Available online: March 16, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 209–221 (2019)
DOI: 10.31801/cfsuasmas.443732
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

COMMUNICATIONS
SERIES A1

ON COMMUTATIVITY OF PRIME NEAR-RINGS WITH MULTIPLICATIVE GENERALIZED DERIVATION

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Abstract. In the present paper, we shall prove that $3-$prime near-ring N is commutative ring, if any one of the following conditions are satisfied: (i) $f(N) \subseteq Z$, (ii) $f([x,y]) = 0$, (iii) $f([x,y]) = \pm[x,y]$, (iv) $f([x,y]) = \pm(xoy)$, (v) $f([x,y]) = [f(x),y]$, (vi) $f([x,y]) = [x,f(y)]$, (vii) $f([x,y]) = [d(x),y]$, (viii) $f([x,y]) = d(x)oy$,(ix) $[f(x),y] \in Z$ for all $x,y \in N$ where f is a nonzero multiplicative generalized derivation of N associated with a multiplicative derivation d.

1. Introduction

Throughout this paper, N is a left near-ring. A near-ring N is called zero symmetric if $0x = 0$ for all $x \in N$ (recall that left distributive yields $x0 = 0$). Z will represent the multiplicative center of N, that is $Z = \{x \in N \mid yx = xy$ for all $y \in N\}$. A near-ring N is said to be $3-$prime if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x,y \in N$, as usual $[x,y] = xy - yx$ and $xoy = xy + yx$ will denote the well-known Lie and Jordan product, respectively. For terminologies concerning near-rings we refer to G. Pilz [10].

Let R be a ring. An additive mapping $d : R \to R$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x,y \in N$. An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x,y \in R$. Many papers in literature have investigated the commutativity of prime rings satisfying certain functional identities involving derivations or generalized derivations.

Let us introduce the background of investigation about multiplicative derivation. A mapping $d : R \to R$ is called a multiplicative derivation if $d(xy) = xd(y) + d(x)y$ holds for all $x,y \in R$. Of course these maps are not additive. To best of my knowledge, the concept of multiplicative derivation appeared for the first time in the work of Daif [3] motivated by Martindale in [9]. In [7], Goldman and Semrl gave the

Received by the editors: June 16, 2017; Accepted: November 28, 2017.
2010 *Mathematics Subject Classification.* 16Y30, 16W25.
Key words and phrases. Prime near-ring, derivation, multiplicative generalized derivation.

complete description of these maps. Further, Daif and Tammam-El-Sayiad defined the notion of multiplicative generalized derivation in [5]. Further, they extended the notion of multiplicative derivation to multiplicative generalized derivations for rings.

In [4], M. N. Daif and H. E. Bell proved that R is semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x,y]) = \pm[x,y]$ for all $x,y \in U$, then $U \subseteq Z$. Many authors generalized this result replacing derivation d with a generalized derivation or multiplicative derivation. Some of them proved this theorem for some suitable subset of a semiprime ring R.

On the other hand, the study of near-ring with derivation was initiated by H. E. Bell and G. Mason in [1]. During the last thirty years, a lot of work about commutativity of prime near-rings with derivation or generalized derivation had been done (see references for a partial bibliography). In this line of investigation, it is more interesting to study the identities replacing derivation with multiplicative derivation. It is shown first time by Ö. Gölbaşı and Z. Bedir in [6] for near-rings. They obtained the commutative rings $3-$prime near-rings N satisfying some differential identities where d is a multiplicative derivation of N. In the present paper, we shall prove these results for multiplicative generalized derivations of a $3-$prime near-ring N. The results obtained in this paper extend, unify and complement several known results.

