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# EMBEDDING THE COMPLEMENT OF A COMPLETE GRAPH IN A FINITE PROJECTIVE PLANE

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Abstract. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a non-trivial regular finite linear space with $v$ points, $v + k$ lines, $k \geq 3$. We show that if $\mathcal{S}$ contains at least $\binom{k}{2}$ lines of size $b(p) - 2$ and one line size $b(p)$ for some point $p$, then $S$ is embeddable in a unique projective plane $\pi$ of order $b(p) - 1$ and $\pi - s$ is a complete graph of order k , where $b(p) \geq 4$ for some point $p$.

**Key Words**: linear space, projective plane, complete graph.
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## 1. Introduction

Linear spaces lie at the foundation of incidence geometry, and more in particular, of finite geometry. A lot of characterizations of projective and affine spaces use linear spaces. Also, many important diagram geometries related to classes of simple groups are build with linear spaces. Linear spaces with constant block size are called Steiner systems and also play a prominent role in finite geometry. But there are also linear spaces that are not Steiner systems, and yet they appear often naturally. One such class of linear spaces is the class of A-affine linear spaces Let us first recall some definitions and results. For more details, (see [1] , [2]).

A *finite linear space* is a pair $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is a finite set of *points* and $\mathcal{L}$ is a family of proper subsets of $\mathcal{P}$, which are called *lines*, such that

(L1) Any two distinct points lie on exactly one line,
(L2) Any line contains at least two points,
(L3) There exist at least two lines.

It is clear that (L3) could be replaced by an axiom $(L3)'$: There are three lines of $\mathcal{S}$ not incident with a common point. In any case, (L3) and $(L3)'$ are 'non-triviality' conditions. Systems satisfying (L1) and (L2) but not (L3) are called *trivial linear spaces.*

In a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$, $v$ and $b$ denote the total number of points and lines, respectively. The degree $b(p)$ of a point $p$ is the total number of lines through $p$, and the size $v(l)$ of a line $l$ is the total number of points on $l$. Thus; if $v(l) = k$ then $l$ is called a $k-$line. The total number of $k-$lines is denoted by $b_k$.

The integer $n$ defined by $n + 1 = \max\{b(p) : p \in \mathcal{P}\}$ is the *order* of a linear space. It is clear that any line of size $n+1$ meets every other line in a linear space of order $n$.The linear spaces with constant point degree is called regular linear spaces.

The numbers $v$, $b$, $v(l)$ and $b(p)$ will be called the *parameters* of $\mathcal{S}$.

A *projective plane* $\pi$ is a linear space in which all lines meet and in which all points are on $n + 1$ lines, $n \geq 2$. The number $n$ is called the order of $\pi$.

An *affine plane* $\mathbb{A}$ is a linear space in which, for any point $p$ not on a line $l$, there is a unique line on $p$ missing $l$, and in which all points are on $n + 1$ lines, $n \geq 2$.

A *k-arc* in a projective plane of order n is a set of k points no three of which are colinear. A k-arc can be thought of as a *complete graph* embedded in the projective plane.

An *hyperoval* is an $(n + 2)-$arc in a projective plane of even order n.

For any line $l$ of a linear space $\mathcal{S}$ of order $n$, the difference $n + 1 - v(l)$ is called a *deficiency* of $l$, denoted $d(l)$. Since the size of any line cannot exceed $n + 1$, the deficiency of any line is non-negative.

Let $\mu$ and $\lambda$ be the respective minimum and maximum deficiencies among those lines of $\mathcal{S}$ which have size less than $n$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a linear space and let $\mathcal{X}$ be a subset of $\mathcal{P}$. Then we can define the linear space $\mathcal{S}' = (\mathcal{X}, \{l \cap \mathcal{X} : l \in \mathcal{L}, |l \cap \mathcal{X}| \geq 2\})$. If $\mathcal{C} = \mathcal{P} - \mathcal{X}$, then $\mathcal{S}'$ is called the *complement* of $\mathcal{C}$ in $\mathcal{S}$ and we say that $\mathcal{S}'$ is obtained by removing $\mathcal{C}$ from $\mathcal{S}$. We denote the complement of $\mathcal{C}$ in $\mathcal{S}$ by $\mathcal{S} - \mathcal{C}$.

