_g}.$$

Lemma 2. *([2]) Let f be an entire function of regular L^*-growth with non zero finite L^*-order and g be an entire function with $0 \leq \lambda_g \leq \rho_g \leq \infty$. Then*

$$\rho_g^{L^*}(f) = \frac{\lambda_f^{L^*}}{\lambda_g} \text{ and } \lambda_g^{L^*}(f) = \frac{\rho_f^{L^*}}{\rho_g}.$$

4. Main Results

In this section we state the main results of the paper.

Theorem 1. *Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then*

$$\begin{aligned}\left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} &\leq \overline{\sigma}_g^{L^*} \leq \min\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \\ &\leq \max\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \sigma_g^{L^*}(f) \leq \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.\end{aligned}$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_f^{L^*}$ and $\overline{\sigma}_f^{L^*}$, we have for all sufficiently large values of r that

$$M_f(r) \leq \exp\left[(\sigma_f^{L^*} + \varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right], \tag{1}$$

$$M_f(r) \geq \exp\left[(\overline{\sigma}_f^{L^*} - \varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right] \tag{2}$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp\left[(\sigma_f^{L^*} - \varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right] \tag{3}$$

$$M_f(r) \leq \exp\left[(\overline{\sigma}_f^{L^*} + \varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right] \tag{4}$$

Similarly from the definitions of σ_g and $\overline{\sigma}_g$, it follows for all sufficiently large values of r that

$$\begin{aligned} M_g(r) &\leq \exp\left[(\sigma_g + \varepsilon)\cdot r^{\rho_g}\right] \\ i.e., r &\leq M_g^{-1}\left[\exp\left[(\sigma_g + \varepsilon)\cdot r^{\rho_g}\right]\right] \end{aligned}$$

$$i.e., M_g^{-1}(r) \geq \left[\left(\frac{\log r}{(\sigma_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \tag{5}$$

$$\begin{aligned} M_g(r) &\geq \exp\left[(\overline{\sigma}_g - \varepsilon)\cdot r^{\rho_g}\right] \\ i.e., r &\geq M_g^{-1}\left[\exp\left[(\overline{\sigma}_g - \varepsilon)\cdot r^{\rho_g}\right]\right] \end{aligned}$$

$$M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \tag{6}$$

and for a sequence of values of r tending to infinity, we obtain that

$$\begin{aligned} M_g(r) &\geq \exp\left[(\sigma_g - \varepsilon)\cdot r^{\rho_g}\right] \\ i.e., r &\geq M_g^{-1}\left[(\sigma_g - \varepsilon)\cdot r^{\rho_g}\right] \end{aligned}$$

$$i.e., M_g^{-1}(r) \leq \left[\left(\frac{\log r}{(\sigma_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right], \tag{7}$$

$$\begin{aligned} M_g(r) &\leq \exp\left[(\overline{\sigma}_g + \varepsilon)\cdot r^{\rho_g}\right] \\ i.e., r &\leq M_g^{-1}\left[\exp\left[(\overline{\sigma}_g + \varepsilon)\cdot r^{\rho_g}\right]\right] \end{aligned}$$

$$i.e., M_g^{-1}(r) \geq \left[\left(\frac{\log r}{(\overline{\sigma}_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right]. \tag{8}$$

Now from (3) and in view of (5), we get for a sequence of values of r tending to infinity that

$$\begin{aligned} M_g^{-1}M_f(r) &\geq M_g^{-1}\left[\exp\left[(\sigma_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\left(\frac{\log\exp\left[(\sigma_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\frac{(\sigma_f^{L^*}-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}} \end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}} \geq \left[\frac{(\sigma_f^{L^*}-\varepsilon)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{9}$$

As $\varepsilon(>0)$ is arbitrary, in view of Lemma 1 it follows that

$$\lim_{r\to\infty}\sup\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \geq \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \sigma_g^{L^*}(f) \geq \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} \tag{10}$$

Analogously from (2) and in view of (8),for a sequence of values of r tending to infinity we get that

$$\begin{aligned} M_g^{-1}M_f(r) &\geq M_g^{-1}\left[\exp\left[(\overline{\sigma}_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\left(\frac{\log\exp\left[(\overline{\sigma}_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\sigma}+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\frac{(\overline{\sigma}_f^{L^*}-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}} \end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}} \geq \left[\frac{(\overline{\sigma}_f^{L^*}-\varepsilon)}{(\overline{\sigma}_g+\varepsilon)}\right]^{\frac{1}{\rho_g}} \tag{11}$$

Since $\varepsilon(>0)$ is arbitrary, we get from above and Lemma (1) that

$$\lim_{r\to\infty}\sup\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}\geq\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \sigma_g^{L^*}(f)\geq\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{12}$$