2. Results

Definition 1. *Let N be a near-ring and d is a map of N. A mapping $f : N \to N$ (not necessarily additive) is said to be a right multiplicative generalized derivation of N associated with d if*

$$f(xy) = f(x)y + xd(y) \quad \text{for all } x,y \in N$$

and f is said to be a left multiplicative generalized derivation of N associated with d if

$$f(xy) = d(x)y + xf(y) \quad \text{for all } x,y \in N.$$

Finally, f is said to be a multiplicative generalized derivation of N associated with d if it is both a left and right multiplicative generalized derivation of N associated with d.

Lemma 1. [2, Lemma 1.2] *Let N be a $3-$prime near-ring.*

(*i*) *If $z \in Z\setminus\{0\}$, then z is not a zero divisor.*

(*ii*) *If Z contains a nonzero element z for which $z + z \in Z$, then $(N, +)$ is abelian.*

(*iii*) *If $z \in Z\setminus\{0\}$ and $x \in N$* such that $xz \in Z$ or $zx \in Z$, *then $x \in Z$.*

Lemma 2. [2, Lemma 1.5] *Let N be a $3-$prime near ring. If Z contains a nonzero semigroup ideal of N, then N is commutative ring.*

Lemma 3. [8, Lemma 2.1] *A near-ring N admits a multiplicative derivation if and only if it is zero symmetric.*

Lemma 4. [6, Lemma 3] *Let N be a $3-$prime near-ring and $a \in N$. If N admits a nonzero multiplicative derivation d such that $d(N)a = 0$ (or $ad(N) = 0$), then $a = 0$.*

Lemma 5. *Let N be a near-ring and f a multiplicative generalized derivation of N associated with a map d. Then*

$$(d(x)y + xf(y))z = d(x)yz + xf(y)z$$

for all $x, y, z \in N$.

Proof. For all for $x, y, z \in N$, we get

$$f((xy)z) = f(xy)z + xyd(z)$$

and

$$\begin{aligned} f(x(yz)) &= d(x)yz + xf(yz) \\ &= d(x)yz + xf(y)z + xyd(z). \end{aligned}$$

From two expressions of $f(xyz)$, we have

$$f(xy)z = d(x)yz + xf(y)z, \quad \text{for all } x, y, z \in N,$$

and so

$$(d(x)y + xf(y))z = d(x)yz + xf(y)z, \quad \text{for all } x, y, z \in N.$$

□

Lemma 6. *Let N be a $3-$prime near-ring, f a multiplicative generalized derivation of N associated with a map d and $a \in N$. If $af(N) = 0$, then $a = 0$ or $d = 0$.*

Proof. For all $x, y \in N$, by the assumption

$$af(xy) = 0.$$

Expanding this equation and using the hypothesis, we get

$$aNd(y) = 0, \quad \text{for all } y \in N.$$

By the $3-$primeness of N gives that $a = 0$ or $d = 0$. □

Lemma 7. *Let N be a $3-$prime near-ring, f a multiplicative generalized derivation of N associated with a map d and $a \in N$. If $f(N)a = 0$, then $a = 0$ or $d = 0$.*

Proof. By the hypothesis, for all $x, y \in N$,

$$f(xy)a = 0.$$

That is

$$(d(x)y + xf(y))a = 0, \quad \text{for all } x, y \in N.$$

Using Lemma 5 and the hypothesis, we get

$$d(x)Na = 0, \quad \text{for all } x \in N.$$

Since N is $3-$prime, we conclude that $a = 0$ and $d = 0$. □

Theorem 1. *Let N be a $3-$prime near-ring and f a nonzero multiplicative generalized derivation of N associated with a multiplicative derivation d. If $f(N) \subseteq Z$, then N is a commutative ring.*

Proof. Assume that $d(Z) \neq (0)$. Choose $z \in Z$ such that $d(z) \neq 0$. By the assumption, we have

$$f(zx)y = yf(zx), \quad \text{for all } x, y \in N,\ z \in Z.$$

That is

$$(d(z)x + zf(x))\, y = y\,(d(z)x + zf(x)), \quad \text{for all } x, y \in N,\ z \in Z.$$