Let $\mathcal{X}$ be a set of points in a projective plane $\pi$ of order $n$. Suppose that we remove $\mathcal{X}$ from $\pi$. We obtain a linear space $\pi - \mathcal{X}$ having certain parameters (i.e., the number of points, the number of lines, the point-degrees and line-degrees) (see [1]).

We call any linear space, which has the same parameters as $\pi - \mathcal{X}$, a *pseudo-complement* of $\mathcal{X}$ in $\pi$.

We have already encountered the notation of a pseudo-complement, namely the *pseudo-complement of one line*. This is a linear space with $n^2$ points, $n^2 + n$ lines in which any point has degree $n + 1$ and any line has degree $n$. We know that this is an affine plane, which is a structure embeddable in a projective plane of order $n$.

A linear space with $n^2 + n - m^2 - m$ points, $b = n^2 + n + 1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m + 1$ lines of size $n - m$ will be called the *pseudo-complement of a projective subplane of order* $m$ in a projective plane of order $n$. It is clear that $m < n$.

A linear space with $n^2 + n + 1 - m^2$ points, $b = n^2 + n + 1$ lines, constant point-degree $n + 1$ and containing at least $m^2 + m$ lines of size $n + 1 - m$ will be called the *pseudo-complement of an affine subplane order* $m$ in a projective plane of order $n$. It is clear that $m < n$.

A linear space with $n^2 + n + 1 - k$ points, $b = n^2 + n + 1$ lines, constant point-degree $n+1$ and lines of size $n+1$, $n$ and $n-1$ will be called the *pseudo-complement of a k-arc* in a projective plane of order n.

Two lines $l$ and $l'$ are *parallel* if $l = l'$ or $l \cap l' = \phi$. Two lines $l$ and $l'$ are *disjoint* if $l \cap l' = \phi$.

A *parallel class* in the linear space $(\mathcal{P}, \mathcal{L})$ is a subset of $\mathcal{L}$ with the property that each point of $\mathcal{P}$ is on a unique element of this subset.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ and $\mathcal{S}' = (\mathcal{P}', \mathcal{L}')$ be two finite linear spaces. We say that $\mathcal{S}$ can be *embedded* in $\mathcal{S}'$ if $\mathcal{P} \subseteq \mathcal{P}'$ and $\mathcal{L} = \{l' \cap \mathcal{P} : l' \in \mathcal{L}'$ and $|l' \cap \mathcal{P}| \geq 2\}$. Hall proved in [10] that every finite linear space can be embedded in an infinite projective plane.

The complementation problem with respect to a projective plane is the following: Remove a certain subset of points and lines from the projective plane. Determine the parameters of the resulting space. Now assume that you are starting with a space having these parameters. Does this somehow force this subset to reappear, thus giving an embedding in the original projective plane? A number of people have considered complementation problems $([1], [2], [3], ..., [13])$. In 1970, Dickey solved the problem for the case where the configuration removed was a unital [7]. Batten [2] gave characterizations of linear spaces which are the complement of affine or projective subplanes of finite projective planes.

In this paper, We show that if $\mathcal{S}$ contains at least $\binom{k}{2}$ lines of size $b(p) - 2$ and one line size $b(p)$ for some point $p$, then $S$ is embeddable in a projective plane $\pi$ of order $b(p) - 1$ and $\pi - s$ is a complete graph of order k , where $b(p) \geq 4$ for some point $p$.