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$\begin{aligned}
M_g^{-1}M_f(r) &\leq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]\\
i.e., M_g^{-1}M_f(r) &\leq \left[\left(\frac{\log\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{\left(\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}\right]\\
i.e., M_g^{-1}M_f(r) &\leq \left[\frac{\left(\sigma_f^{L^*}+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}
\end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}}\leq\left[\frac{\left(\sigma_f^{L^*}+\varepsilon\right)}{\left(\overline{\sigma}_g-\varepsilon\right)}\right]^{\frac{1}{\rho_g}}. \tag{13}$$

As $\varepsilon(>0)$ is arbitrary, in view of Lemma 1 it follows that

$$\lim_{r\to\infty}\sup\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}\leq\left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \sigma_g^{L^*}(f)\leq\left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}} \tag{14}$$

Again from (2) and in view of (5), we get for all sufficiently large values of r that

$$\begin{aligned} M_g^{-1}M_f(r) &\geq M_g^{-1}\left[\exp\left[\left(\overline{\sigma}_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_f^{L^*}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\sigma_g+\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \\ i.e., M_g^{-1}M_f(r) &\geq \left[\frac{\left(\overline{\sigma}_f^{L^*}-\varepsilon\right)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}} \end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}} \geq \left[\frac{\left(\overline{\sigma}_f^{L^*}-\varepsilon\right)}{(\sigma_g+\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{15}$$

As $\varepsilon(>0)$ is arbitrary, it follows from above and Lemma 1 that

$$\lim_{r\to\infty}\inf\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}} \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \sigma_g^{L^*}(f) \geq \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} \tag{16}$$

Also in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$\begin{aligned} M_g^{-1}M_f(r) &\leq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right] \\ i.e., M_g^{-1}M_f(r) &\leq \left[\left(\frac{\log\exp\left[\left(\sigma_f^{L^*}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\sigma_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right] \\ i.e., M_g^{-1}M_f(r) &\leq \left[\frac{\left(\sigma_f^{L^*}+\varepsilon\right)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}} \end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}} \leq \left[\frac{\left(\sigma_f^{L^*}+\varepsilon\right)}{(\sigma_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}. \tag{17}$$

Since $\varepsilon(>0)$ is arbitrary, we get from Lemma (1) and above that

$$\lim_{r\to\infty}\sup\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}\leq\left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \overline{\sigma}_g^{L^*}(f)\leq\left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}. \tag{18}$$

Similarly from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that

$$\begin{aligned}
M_g^{-1}M_f(r) &\leq M_g^{-1}\left[\exp\left[(\overline{\sigma}_f^{L^*}+\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]\\
i.e., M_g^{-1}M_f(r) &\leq \left[\left(\frac{\log\exp\left[(\overline{\sigma}_f^{L^*}+\varepsilon)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\sigma}_g-\varepsilon)}\right)^{\frac{1}{\rho_g}}\right]\\
i.e., M_g^{-1}M_f(r) &\leq \left[\frac{(\overline{\sigma}_f^{L^*}+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}}\cdot\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}
\end{aligned}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}}\leq\left[\frac{(\overline{\sigma}_f^{L^*}+\varepsilon)}{(\overline{\sigma}_g-\varepsilon)}\right]^{\frac{1}{\rho_g}} \tag{19}$$

Since $\varepsilon(>0)$ is arbitrary, we get from Lemma (1) and above that

$$\lim_{r\to\infty}\sup\frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\rho_g^{L^*}(f)}}\leq\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$

$$i.e., \overline{\sigma}_g^{L^*}(f)\leq\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}. \tag{20}$$

Thus the theorem follows from (10), (12), (14), (16), (18) and (20). □

Theorem 2. *Let f be an entire function of regular L^*-growth with non zero finite L^*-order and g be an entire function with $0\leq\lambda_g\leq\rho_g\leq\infty$. Then*

$$\begin{aligned}
\left[\frac{\tau_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} &\leq \overline{\sigma}_g^{L^*}(f)\leq\min\left\{\left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\}\\
&\leq \max\left\{\left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}},\left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\}\leq\sigma_g^{L^*}(f)\leq\left[\frac{\overline{\tau}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}.
\end{aligned}$$

Proof. Suppose $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\overline{\tau}_f^{L^*}$ and $\tau_f^{L^*}$, we have for all sufficiently large values of r that

$$\begin{aligned} M_f(r) &\leq \exp\left[(\overline{\tau}_f^{L^*}+\varepsilon)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right], \\ M_f(r) &\geq \exp\left[(\tau_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right] \end{aligned}$$

and also for a sequence of values of r tending to infinity, we get that

$$\begin{aligned} M_f(r) &\geq \exp\left[(\overline{\tau}_f^{L^*}-\varepsilon)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right], \\ M_f(r) &\leq \exp\left[(\tau_f^{L^*}+\varepsilon)\left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right]. \end{aligned}$$