Using Lemma 5, this can be written

$$d(z)xy + zf(x)y = yd(z)x + yzf(x), \quad \text{for all } x, y \in N,\ z \in Z.$$

Since $z, f(x) \in Z$, we get

$$d(z)xy + f(x)zy = yd(z)x + f(x)zy$$

and so

$$d(z)xy = yd(z)x, \quad \text{for all } x, y \in N,\ z \in Z. \tag{2.1}$$

Taking xt instead of x in this equation, we have

$$d(z)xty = yd(z)xt$$
$$d(z)xty = d(z)xyt$$

and so

$$d(z)N\,[t, y] = 0, \quad \text{for all } y, t \in N,\ z \in Z.$$

By the $3-$primeness of N, we obtain that

$$d(z) = 0,\ z \in Z \text{ or } y \in Z, \quad \text{for all } y \in N. \tag{2.2}$$

Since $d(z) \neq 0$, we obtain that $y \in Z$, for all $y \in N$, and so $N \subseteq Z$. Hence we get N is commutative ring by Lemma 2.

Now we assume $d(Z) = (0)$. Again by the hypothesis, we have

$$f(xy)k = kf(xy), \quad \text{for all } x, y, k \in N.$$

That is

$$(d(x)y + xf(y))\, k = k\,(d(x)y + xf(y)), \quad \text{for all } x, y, k \in N$$

and therefore

$$d(x)yk + xf(y)k = kd(x)y + kxf(y), \quad \text{for all } x, y, k \in N.$$

Using the our hypothesis, we have

$$-kd(x)y + d(x)yk = f(y)\,[k,x]\,, \quad \text{for all } x,y,k \in N.$$

Replacing x by k, we have

$$d(x)yx = xd(x)y, \quad \text{for all } x,y \in N.$$

Writing yt instead of y in this equation and using this, we get

$$d(x)y\,[t,x] = 0$$

and so

$$d(x)N\,[t,x] = 0, \quad \text{for all } x,y,t \in N.$$

By the 3$-$primeness of N, we obtain that

$$d(x) = 0 \ \text{ or } \ x \in Z. \tag{2.3}$$

If $x \in Z$, then $d(x) \in d(Z) = (0)$, and so, we have $d\,(x) = 0$. Thus we arrive at $d(x) = 0$ for both cases. That is $d = 0$, a contradiction. Hence our proof is completed. □

Theorem 2. *Let N be a 3$-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y]) = 0$, for all $x,y \in N$, then N is a commutative ring.*

Proof. Assume that $d\,(Z) \neq (0)$. Choose $z \in Z$ such that $d\,(z) \neq 0$. By the hypothesis

$$f([x,y]) = 0, \ \text{for all } x,y \in N. \tag{2.4}$$

Putting zy instead of y in (2.4), $z \in Z$, we have

$$f([x,zy]) = 0, \ \text{for all } x,y \in N, z \in Z.$$

That is

$$f(z\,[x,y]) = 0, \ \text{for all } x,y \in N, \ z \in Z.$$

Expanding the last equation, we have

$$d(z)\,[x,y] + zf([x,y]) = 0, \ \text{for all } x,y \in N, z \in Z.$$

By the hypothesis, we arrive at

$$d(z)\,[x,y] = 0, \ \text{for all } x,y \in N, \ z \in Z. \tag{2.5}$$

and so

$$d(z)xy = d(z)yx, \ \text{for all } x,y \in N, \ z \in Z.$$

Writing yt instead of y in last equation and using it again, we get

$$d(z)N\,[x,t] = 0, \ \text{for all } x,t \in N, \ z \in Z.$$

Hence, we obtain that

$$d(z) = 0, \ z \in Z \text{ or } x \in Z, \quad \text{for all } x \in N.$$

Since $d(z) \neq 0$, we obtain that $x \in Z$, for all $x \in N$, and so $N \subseteq Z$. Therefore we conclude that N is commutative ring by Lemma 2.