## 2. Main Results

**Theorem 2.1.** *If $S$ is a $(n+1)-$regular linear space with $v = n^2 + n + 1 - k$ points, $b = n^2 + n + 1$ lines and contains exactly $k\,(n + 2 - k) > 0$ lines of size $n$, $S$ is uniquely embeddable in a projective plane $\pi$ of order $n$*

*Proof.* Fix an $n-$line $l$. Then the number induces a parallel class of $n + 1$ lines. Let a be the number of $n-$lines in a fixed parallel class. Then

$$an + (n + 1 - a)(n - 1) = n^2 + n + 1 - k$$

It requires that the number of $n-$lines in a parallel class is $n + 2 - k$. Since $b_n = k(n + 2 - k)$, the number of distinct parallel classes is $k$. Consider the structure $S^\star = (P^\star, L^\star)$ where $P^\star$ is $P$ along with the parallel classes and $L^\star$ consist the lines of $L$ extended by those parallel classes to which they belong. We shall prove that $S^\star$ is a linear space: It is clear that two old points (points of $P^\star$) or an old and a new point are one unique line of $L^\star$, since $S$ is a linear space. Let $p$ and $q$ be two new distinct points. We must show that thet determine a unique line of $L^\star$. Let $l_p$ and $l_q$ be $n-$lines which determine the parallel classes corresponding to $p$ and $q$, respectively. If $l_p \cap l_q = \emptyset$, $p = q$ which is a contradiction. So $l_p$ and $l_q$ meet. Each point of $l_q$ is on a unique line of the parallel classes determined by $l_p$. Thus $l_q$ does not meet precisely one line to the parallel class determined by $l_p$. This leaves precisely one line $d$ to parallel to both $l_p$ and $l_q$ such that $p, q \in d$. Thus $S^\star$ is a projective plane of order $n$. Therefore, $S$ can be embedded in a projective plane $\pi$ of order $n$

□

**Theorem 2.2.** *Let $S = (P, L)$ be a non-trivial regular finite linear space with $v$ points and $b$ lines, $3 \leq b - v = k$. If $S$ contains at least $\binom{k}{2}$ lines of size $b(p) - 2$,then $S$ can be embedded in a projective plane $\pi$ of order $b(p) - 1$ and $\pi - S$ is a complete graph of order $k$ embedded a finite projective plane $\pi$ of order $b(p) - 1$ for some point $p \in P$.*

*Proof.* Let $b(p) = n+1$, $b - v = k \geq 3$ for some point $p$ of $S$. By all hypothesis of theorem, $n \geq 2$ and $S$ is a non trivial $(n+1)-$regular linear space with $n^2+n+1-k$ points and $n^2+n+1$ lines. Let $b_i$ be the number of all $i-$lines of $S$. Then also, by simple counting methots,

i) $\sum\limits_i b_i = n^2+n+1$

ii) $\sum\limits_i ib_i = (n+1)(n^2+n+1-k)$

iii) $\sum\limits_i i(i-1)b_i = (n^2+n-k)(n^2+n+1-k)$

iv) $\sum (n-i)\,(n+i-1)b_i = \binom{k}{2}$

Hence However, $S$ has at least $\binom{k}{2}$ lines of size $n-1$, and each of them contributes 2 to the left hand side of the equality iv). Thus $b_i = 0$, $i \neq n+1,\ n,\ n-1$. Therefore, by i)-iv), the lines of $S$ consist of $\binom{k}{2}$ lines of size $n-1$, $k(n+2-k)$ lines of size $n$ and $n^2+n+1+k^2-\binom{k}{2}-(n+2)k$ lines of size $n+1$.

**Case 1.** Let $k < n+2$. In this case, $S$ is the pseudo-complement of a $k-arc$ in a finite peojective plane of order $n$ and $k \leq n+2$ since $b_n \geq 0$, $k \leq n+2$. Therefore by theorem 1, $S$ can be embeded in a projective plane of order $n$. Then $k \leq n+2$