Similarly from the definitions of $\overline{\tau}_g$ and τ_g, it follows for all sufficiently large values of r that

$$\begin{aligned} M_g(r) &\leq \exp\left[(\overline{\tau}_g+\varepsilon)\cdot r^{\lambda_g}\right] \\ i.e., r &\leq M_g^{-1}\left[\exp\left[(\overline{\tau}_g+\varepsilon)\cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\overline{\tau}_g+\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right], \end{aligned}$$

$$\begin{aligned} M_g(r) &\geq \exp\left[(\tau_g-\varepsilon)\cdot r^{\lambda_g}\right] \\ i.e., r &\geq M_g^{-1}\left[\exp\left[(\tau_g-\varepsilon)\cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\tau_g-\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \end{aligned}$$

and for a sequence of values of r tending to infinity, we obtain that

$$\begin{aligned} M_g(r) &\geq \exp\left[(\overline{\tau}_g-\varepsilon)\cdot r^{\lambda_g}\right], \\ i.e., r &\geq M_g^{-1}\left[\exp\left[(\overline{\tau}_g-\varepsilon)\cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\leq \left[\left(\frac{\log r}{(\overline{\tau}_g-\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \end{aligned}$$

$$\begin{aligned} M_g(r) &\leq \exp\left[(\tau_g+\varepsilon)\cdot r^{\lambda_g}\right] \\ i.e., r &\leq M_g^{-1}\left[\exp\left[(\tau_g+\varepsilon)\cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\tau_g+\varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]. \end{aligned}$$

Now using the same technique of Theorem (1), one can easily prove the conclusion of the present theorem by the help of Lemma (2) and the above inequalities. Therefore the remaining part of the proof of the present theorem is omitted. □

Theorem 3. *Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then*

$$\begin{aligned} \left[\frac{\tau_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}} &\leq \tau_g^{L^*}(f) \leq \min\left\{\left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}, \left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \\ &\leq \max\left\{\left[\frac{\tau_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_{g(m,p)}}}, \left[\frac{\overline{\tau}_f^{L^*}}{\overline{\tau}_g}\right]^{\frac{1}{\lambda_g}}\right\} \leq \overline{\tau}_g^{L^*}(f) \leq \left[\frac{\overline{\tau}_f^{L^*}}{\tau_g}\right]^{\frac{1}{\lambda_g}}. \end{aligned}$$

Theorem 4. *Let f be an entire function of regular L^*-growth with with non zero finite L^*-order and g be an entire function with $0 \leq \lambda_g \leq \rho_g \leq \infty$. Then*

$$\begin{aligned} \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}} &\leq \tau_g^{L^*}(f) \leq \min\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \\ &\leq \max\left\{\left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}, \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}\right\} \leq \overline{\tau}_g^{L^*}(f) \leq \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\lambda_g}}. \end{aligned}$$

References

[1] Bernal , L. : Orden relative de crecimiento de funciones enteras , Collect.Math., Vol. 39(1988), pp.209-229.

[2] Datta, S. K., Biswas, T. and Sen, P.: Measure of growth of entire functions using relative L^* order, International Journal of Mathematical Science and Engineering Applications (IJMSEA), Vol. 8, No. IV (July, 2014), pp. 125-133.

[3] Datta, S. K, Biswas, T. and Dutta, D. : Generalized relative type and Generalized relative weak type of entire function, Journal of Complex Analysis, Volume 2016, Article ID 3468354, 1 pages, http://ds.doi.org/10.1155/20163468354.

[4] Datta, S. K, Biswas : Measure of growths of entire functions from the view point of their relative L^*-type and relative L^*-weak type, Palestine Journal of Mathematics; Vol 6 (special Issue: II), pp. 149-156.

[5] Datta , S. K. and Biswas, T. : Growth of entire functions based on relative order, International Journal of Pure and Applied Mathematics (IJPAM), Vol.51, No.1 (2009), pp. 49-58.

[6] Datta , S. K. and Jha, A. : On the weak type of meromorphic functions, Int. Math. Forum, Vol. 4, No. 12 (2009), pp. 569-579.

[7] Datta , S. K. and Kar, S. : On the L-order of meromorphic functions based on relative sharing, International Journal of Pure and Applied Mathematics (IJPAM), Vol.56, No.1 (2009), pp. 43-47.

[8] Singh, S. K. and Barker, G. P. : Slowly changing functions and their applications, Indian J. Math., Vol. 19 , No. 1(1977), pp 1-6.

[9] Somasundaram, D. and Thamizharasi, R. : A note on the entire functions of L-bounded index and L type, Indian J. Pure Appl. Math., Vol. 19 , No. 3(March 1988) , pp. 284-293.

[10] Valiron, G. : Lectures on the General Theory of Integral Functions, Chelsea Publishing Company,1949.

Current address: Sanjib Kumar Datta: Department of Mathematics, University of Kalyani P.O.-Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

E-mail address: sanjib_kr_datta@yahoo.co.in

Current address: Tanmay Biswas: Rajbari, Rabindrapalli, R. N. Tagore Road P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.

E-mail address: tanmaybiswas_math@rediffmail.com

ORCID Address: http://orcid.org/0000-0001-6984-6897