Assume now that $d(Z) = (0)$ holds. Writing xy instead of y in (2.4) and using this, we get

$$f(x[x,y]) = 0, \quad \text{for all } x,y \in N.$$

Expanding this equation, we have

$$d(x)[x,y] + xf([x,y]) = 0, \text{ for all } x,y \in N.$$

Using our hypothesis, we obtain that

$$d(x)[x,y] = 0, \text{ for all } x,y \in N. \tag{2.6}$$

The last expression gives that

$$d(x)xy = d(x)yx, \quad \text{for all } x,y \in N. \tag{2.7}$$

In (2.7), we replace y by yk and use (2.7), we get

$$d(x)N[x,k] = 0, \quad \text{for all } x,k \in N.$$

Since N is a $3-$prime near-ring, we obtain that

$$d(x) = 0 \text{ or } [x,k] = 0, \quad \text{for all } k \in N.$$

If $x \in Z$, then $d(x) \in d(Z) = (0)$, and so we have $d(x) = 0$. Hence, we arrive at $d(x) = 0$ for both cases. That is $d = 0$, a contradiction. The proof is completed. □

Theorem 3. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y]) = \pm[x,y]$, for all $x,y \in N$, then N is a commutative ring.*

Proof. Let $d(Z) \neq (0)$. Choose $z \in Z$ such that $d(z) \neq 0$. By the hypothesis, we get

$$f([x,y]) = \pm[x,y], \quad \text{for all } x,y \in N. \tag{2.8}$$

Taking zy instead of y in this equation, we have

$$f(z[x,y]) = \pm z[x,y], \quad \text{for all } x,y \in N, \; z \in Z.$$

That is

$$d(z)[x,y] + zf([x,y]) = \pm z[x,y], \text{ for all } x,y \in N, z \in Z.$$

Applying the hypothesis in the above equation, we obtain that

$$d(z)[x,y]) = 0, \quad \text{for all } x,y \in N, \; z \in Z.$$

Using the same methods after the equation (2.5) in the proof of Theorem 2, we conclude that N is commutative ring.

Now we assume that $d(Z) = (0)$. Writing xy instead of y in the hypothesis, we have

$$f(x[x,y]) = \pm x[x,y], \quad \text{for all } x,y \in N.$$

Expanding this equation and using the hypothesis, we find that

$$d(x)\,[x,y]=0,\quad \text{for all } x,y\in N.$$

The same argument can be adapted after the equation (2.6) in the proof of Theorem 2. This proves the theorem completely. □

Theorem 4. *Let N be a 3−prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y])=\pm(x\circ y)$, for all $x,y\in N$, then N is a commutative ring.*

Proof. Let assume that $d(Z)\neq(0)$. Choose $z\in Z$ such that $d(z)\neq 0$. Our hypothesis is

$$f([x,y])=\pm(x\circ y),\quad \text{for all } x,y\in N. \tag{2.9}$$

Substituting y by zy, $z\in Z$ in this equation, we get

$$f(z\,[x,y])=\pm z(x\circ y),\quad \text{for all } x,y\in N,\ z\in Z.$$

Since f is a multiplicative generalized derivation of N, we have

$$d\,(z)\,[x,y]+zf\,([x,y])=\pm z(x\circ y),\quad \text{for all } x,y\in N,\ z\in Z.$$

By (2.9), we obtain that

$$d(z)\,[x,y]=0,\quad \text{for all } x,y\in N,\ z\in Z.$$

Applying the same methods in the proof of Theorem 2, we conclude that N is commutative ring.