**Case 2.** Let $k = n+2$. In this case, every point is contained in $\frac{n+2}{2}$ lines of size $n+1$ and in $\frac{n}{2}$ lines of size $n-1$. The number of lines size $n-1$ is $\frac{1}{2}(n+2)(n+1)$ and the number of lines of size $n+1$ is $\frac{1}{2}n(n-1)$. Further more a short line of size $n-1$ is parallel to $2n$ other $(n-1)-$lines and $a(n+1)-$lines meets ever other line Fix $a(n-1)-$line $l$ and denote by $\pi(l)$ the set of the $2n$ lines parallel to $l$. It follows from proposition 1.1 that if $\pi(l)$ were to contain a triangle then $n \leq 6$ this case contradiction to $n > 6$. Let $l_1$ and $l_2$ be intersecting lines of $\pi(l)$; denote by $M_1$ the set of lines of $\pi(l)$ which meet $l_2$ and by $M_2$ the set of lines of $\pi(l)$ which meet $l_i$ since $\pi(l)$ contains no triangle, $M_1$ and $M_2$ consists of mutually parallel lines. We have $|M_j| = n-1$ and $l_j \in M_j$. Furthermore $M_1 \cap M_2 =$ (because $\pi(l)$ contains no triangle). Let $d_1$ and $d_2$ be two lines of $\pi(l) - (M_1 \cup M_2)$. We claim that each line of $M_1$ is parallel to at $n-1$ other lines of $\pi(l)$. Then every line of $M_1$ meets at least $n-3$ lines of $M_2$. Therefore, $\pi(l) = (M_1 \cup M_2) \cup \{d_1 \cup d_2\}$ and $M_1$ consist of mutually parallel lines.

$a(n-2)+(n+1-a)n = n^2-2$ and $a = \frac{n+2}{2}$.

$S$ince $a \in \mathbb{Z}$, $n$ is even integer.

The line $d_1$ meets $n-1$ other lines of $\pi(l)$. One of these lines may be $d_2$ but at least $n-2$ of them are in $M_1 \cup M_2$. Therefore without ........of generality, $d_1$ meets at least $\frac{1}{2}(n-2) > 2$, lines of $M_2$. Hence, İf $h$ is an arbitrary line of $M_1$, then $h$ meets a line of $M_2$, which also meets $d_1$. Since $\pi(l)$ has no trianles, two implies that $d_1$ is parallel to $h$. So $d_1$ is parallel to every line of $M_1$. Consequently, $\pi_i$ $M_1 \cup \{l, d_1\}$ is a set of mutually parallel lines with $|\pi_i| = n+1$. In wiew of $v = n^2-1 = |\pi_i| \,.\, (n-1)$, $\pi_i$ is a parallel class. Therefore, $\pi_1 \cap \pi_2 = \{l\}$. If $\alpha$ is the totall number of parallel classes, $\alpha = \frac{2b_{n-1}}{n+1} = n+2$. Thus extension of $S$ is a projective planes of order $n$ and $S$ can be embedded into a projective plane of order $n$ as the complement of a hyperoval.

□

In fact, this case was originally proved by R. C Bose and S. S. Shrikhandle (1973) and then generalized greatly, allowing $n \geq 2$ by P.de Witte (1977)

**Corollary 2.1.** *If $S$ is a non-trivial regular linear space with $b$ lines, $v$ points, $b-v \geq 3$ and at least $\binom{b-v}{2}$ lines of size $\frac{\sqrt{4b-3}-3}{2}$ and at least one point of degree $\frac{\sqrt{4b-3}+1}{2}$, $S$ can be embedded in a projective plane $\pi$ of order $\frac{\sqrt{4b-3}-1}{2}$ and is the pseudo-complement of a $(b-v)-arc$ in a projective plane of order $\frac{\sqrt{4b-3}-1}{2}$*

**Corollary 2.2.** *If $S$ is a non -trivial regular linear space with $v$ points, $b$ lines, $b-v \geq 3$, at least $\binom{b-v}{2}$ lines of size $b(p)-2$ , $S$ can be embedded in a projective plane $\pi$ of order $b(p)-1$ and is the pseudo-complement of a $(b-v)-arc$ in a projective plane of order $b(p)-1$*

**Theorem 2.3.** *Let $S=(P,L)$ be a non-trivial $n+1-$regular linear space having properties follows:*

i) $|P| = n^2+n+1-k,\ |L| = n^2+n+1,\ k \geq 3,\ n \geq 2$
ii) $v(l) \in \{n+1, n, n-1\}$ for each line $l$.

Then $S$ can be embeded in a finite projective plane $\pi$ of order $n$ and $\pi - S$ is the $k-arc$

*Proof.* The proof of this theorem is completely similar to theorem 2.2. □

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