Let $d\,(Z)=(0)$. Taking xy instead of y in the hypothesis, we obtain that

$$f(x\,[x,y])=\pm x(x\circ y),\quad \text{for all } x,y\in N.$$

Expanding this equation and using the hypothesis, we have

$$d(x)\,[x,y])=0,\quad \text{for all } x,y\in N.$$

This equation is the same as (2.6) in the proof of Theorem 2. Using the same arguments, we conclude that $d=0$, a contradiction. This completes our proof. □

Theorem 5. *Let N be a 3−prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y])=[f(x),y]$, for all $x,y\in N$, then N is a commutative ring.*

Proof. Firstly, we assume that $d\,(Z)\neq(0)$. Choose $z\in Z$ such that $d\,(z)\neq 0$. By the hypothesis, we have

$$f([x,y])=[f(x),y]\,,\quad \text{for all } x,y\in N. \tag{2.10}$$

Writing zy instead of y in this equation, $z\in Z$, we get

$$f([x,zy])=[f(x),zy]\,,\quad \text{for all } x,y\in N,z\in Z.$$

That is

$$f(z\,[x,y])=z\,[f(x),y]\,,\quad \text{for all } x,y\in N,z\in Z.$$

Expanding the above equation, we arrive at

$$d(z)[x,y] + zf([x,y]) = z[f(x),y], \quad \text{for all } x,y \in N, z \in Z.$$

Using the hypothesis, we obtain

$$d(z)[x,y] = 0, \quad \text{for all } x,y \in N, z \in Z.$$

Using the same methods after the equation (2.5) in the proof of Theorem 2, we conclude that N is commutative ring.

Now we assume that $d(Z) = (0)$. Replace y by xy in the hypothesis, we get

$$f([x,xy]) = [f(x),xy],$$
$$f(x[x,y]) = [f(x),xy], \quad \text{for all } x,y \in N.$$

Expanding the above equation and again using the hypothesis, we find that

$$d(x)[x,y] + xf([x,y]) = f(x)xy - xyf(x)$$
$$d(x)[x,y] + x[f(x),y] = f(x)xy - xyf(x)$$
$$d(x)[x,y] + xf(x)y - xyf(x) = f(x)xy - xyf(x),$$

and so

$$d(x)[x,y] + xf(x)y = f(x)xy, \quad \text{for all } x,y \in N. \tag{2.11}$$

On the other hand, replacing $y = 0$ in the hypothesis and using Lemma 3, we arrive at $f(0) = 0$. Again taking x instead of y in (2.10), we get $[f(x),x] = f(0)$. Hence we get $[f(x),x] = 0$, for all $x \in N$. That is

$$f(x)x = xf(x), \quad \text{for all } x \in N.$$

Using this equation in (2.11), we arrive at

$$d(x)[x,y] = 0, \quad \text{for all } x,y \in N.$$

Applying the same techniques in the proof of Theorem 2, we get the required result. □

Theorem 6. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y]) = [x,f(y)]$, for all $x,y \in N$, then N is a commutative ring.*

Proof. We are assuming that $d(Z) \neq (0)$. Choose $z \in Z$ such that $d(z) \neq 0$. By the hypothesis

$$f([x,y]) = [x,f(y)], \quad \text{for all } x,y \in N. \tag{2.12}$$

Taking x by zx in the above equation, $z \in Z$, we have

$$f(z[x,y]) = z[x,f(y)], \quad \text{for all } x,y \in N, z \in Z.$$

Expanding this equation and using the hypothesis, we get

$$d(z)[x,y] = 0, \quad \text{for all } x,y \in N, z \in Z.$$

Applying similar approach with necessary variations in the proof of Theorem 2, we obtain that N is a commutative ring.

Now assuming that $d(Z)=(0)$. Replacing yx instead of x in (2.12), we obtain that

$$f(y[x,y])=[yx,f(y)], \quad \text{for all } x,y\in N.$$

Expanding the above expression and using the hypothesis, we arrive at

$$d(y)[x,y]+yxf(y)-yf(y)x=yxf(y)-f(y)yx, \quad \text{for all } x,y\in N. \tag{2.13}$$

Respectively writing y instead of x in (2.12), we get that $[y,f(y)]=f(0)$ and taking $x=0$ in (2.12), we obtain that $f(0)=0$ by Lemma 3. Therefore, we have $[y,f(y)]=0$, so that

$$f(y)y=yf(y), \quad \text{for all } y\in N.$$

Returning to the equation (2.13) and using this, we obtain

$$d(y)[x,y]=0, \quad \text{for all } x,y\in N.$$

Applying the same methods in the proof of Theorem 2, we get the required result. □

Theorem 7. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x,y])=[d(x),y]$, for all $x,y\in N$, then N is a commutative ring.*

Proof. At first, assume that $d(Z)\neq(0)$. Choose $z\in Z$ such that $d(z)\neq 0$. By the our hypothesis

$$f([x,y])=[d(x),y], \quad \text{for all } x,y\in N. \tag{2.14}$$

Substituting y by zy in (2.14), $z\in Z$, we have

$$f([x,zy])=[d(x),zy]$$

and so

$$f(z[x,y])=z[d(x),y] \text{ for all } x,y\in N, z\in Z.$$

Since f is a multiplicative generalized derivation of N and using our hypothesis, we have

$$d(z)[x,y]=0, \quad \text{for all } x,y\in N, z\in Z.$$

This expression is the same as expression (2.5). Using the same arguments as we used in the proof of Theorem 2, we conclude that N is a commutative ring.

Let $d(Z)=(0)$.Writing xy instead of y in (2.14), we get

$$f(x[x,y])=[d(x),xy], \quad \text{for all } x,y\in N. \tag{2.15}$$

On the other hand, replacing y by x in (2.14), we have $[d(x),x]=f(0)$ and also writing $y=0$ in (2.14), we obtain that $f(0)=0$. Combining these two expressions, we obtain that

$$d(x)x=xd(x) \text{ for all } x\in N.$$

Using the last equation and our hypothesis, we get

$$\begin{aligned}[d(x), xy] &= d(x)\,xy - xyd(x) \\ &= xd(x)\,y - xyd(x) \\ &= x\,[d(x)\,, y] \\ &= xf([x, y])\end{aligned}$$

and so

$$[d(x), xy] = xf([x, y]), \quad \text{for all } x, y \in N. \tag{2.16}$$

Now, (2.15) and (2.16) together imply that

$$\begin{aligned}f(x\,[x, y]) &= xf([x, y]) \\ d(x)[x, y] + xf([x, y]) &= xf([x, y])\end{aligned}$$

and so

$$d(x)\,[x, y] = 0, \quad \text{for all } x, y \in N.$$

Using the same methods in the proof of Theorem 2, we find that $d = 0$, a contradiction. This completes the proof. □

Theorem 8. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $f([x, y]) = d(x)oy$, for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that $d\,(Z) \neq (0)$. Choose $z \in Z$ such that $d\,(z) \neq 0$. By the our hypothesis

$$f([x, y]) = d(x)oy, \quad \text{for all } x, y \in N. \tag{2.17}$$

Taking y by zy in (2.17), $z \in Z$, we have

$$f([x, zy]) = d(x)ozy.$$

Since $z \in Z$, we have

$$f(z\,[x, y]) = z\,(d(x)oy) \ \text{ for all } x, y \in N, z \in Z.$$

Expanding this equation and using equation (2.17), we obtain

$$d\,(z)\,[x, y] = 0, \quad \text{for all } x, y \in N, z \in Z.$$

This expression is the same as expression (2.5). Using the same arguments as we used in the proof of Theorem 2, we conclude that N is a commutative ring.

Now, let $d\,(Z) = (0)$ .Writing xy instead of y in (2.17), we get

$$f(x\,[x, y]) = d(x)oxy, \quad \text{for all } x, y \in N.$$

Hence, we have

$$d\,(x)\,[x, y] + xf([x, y]) = d(x)oxy, \quad \text{for all } x, y \in N.$$

By our hypothesis

$$d\,(x)\,[x, y] + x\,(d(x)oy) = d(x)oxy, \quad \text{for all } x, y \in N.$$

Expanding this equation, we get

$$d(x)xy - d(x)yx + xd(x)y + xyd(x) = d(x)xy + xyd(x)$$

and so

$$xd(x)y = d(x)yx, \text{ for all } x, y \in N. \tag{2.18}$$

Substituting y by yt in (2.18), we have

$$xd(x)yt = d(x)ytx$$

Using (2.18) in last equation, we arrive at

$$d(x)N[x,t] = 0, \quad \text{for all } x, t \in N.$$

Since N is a $3-$prime near-ring, we obtain

$$d(x) = 0 \text{ or } [x,t] = 0, \quad \text{for all } t \in N.$$

Using the same methods as we used in the last paragraph of the proof of Theorem 2, we find that $d = 0$, a contradiction. This completes the proof. □

Theorem 9. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N associated with a multiplicative derivation of d. If $[f(x), y] \in Z$, for all $x, y \in N$, then N is a commutative ring.*

Proof. Assume that

$$[f(x), y] \in Z, \quad \text{for all } x, y \in N. \tag{2.19}$$

Writing $f(x)y$ instead of y, we get

$$f(x)[f(x), y] \in Z.$$

Since $[f(x), y] \in Z$ by Lemma 1 (iii), we have

$$f(x) \in Z \text{ or } [f(x), y] = 0, \quad \text{for all } x, y \in N.$$

If $[f(x), y] = 0$ for all $y \in N$, then we obtain that $f(x) \in Z$. It gives that $f(N) \subseteq Z$ for any cases. Hence the conclusion is obtained by Theorem 1. □

Theorem 10. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N such that $f(x)y = xf(y)$, for all $x, y \in N$, then $d = 0$.*

Proof. By the hypothesis, we have

$$f(x)y = xf(y), \text{ for all } x, y \in N. \tag{2.20}$$

Writing xz instead of x in (2.20), we obtain that

$$f(xz)y = xzf(y), \text{ for all } x, y, z \in N.$$

That is

$$(d(x)z + xf(z))\,y = xzf(y)$$

and so

$$d(x)zy + xf(z)y = xzf(y), \text{ for all } x, y, z \in N. \tag{2.21}$$

Replace x by z in the our hypothesis, we find that $f(z)y = zf(y)$,for all $x, y \in N$. Returning to the equation (2.21) and using last expression, we obtain that

$$d(x)zy + xf(z)y = xf(z)y, \text{ for all } x, y, z \in N.$$

and so

$$d(x)Ny = (0), \text{ for all } x, y \in N.$$

Since N is a $3-$prime near-ring, yields that

$$d(x) = 0 \text{ or } y = 0, \text{ for all } x, y \in N.$$

Hence, we obtain that $d(x) = 0$, for all $x \in N$. That is $d = 0$. This completes the proof. □

Theorem 11. *Let N be a $3-$prime near-ring and f a multiplicative generalized derivation of N such that $f(x)y = xd(y)$, for all $x, y \in N$, then $d = 0$.*

Proof. By our hypothesis, we get

$$f(x)y = xd(y) \text{ for all } x, y \in N. \tag{2.22}$$

Replacing y by yz in (2.22), we arrive at

$$\begin{aligned} f(x)yz &= xd(yz) \\ &= xyd(z) + xd(y)z \text{ for all } x, y, z \in N. \end{aligned}$$

Using equation (2.22), we obtain that

$$f(x)yz = xyd(z) + f(x)yz \text{ for all } x, y, z \in N.$$

This implies that

$$xNd(z) = 0 \text{ for all } x, z \in N.$$

Since N is a $3-$prime near-ring, we find that

$$x = 0 \text{ or } d(z) = 0, \text{ for all } x, z \in N.$$

Thus, we get $d = 0$. This completes the proof. □

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